1. Let $A$ denote any “trivial” finite affine plane. You may assume $|L| \geq 1$ $\forall L \in \mathcal{L}$.

   Assume $A$ contains no lines Then $A$ has 0 or 1 point.

   Assume $A$ contains precisely one line $L$. Assume there exists a point $P$ not on $L$. But now
   by axiom A5 there exists a line $M$ through $P$ parallel to $L$, but this is a contradiction. So all
   the points are on $L$ and there are a finite number of such points.

   Assume $A$ contains two or more lines. If $A$ contains one point $P$, then there are $n$ lines
   through the point $P$. If $A$ contains two or more points, then the points are collinear or there are
   three non-collinear points. If all the points are on $L$ then $L$, or all the other lines are parallel
   and intersect $L$ at a distinct point. If there are three non-collinear points $P_1, P_2, P_3$ then there
   exists lines $P_1P_2, P_1P_3$ etc. and lines $L_1$ parallel to $P_1P_2$ and $L_2$ parallel to $P_1P_3$ such that $P_3$ is
   on $L_1$ and $P_2$ is on $L_2$. Assume $L_1$ is parallel to $L_2$, but this implies there are two lines parallel
   to $L_2$, which is impossible. Therefore $L_1$ and $L_2$ intersect in a point say $P_4$, but now we have a
   triangle. So this case does not arise.

2. 

   (i) $|\phi(P_1)| = |P_2| = |P_1| = n^2$. Therefore $\phi$ is one–to–one on the set of points. Similarly for
   the set of lines.

   (ii) Here we must show that $\phi$ preserves non–incidence. Consider

   \[
   \phi^* : (P_1, L_1) \rightarrow (\phi(P_1), \phi(L_1)).
   \]

   The proof is completed as in Question 2 on Sheet 4.

   (iii) We have to show that $\phi^{-1}$ preserves incidence. Note that if $\phi^{-1}(P)$ is not on $\phi^{-1}(L)$ in $\alpha_1$
   then $\phi(\phi^{-1}(P))$ is not on $\phi(\phi^{-1}(L))$ in $\alpha_2$. So $P$ is not on $L$ in $\alpha_2$. Therefore $\phi^{-1}$ preserves
   incidence and so it must be an isomorphism.

3. $\phi(PQ) = \phi(P)\phi(Q)$ since $\phi$ preserves incidence and is one–to–one. Since $\phi(P) = P$ and
   $\phi(Q) = Q$ it follows that $\phi(PQ) = PQ$.

4. If $\phi$ fixes a pair of intersecting lines $L$ and $M$ of $\alpha$ then $\phi$ fixes the point $L \cap M$ of $\alpha$.

   $\phi(L \cap M) = \phi(L) \cap \phi(M)$ since $\phi$ preserves incidence and is one–to–one and so $\phi(L \cap M) =
   L \cap M$.

5. Let $\pi$ be a projective plane denoted by $(\mathcal{P}, \mathcal{L}, \mathcal{I})$. Let the set $\mathcal{I}$ be the set of ordered pairs
   $(P, L)$ where point $P$ is on line $L$. Go back to lecture notes and revise the definition of the dual of
   $\pi$. Then $\pi^d$ is a projective plane denoted by $(\mathcal{L}, \mathcal{P}, \mathcal{I}^d)$, where $(L, P) \in \mathcal{I}^d \leftrightarrow (P, L) \in \mathcal{I}$. Define
   $\pi_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$ where $\mathcal{I}_0 \subseteq \mathcal{I} \cap (\mathcal{P}_0 \times \mathcal{L}_0)$ and $\pi_0 \subset (\mathcal{L}_0, \mathcal{P}_0, \mathcal{I}_0)$ where $(L, P) \in \mathcal{I}_0 \leftrightarrow (P, L) \in \mathcal{I}_0$.

   Consider $P \in \mathcal{P}_0$, $L \in \mathcal{L}_0$. Suppose $(L, P) \in \mathcal{I}_0$. Then $(P, L) \in \mathcal{I}_0$ and so $(P, L) \in \mathcal{I}$, thus
   $(L, P) \in \mathcal{I}^d$. But $(L, P) \in \mathcal{L}_0 \times \mathcal{P}_0$ and so $(L, P) \in \mathcal{I}^d \cap \mathcal{L}_0 \times \mathcal{P}_0$. Thus $(L, P) \in \mathcal{I}_0^d$ implies
   $(L, P) \in \mathcal{I}^d \cap (\mathcal{L}_0 \times \mathcal{P}_0)$ and so $\mathcal{I}_0^d \subseteq \mathcal{I}^d \cap (\mathcal{L}_0 \times \mathcal{P}_0)$. Thus $\pi_0^d$ is a substructure of $\pi^d$. Now $\pi_0$
   is a projective plane and so $\pi_0^d$ is a projective plane. So $\pi_0^d$ is a projective subplane of $\pi^d$. 

6.(i) The projective plane is isomorphic to

\[
L_1 = \{1, 2, 3, c_1\} \quad L_2 = \{4, 5, 6, c_1\} \quad L_3 = \{7, 8, 9, c_1\} \\
L_4 = \{1, 4, 7, c_2\} \quad L_5 = \{2, 5, 8, c_2\} \quad L_6 = \{3, 6, 9, c_2\} \\
L_7 = \{1, 5, 9, c_3\} \quad L_8 = \{2, 6, 7, c_3\} \quad L_9 = \{3, 4, 8, c_3\} \\
L_{10} = \{1, 6, 8, c_4\} \quad L_{11} = \{2, 4, 9, c_4\} \quad L_{12} = \{3, 5, 7, c_4\} \\
L_{13} = \{c_1, c_2, c_3, c_4\}
\]

If we take the quadrangle \(Q = \{1, 2, 4, 5\}\) then these points together with the \(Q\)-secants form an affine subplane of order 2.

(ii) Note that

- \(c_1\) is on lines 12 and 45
- \(c_2\) is on lines 14 and 25
- \(c_3\) is on line 15
- \(c_4\) is on line 24
- 3 is on line 12
- 6 is on line 45
- 7 is on line 14
- 8 is on line 25
- 9 is on line 24

Every point of \(\pi\) is on at least one line of \(\alpha\). So the set of exterior points of the line set of \(\alpha\) is empty. The set of tangent points of the line set of \(\alpha\) is not empty since, for example, it contains the point 3.

7. Since \(m < n\), for any line \(L\) of \(\pi_0\) there exists a point \(P\) not in \(\mathcal{P}_0\) such that \(P\) is on \(L\) in \(\pi\). Assume that there exists a second line \(M\) on \(P\) which meets \(\pi_0\) in two or more points. Then \(L\) and \(M\) meet in a point of \(\pi_0\). But since two lines intersect in one point this implies that \(P\) is in \(\pi_0\) which is a contradiction. So all lines on \(P\) except \(L\) intersect \(\pi_0\) in at most one point. However, each point of \(\pi_0\) is joined by a line to \(P\), and there are \((m^2 + m + 1) - (m + 1) = m^2\) such points not on \(L\). Therefore

\[
n + 1 \geq m^2 + 1 \quad \text{or} \quad n \geq m^2.
\]

If \(n + 1 > m^2 + 1\), there is a line \(N\) on \(P\) not meeting \(\pi_0\) at all. Since all lines of \(\pi_0\) must intersect in \(\pi_0\) the \(m^2 + m + 1\) lines of \(\pi_0\) meet \(N\) in distinct points. Therefore \(n + 1 \geq m^2 + m + 1\) or

\[
n \geq m^2 + m.
\]

8. \(\sqrt{4} = 2\)

9. Using Question 8 of Sheet 3, we know the diagonal elements of any quadrangle must be colinear. Using Question 3 of Problem Sheet 4 the substructure defined by this quadrangle must be a projective plane of order 2 and \(2^2 = 4\) so each quadrangle is a contained in a Bear subplane.