MATHEMATICS DEPARTMENT: MATH3301 GEOMETRY

Problem Sheet 9: SOLUTIONS

1. Let A denote any "trivial" finite affine plane. You may assume $|L| \geq 1 \; \forall L \in \mathcal{L}$.

Assume A contains no lines Then A has 0 or 1 point.

Assume A contains precisely one line L. Assume there exists a point P not on L. But now by axiom A5 there exists a line M through P parallel to L, but this is a contradiction. So all the points are on L and there are a finite number of such points.

Assume A contains two or more lines. If A contains one point P, then there are n lines through the point P. If A contains two or more points, then the points are collinear or there are three non-collinear points. If all the points are on L then L, or all the other lines are parallel and intersect L at a distinct point. If there are three non–collinear points $P_1,\ P_2,\ P_3$ then there exists lines P_1P_2, P_1P_3 etc. and lines L_1 parallel to P_1P_2 and L_2 parallel to P_1P_3 such that P_3 is on L_1 and P_2 is on L_2. Assume L_1 is parallel to L_2, but this implies there are two lines parallel to L_2, which is impossible. Therefore L_1 and L_2 intersect in a point say P_4, but now we have a triangle. So this case does not arise.

2.

(i) $|\phi(\mathcal{P}_1)| = |\mathcal{P}_2| = |\mathcal{P}_1| = n^2$. Therefore ϕ is one–to–one on the set of points. Similarly for the set of lines.

(ii) Here we must show that ϕ preserves non–incidence. Consider

$$\phi^* \ : \ (P_1, L_1) \to (\phi(P_1), \phi(L_1)).$$

The proof is completed as in Question 2 on Sheet 4.

(iii) We have to show that ϕ^{-1} preserves incidence. Note that if $\phi^{-1}(P)$ is not on $\phi^{-1}(L)$ in α_1 then $\phi(\phi^{-1}(P))$ is not on $\phi(\phi^{-1}(L))$ in α_2. So P is not on L in α_2. Therefore ϕ^{-1} preserves incidence and so it must be an isomorphism.

3. $\phi(PQ) = \phi(P)\phi(Q)$ since ϕ preserves incidence and is one–to–one. Since $\phi(P) = P$ and $\phi(Q) = Q$ it follows that $\phi(PQ) = PQ$.

4. If ϕ fixes a pair of intersecting lines L and M of α then ϕ fixes the point $L \cap M$ of α.

$\phi(L \cap M) = \phi(L) \cap \phi(M)$ since ϕ preserves incidence and is one–to–one and so $\phi(L \cap M) = L \cap M$.

5. Let π be a projective plane denoted by $(\mathcal{P}, \mathcal{L}, \mathcal{I})$. Let the set $\mathcal{I}$ be the set of ordered pairs (P, L) where point P is on line L. Go back to lecture notes and revise the definition of the dual of π. Then π^d is a projective plane denoted by $(\mathcal{L}, \mathcal{P}, \mathcal{I}^d)$, where $(L, P) \in \mathcal{I}^d \leftrightarrow (P, L) \in \mathcal{I}$. Define $\pi_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$ where $\mathcal{I}_0 \subseteq \mathcal{I} \cap (\mathcal{P}_0 \times \mathcal{L}_0)$ and $\pi_0^d = (\mathcal{L}_0, \mathcal{P}_0, \mathcal{I}_0^d)$ where $(L, P) \in \mathcal{I}_0^d \leftrightarrow (P, L) \in \mathcal{I}_0$.

Consider $P \in \mathcal{P}_0$, $L \in \mathcal{L}_0$. Suppose $(L, P) \in \mathcal{I}_0^d$. Then $(P, L) \in \mathcal{I}_0$ and so $(P, L) \in \mathcal{I}$, thus $(L, P) \in \mathcal{I}^d$. But $(L, P) \in \mathcal{L}_0 \times \mathcal{P}_0$ and so $(L, P) \in \mathcal{I}^d \cap \mathcal{L}_0 \times \mathcal{P}_0$. Thus $(L, P) \in \mathcal{I}_0^d$ implies $(L, P) \in \mathcal{I}^d \cap (\mathcal{L}_0 \times \mathcal{P}_0)$ and so $\mathcal{I}_0^d \subseteq \mathcal{I}^d \cap (\mathcal{L}_0 \times \mathcal{P}_0)$. Thus π_0^d is a substructure of π^d. Now π_0 is a projective plane and so π_0^d is a projective plane. So π_0^d is a projective subplane of π^d.

6.(i) The projective plane is isomorphic to

$$\begin{aligned} L_1 &= \{1,2,3,c_1\} & L_2 &= \{4,5,6,c_1\} & L_3 &= \{7,8,9,c_1\} \\ L_4 &= \{1,4,7,c_2\} & L_5 &= \{2,5,8,c_2\} & L_6 &= \{3,6,9,c_2\} \\ L_7 &= \{1,5,9,c_3\} & L_8 &= \{2,6,7,c_3\} & L_9 &= \{3,4,8,c_3\} \\ L_{10} &= \{1,6,8,c_4\} & L_{11} &= \{2,4,9,c_4\} & L_{12} &= \{3,5,7,c_4\} \\ & & & & L_{13} &= \{c_1,c_2,c_3,c_4\} \end{aligned}$$

If we take the quadrangle $\mathcal{Q} = \{1,2,4,5\}$ then these points together with the $\mathcal{Q}$–secants form an affine subplane of order 2.

(ii) Note that

c_1 is on lines 12 and 45	c_2 is on lines 14 and 25	c_3 is on line 15
c_4 is on line 24	3 is on line 12	6 is on line 45
7 is on line 14	8 is on line 25	9 is on line 24

Every point of π is on at least one line of α. So the set of exterior points of the line set of α is empty. The set of tangent points of the line set of α is not empty since, for example, it contains the point 3.

7. Since $m < n$, for any line L of π_0 there exists a point P not in $\mathcal{P}_0$ such that P is on L in π. Assume that there exists a second line M on P which meets π_0 in two or more points. Then L and M meet in a point of π_0. But since two lines intersect in one point this implies that P is in π_0 which is a contradiction. So all lines on P except L interesect π_0 in at most one point. However, each point of π_0 is joined by a line to P, and there are $(m^2+m+1)-(m+1) = m^2$ such points not on L. Therefore

$$n+1 \geq m^2+1 \quad \text{or} \quad n \geq m^2.$$

If $n+1 > m^2+1$, there is a line N on P not meeting π_0 at all. Since all lines of π_0 must intersect in π_0 the m^2+m+1 lines of π_0 meet N in distinct points. Therefore $n+1 \geq m^2+m+1$ or

$$n \geq m^2+m.$$

8. $\sqrt{4} = 2$

9. Using Question 8 of Sheet 3, we know the diagonal elements of any quadrangle must be colinear. Using Question 3 of Problem Sheet 4 the substructure defined by this quadrangle must be a projective plane of order 2 and $2^2 = 4$ so each quadrangle is a contained in a Bear subplane.