A free boundary problem for aggregation by short range sensing and differentiated diffusion

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Abstract.

On the d-dimensional torus we consider the drift-diffusion equation corresponding to the mean field limit of a stochastic model for direct aggregation which features a diffusion coefficient that depends on the locally measured empirical density.

In particular, we consider the situation where the diffusion coefficient may take one of two possible values depending on whether the locally sensed density is below or above a given threshold. This can be interpreted as an aggregation model for particles like insect populations or freely diffusing proteins which slow down their dynamics within dense aggregates. This leads to a free boundary model where the free boundary separates densely packed aggregates from areas with a loose particle concentration.

The paper has a rigorous part and a formal part. In the rigorous part we prove existence of solutions to the distributional formulation of the model. In the second, formal, part we derive the strong formulation of the model including the free boundary conditions and characterize stationary solutions giving necessary conditions for the emergence of stationary plateaus. We conclude that stationary aggregation plateaus in this situation are either spherical, complements of sphericals or stripes, which has implications for biological applications.

Finally, numerical simulations in one and two dimensions are used to give evidence for the long time convergence to stationary states which feature aggregations.

Key words: Parabolic equation with discontinuous coefficients, piecewise constant volatility, aggregation, differentiated diffusion, parabolic free boundary problem.
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1 Introduction

In this paper we study the free-boundary problem induced by the nonlinear parabolic equation with discontinuous coefficients for the mass density $\varrho = \varrho(t, \mathbf{x})$,

$$\frac{\partial \varrho}{\partial t} = \Delta \left(F(\varrho * \mathcal{W})\varrho \right) , \tag{1.1}$$

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where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a piecewise constant, non-increasing function (step function). For simplicity and without loss of generality, we restrict ourselves to the case

$$F(s) := \begin{cases} \overline{F} & \text{for } s \leq \mathcal{C} \,, \\ \underline{F} & \text{for } s > \mathcal{C} \,, \end{cases} \tag{1.2}$$

for the constants $\mathcal{C} > 0$ and $\overline{F} > \underline{F} > 0$. We consider the equation (1.1) on the d-dimensional torus $\mathbb{T}^d$, which we identify with the product of d unit intervals $(0,1) \times \ldots \times (0,1)$ with periodic boundary conditions for ϱ. This is mainly due to mathematical convenience for the analysis below; with appropriate modifications, our approach also works in the full-space setting or on a bounded domain with suitable boundary conditions. The kernel $\mathcal{W} = \mathcal{W}(\mathbf{x}) : \mathbb{T}^d \to \mathbb{R}^+$ is assumed to be smooth enough; moreover, in this paper we only consider radially symmetric kernels $\mathcal{W} = \mathcal{W}(|\mathbf{x}|)$. The convolution $\varrho * \mathcal{W}$ in (1.1) is carried out with respect to the x-variables only, i.e.,

$$\varrho * \mathcal{W}(t, \mathbf{x}) := \int_{\mathbb{T}^d} \varrho(t, \mathbf{x} - \mathbf{y}) \mathcal{W}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Observe that the convolution has to be interpreted taking into account the periodicity of the torus $\mathbb{T}^d$. The problem (1.1) can be seen as a limit of a family of nonlinear drift-diffusion equations which act as a regularization,

$$\frac{\partial \varrho^\varepsilon}{\partial t} = \Delta \left(F^\varepsilon(\varrho^\varepsilon * \mathcal{W}) \varrho^\varepsilon \right), \tag{1.3}$$

where $\{F^\varepsilon\}_{\varepsilon>0}$ is a family of smoothed versions of the function F given by (1.2). In particular, we assume that the functions F^ε are continuously differentiable, monotonically decreasing, bounded in the sense that $0 < \underline{F} \leq F^\varepsilon \leq \overline{F}$ and that they converge pointwise to F as $\varepsilon \to 0$. The usual way to construct the family $\{F^\varepsilon\}_{\varepsilon>0}$ is by convolution of F with the standard mollifier; however, we do not require this particular construction for our forthcoming analysis. Note that a direct dependence of F^ε on the density ϱ^ε would imply negative diffusivity for certain values of ϱ^ε, leading to an ill-posed model.

In [4] the authors studied the problem (1.3) and proved the existence of weak solutions ϱ^ε. Here we will rigorously carry out the limit as $\varepsilon \to 0$ to obtain a weak solution of a distributional formulation of (1.1). Under regularity assumptions on the solution ϱ, we will show that it satisfies a weak formulation of the free boundary problem based on the partition of $[0,T) \times \mathbb{T}^d$ into open, connected sets S_i, $i \in \mathcal{I}$ and D_j, $j \in \mathcal{J}$,where $\mathcal{I}$ and $\mathcal{J}$ are index sets, such that

$$\varrho * \mathcal{W} < \mathcal{C} \text{ on } \bigcup_{i \in \mathcal{I}} S_i \,, \qquad \varrho * \mathcal{W} > \mathcal{C} \text{ on } \bigcup_{j \in \mathcal{J}} D_j \,.$$

Consequently, the sets S_i will be called "sparse plateaus" and the sets D_j "dense plateaus". As a consequence it holds that the limit of $F^\varepsilon(\varrho^\varepsilon * \mathcal{W})$, denoted by $\mathcal{F}$, satisfies $\mathcal{F} \equiv \overline{F}$ on any

S_i and $\mathcal{F} \equiv \underline{F}$ on any D_j. At the interfaces between the plateaus it holds that $\varrho * \mathcal{W} = \mathcal{C}$. The free boundary problem consists of the system

$$\begin{cases} \partial_t \varrho_S = \overline{F} \Delta \varrho_S & \text{on every } S_i,\ i \in \mathcal{I}\ , \\ \partial_t \varrho_D = \underline{F} \Delta \varrho_D & \text{on every } D_j,\ j \in \mathcal{J}\ , \end{cases} \tag{1.4}$$

where we use the symbol ϱ_S when we evaluate ϱ on any sparse plateau S_i and ϱ_D when we evaluate ϱ on any dense plateau D_j. The heat equations on these plateaus are coupled by interface conditions on their mutual interfaces ∂D_j and ∂S_i,

$$\begin{cases} \overline{F} \varrho_S = \underline{F} \varrho_D\ , \\ \vec{n}_j \cdot \begin{pmatrix} \varrho_D \\ -\underline{F} \nabla_x \varrho_D \end{pmatrix} = \vec{n}_j \cdot \begin{pmatrix} \varrho_S \\ -\overline{F} \nabla_x \varrho_S \end{pmatrix}\ , \\ \varrho * \mathcal{W} = \mathcal{C}\ , \end{cases} \tag{1.5}$$

where $\vec{n}_j$ is an outer unit normal vector of D_j, antiparallel to the outer unit normal vector $\vec{n}_i$ of the adjacent sparse plateau S_i. While the last equation in (1.5) corresponds to the partitioning of the domain into level sets, the jump conditions emerge from a weak formulation after applying the divergence theorem. Most notably the second equation corresponds to a Rankine-Hugoniot condition in which the propagation of the interface in time is contained in the outer unit normal.

In [4] the equation (1.3) has been formally derived as the mean field limit of a discrete particle system modeling direct aggregation of biological species: Particle locations are subject to an average density-dependent random walk,

$$\begin{cases} \mathrm{d}\mathbf{X}_i(t) = G(\vartheta_i(t))\, \mathrm{d}\mathbf{B}_i^t\,, \\ \mathbf{X}_i(0) = \mathbf{X}_i^0\ , \qquad\qquad i = 1, \ldots, N\ , \end{cases} \tag{1.6}$$

where $\mathbf{B}_i^t$ are independent d-dimensional Brownian motions, $\mathbf{X}_i^0 \in \mathbb{R}^d$ are independent, identically distributed random variables with law ϱ_0, and $G : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded, smooth, non-increasing and non-negative function. The locally sensed density ϑ_i is given by

$$\vartheta_i(t) = \frac{1}{N} \sum_{j \neq i} \mathcal{W}(\mathbf{X}_i(t) - \mathbf{X}_j(t))\,, \tag{1.7}$$

where $\mathcal{W} = \mathcal{W}(\mathbf{x})$ is the convolution kernel defined above. The model (1.3) with $F^\varepsilon = G^2/2$ represents the mean-field limit of (1.6)–(1.7). This has been shown on the formal level in [4] using a generalized BBGKY-hiearchy approach. A rigorous passage to the mean-field limit can be carried out, based on the methodology developed mainly in [9, 8] and recently extended in [2], however, is out of scope of this paper.

The stochastic model (1.6), (1.7) as well as its macroscopic counterpart (1.3) and the limit model (1.4), (1.5) which is the subject of this study have applications in the modeling of particle dynamics and pattern formation by diffusion sorting in biology and molecular biology. Examples are the modeling of self-organized aggregation in cockroaches for which a detailed agent-based numerical model [6] has been developed. The stochastic model (1.6), (1.7) can be interpreted as a mathematical idealization of the agent-based model [6] which still exhibits spontaneous aggregations and which at the same time facilitates the mathematical study of this phenomenon.

Furthermore these models and most notably the model (1.4), (1.5) which we consider in this study can be used to describe P Granules in *C. elegans* embryos as protein aggregates and thus to model their condensation/decondensation which leads to their preferred localization in the posterior part of the cell upon symmetry breaking. This is motivated by the fact that P Granules have been shown to behave like droplike liquid phases that are in a dynamic equilibrium with soluble components within the cytoplasm [3].

In [4] it has been shown that with appropriate choice of parameters, the constant steady state for (1.3) is unstable and the dynamics leads to formation of well localized clusters. This has also been illustrated with numerical simulations in the spatially 1D and 2D settings. In this paper we show that the limit $\varepsilon \to 0$ in (1.3) leads to a generalized formulation of (1.1), which in the case of strong solutions can be formulated as the free boundary problem (1.4). A study of steady state solutions shows that the dynamics typically leads to creation of plateaus with constrained geometries; for instance, in 2D, the plateaus are circles or stripes. We illustrate this kind of aggregation behavior with numerical simulations of (1.1) in 1D and 2D, and, moreover, with simulations of the discrete particle system (1.6)–(1.7) with a discontinuous response function G.

Our paper is organized as follows: In Section 2 we prove rigorous results, namely the existence of weak solutions to (1.1). This is complemented by formal results presented in the subsequent sections. In Section 3 we provide the strong formulation of (1.1) as a free boundary problem and study its properties, and in Section 4 we characterize its stationary solutions. Finally, in Section 5 we provide results of numerical simulations of the problem in the one- and two-dimensional spatial setting.

2 Existence of solutions

In this Section we prove the existence of weak solutions to (1.1) subject to the initial condition $\varrho(t=0) = \varrho_I \in L^2(\mathbb{T}^d)$. To this end we adopt the following set of assumptions on the family of functions $\{F^\varepsilon\}_{\varepsilon>0}$ and the kernel $\mathcal{W}$, that we call *generic assumptions* in the sequel:

- For all $\varepsilon > 0$, the functions $F^\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ are bounded, non-increasing and continuously differentiable with Lipschitz continuous derivative.
- As $\varepsilon \to 0$, F^ε converge pointwise, almost everywhere on $\mathbb{R}^+$, to F given by (1.2).

- The kernel $\mathcal{W}$ is radially symmetric, $\mathcal{W} \in W^{1,\infty}(\mathbb{T}^d) \cap H^2(\mathbb{T}^d)$ and is normalized as

$$\int_{\mathbb{T}^d} \mathcal{W}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1\,.$$

Our proof relies on the main result of [4] concerning the existence of weak solutions to (1.3):

Theorem 1 (Theorem 1 of [4]) *Fix $\varrho_I \in L^2(\mathbb{T}^d)$, $T > 0$ and $\varepsilon > 0$. Under the above set of generic assumptions, there exists*

$$\varrho^\varepsilon \in L^\infty(0,T;L^2(\mathbb{T}^d)) \cap H^1(0,T;H^{-1}(\mathbb{T}^d))$$

such that $\sqrt{F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)}\varrho^\varepsilon \in L^2(0,T;H^1(\mathbb{T}^d))$ and ϱ^ε is a weak solution of (1.3) in the sense that for every smooth, compactly supported test function $\varphi \in C_c^\infty([0,T) \times \mathbb{T}^d)$ we have

$$\int_0^T \int_{\mathbb{T}^d} \varrho^\varepsilon \frac{\partial \varphi}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{T}^d} \varrho_I\, \varphi(t=0)\,\mathrm{d}\mathbf{x} = \int_0^T \int_{\mathbb{T}^d} \nabla(F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)\varrho^\varepsilon) \cdot \nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,. \qquad (2.1)$$

In the following Lemma we prove an ε-uniform bound on the family of weak solutions ϱ^ε, constructed in Theorem 1. This will be the main tool for passing to the limit as $\varepsilon \to 0$ in (1.3).

Lemma 1 *Let ϱ^ε denote the weak solution of (1.3) in the sense of Theorem 1. Then the family $\{\varrho^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0,T;\mathcal{M}_+(\mathbb{T}^d))$, where $\mathcal{M}_+(\mathbb{T}^d)$ denotes the set of finite positive Radon measures on $\mathbb{T}^d$, and in $L^2(0,T;L^2(\mathbb{T}^d)) \cap L^\infty(0,T;H^{-1}(\mathbb{T}^d))$.*

Proof: Clearly, (1.3) conserves mass, so that there exists a constant $M > 0$ such that $\int_{\mathbb{T}^d} \varrho^\varepsilon(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = M$ almost everywhere in $(0,T)$ for all $\varepsilon > 0$. This implies the uniform boundedness of ϱ^ε in $L^\infty(0,T;\mathcal{M}_+(\mathbb{T}^d))$.

To derive the other bounds, we need a slight modification of the weak formulation (2.1). First, by a standard density argument, we extend the validity of (2.1) to test functions $\varphi \in L^2(0,T;H^2(\mathbb{T}^d)) \cap H^1(0,T;H^1(\mathbb{T}^d))$. Indeed, the terms on the left-hand side of (2.1) are well defined since $\varrho^\varepsilon \in H^1(0,T;H^{-1}(\mathbb{T}^d))$ and due to the embedding $H^1(0,T;H^1(\mathbb{T}^d)) \hookrightarrow C([0,T];H^1(\mathbb{T}^d))$. For the term on the right-hand side we note that $\nabla(F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)\varrho^\varepsilon) \in L^2(0,T;H^{-1}(\mathbb{T}^d))$ due to $\sqrt{F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)}\varrho^\varepsilon \in L^2(0,T;H^1(\mathbb{T}^d))$ and the uniform boundedness of F^ε. The term then makes sense since $\nabla\varphi \in L^2(0,T;H^1(\mathbb{T}^d))$. Consequently, for any fixed $s \in (0,T)$, we can use test functions of the form $\varphi_\delta(t,x) := \varphi(t,x)\chi^\delta_{[0,s]}(t)$, where $\chi^\delta_{[0,s]} \in C^\infty(0,T)$ is a smoothed version of the characteristic function of the interval $[0,s]$, that converges pointwise to $\chi_{[0,s]}$ as $\delta \to 0$. It is then a simple exercise to prove that for any $\varrho \in C^\infty(0,T;H^{-1}(\mathbb{T}^d))$,

$$\int_0^T \int_{\mathbb{T}^d} \varrho \frac{\partial \varphi_\delta}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;\to\; \int_0^s \int_{\mathbb{T}^d} \varrho \frac{\partial \varphi}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_{\mathbb{T}^d} \varrho(s,\mathbf{x})\varphi(s,\mathbf{x})\,\mathrm{d}\mathbf{x} \qquad \text{as } \delta \to 0,$$

and extend by density to $\varrho \in H^1(0,T;H^{-1}(\mathbb{T}^d))$. We conclude that (2.1) implies, for any $s \in (0,T)$,

$$
\begin{aligned}
\int_0^s \int_{\mathbb{T}^d} \varrho^\varepsilon \frac{\partial \varphi}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_{\mathbb{T}^d} \varrho^\varepsilon(s,\mathbf{x})\,\varphi(s,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{T}^d} \varrho_I(\mathbf{x})\,\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} = \\
= \int_0^s \int_{\mathbb{T}^d} \nabla(F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)\varrho^\varepsilon) \cdot \nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,.
\end{aligned}
\tag{2.2}
$$

Now, for almost every $t \in (0,T)$ we construct the function $\varphi^\varepsilon(t) \in H^2(\mathbb{T}^d)$ as the unique solution of

$$
\begin{aligned}
\Delta\varphi^\varepsilon = \varrho^\varepsilon - M \qquad & \text{in } \mathbb{T}^d, \\
\nabla\varphi^\varepsilon \cdot \mathrm{n} = 0 \qquad & \text{on } \partial\mathbb{T}^d, \\
\int_{\mathbb{T}^d} \varphi^\varepsilon \,\mathrm{d}\mathbf{x} = 0. &
\end{aligned}
\tag{2.3}
$$

Note that here we interpret $\mathbb{T}^d$ as the domain $(0,1)^d$ with boundary $\partial\mathbb{T}^d$ and outer normal vector $\mathrm{n}(\mathbf{x})$. The regularity properties of ϱ^ε imply $\varphi^\varepsilon \in L^\infty(0,T;H^2(\mathbb{T}^d)) \cap H^1(0,T;H^1(\mathbb{T}^d))$, so that φ_ε is an admissible test function in (2.2). We then calculate

$$
\int_0^s \int_{\mathbb{T}^d} \varrho^\varepsilon \frac{\partial \varphi^\varepsilon}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^s \int_{\mathbb{T}^d} (\Delta\varphi^\varepsilon + M)\frac{\partial \varphi^\varepsilon}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(s,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} - \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(0,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x},
$$

where the second equality is due to integration by parts in the $\mathbf{x}$-variables and the construction of φ^ε (2.3). Analogous calculations in the other terms of (2.2) lead to

$$
-\frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(s,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(0,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = \int_0^s \int_{\mathbb{T}^d} F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)\varrho^\varepsilon(\varrho^\varepsilon - M)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,.
$$

The right-hand side is estimated from below by

$$
\int_0^s \int_{\mathbb{T}^d} F^\varepsilon(W * \varrho^\varepsilon)\varrho^\varepsilon(\varrho^\varepsilon - M)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \geq \underline{F}\int_0^s \int_{\mathbb{T}^d} (\varrho^\varepsilon)^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \overline{F}\,M^2\,,
$$

which implies

$$
\underline{F}\int_0^T \int_{\mathbb{T}^d} (\varrho^\varepsilon)^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(0,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \overline{F}\,M^2\,.
$$

Consequently, the family $\{\varrho^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^d))$. Moreover, the estimate

$$
\frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(s,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \leq \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(0,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \overline{F}\,M^2 \qquad \text{for a.e. } s \in (0,T)
$$

implies the uniform bound on $\{\varrho^\varepsilon\}_{\varepsilon>0}$ in $L^\infty(0,T;H^{-1}(\mathbb{T}^d))$.

■

In the next Theorem we carry out the limit passage $\varepsilon \to 0$, based on the distributional notion of solutions defined below. Note that, trivially, every weak solution in the sense of Theorem 1 is also a distributional solution.

Definition 1 *We call $\varrho \in L^\infty(0,T;\mathcal{M}_+(\mathbb{T}^d))$ a distributional solution of (1.3) subject to the initial datum $\varrho_I \in \mathcal{M}_+(\mathbb{T}^d)$, if for every smooth, compactly supported test function $\varphi \in C_c^\infty([0,\infty) \times \mathbb{T}^d)$,*

$$\int_0^\infty \int_{\mathbb{T}^d} \frac{\partial \varphi}{\partial t} \varrho \,\mathrm{d}\mathbf{x} \,\mathrm{d}t + \int_{\mathbb{T}^d} \varphi(t=0)\varrho_I \,\mathrm{d}\mathbf{x} = -\int_0^\infty \int_{\mathbb{T}^d} (F^\varepsilon(\mathcal{W} * \varrho)\Delta\varphi)\varrho \,\mathrm{d}\mathbf{x} \,\mathrm{d}t\,, \tag{2.4}$$

where we denote by $\varrho \,\mathrm{d}\mathbf{x}$ the integration with respect to the measure $\varrho(t,\cdot) \in \mathcal{M}_+(\mathbb{T}^d)$.

Theorem 2 *Fix the initial datum $\varrho_I \in L^2(\mathbb{T}^d)$ and a family of functions $\{F^\varepsilon\}_{\varepsilon>0}$ satisfying the generic assumptions. Let ϱ^ε denote the weak solutions of (1.3) according to Theorem 1. Then, for a fixed $T > 0$ there exists a sub-sequence of the family $\{\varrho^\varepsilon\}_{\varepsilon>0}$, denoted again by $\{\varrho^\varepsilon\}_{\varepsilon>0}$, and $\varrho \in L^2(0,T;L^2(\mathbb{T}^d)) \cap L^\infty(0,T;H^{-1}(\mathbb{T}^d))$, such that, as $\varepsilon \to 0$,*

$$\begin{aligned} \varrho^\varepsilon \rightharpoonup \varrho \quad &\textit{weakly in} \quad L^2(0,T;L^2(\mathbb{T}^d)),\\ F^\varepsilon(\mathcal{W} * \varrho^\varepsilon) \to \mathcal{F} \quad &\textit{strongly in} \quad L^2(0,T;L^2(\mathbb{T}^d)), \end{aligned}$$

where

$$\mathcal{F}(t,\mathbf{x}) \in \begin{cases} \{\overline{F}\} & \textit{on the set } \{\varrho * \mathcal{W}(t,\mathbf{x}) < \mathcal{C}\},\\ \{\underline{F}\} & \textit{on the set } \{\varrho * \mathcal{W}(t,\mathbf{x}) > \mathcal{C}\},\\ [\underline{F},\overline{F}] & \textit{on the set } \{\varrho * \mathcal{W}(t,\mathbf{x}) = \mathcal{C}\}. \end{cases} \tag{2.5}$$

Moreover, ϱ verifies the distributional formulation

$$\int_0^\infty \int_{\mathbb{T}^d} \frac{\partial \varphi}{\partial t} \varrho \,\mathrm{d}\mathbf{x} \,\mathrm{d}t + \int_{\mathbb{T}^d} \varphi(t=0)\varrho_I \,\mathrm{d}\mathbf{x} = -\int_0^\infty \int_{\mathbb{T}^d} (\mathcal{F}\Delta\varphi)\varrho \,\mathrm{d}\mathbf{x} \,\mathrm{d}t \tag{2.6}$$

with any test function $\varphi \in C_c^\infty([0,T) \times \mathbb{T}^d)$.

Proof: Due to the uniform bound of Lemma 1 there is a weakly converging sub-sequence of $\{\varrho^\varepsilon\}_{\varepsilon>0}$, which we denote again by $\{\varrho^\varepsilon\}_{\varepsilon>0}$, with the limit $\varrho \in L^2(0,T;L^2(\mathbb{T}^d))$. Moreover, due to the assumed uniform boundedness of the family F^ε, $F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)$ is uniformly bounded in $L^\infty((0,T)\times\mathbb{T}^d)$, and we have weak-* convergence of a sub-sequence to some $\mathcal{F} \in L^\infty(0,T)\times\mathbb{T}^d)$. To obtain more information on $\mathcal{F}$ we show that ϱ^ε is uniformly Lipschitz continuous in time with respect to the weak H^{-2} topology on $\mathbb{T}^d$. Indeed, let us construct the test function $\varphi(t,\mathbf{x}) = \alpha(t)\psi(\mathbf{x})$, where $\psi \in C_c^\infty(\mathbb{T}^d)$ and α is a smoothed version of the characteristic function of the interval $[t_1, t_2]$ with $0 < t_1 < t_2 < T$, such that $\alpha(0) = 0$. Using this φ as a test function in (2.4), we obtain

$$\int_0^\infty \int_{\mathbb{T}^d} \frac{\partial \alpha}{\partial t} \psi \varrho^\varepsilon \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = - \int_0^\infty \int_{\mathbb{T}^d} (F^\varepsilon(\mathcal{W} * \varrho^\varepsilon) \Delta \psi) \alpha \varrho^\varepsilon \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \,.$$

Since $\varrho^\varepsilon \in H^1(0, T; H^{-1}(\mathbb{T}^d))$ due to Theorem 1, we may integrate by parts with respect to time on the left-hand side and remove the smoothing in α to obtain

$$\int_{\mathbb{T}^d} [\varrho^\varepsilon(t_2) - \varrho^\varepsilon(t_1)] \psi \, \mathrm{d}\mathbf{x} = \int_{t_1}^{t_2} \int_{\mathbb{T}^d} (F^\varepsilon(\mathcal{W} * \varrho^\varepsilon) \Delta \psi) \varrho^\varepsilon \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \,.$$

Due to the uniform boundedness of the family F^ε, $F^\varepsilon(\mathcal{W} * \varrho^\varepsilon)\varrho^\varepsilon$ is uniformly bounded in $L^2(0, T; L^2(\mathbb{T}^d))$, so that we have

$$\left| \int_{\mathbb{T}^d} [\varrho^\varepsilon(t_2) - \varrho^\varepsilon(t_1)] \psi \, \mathrm{d}\mathbf{x} \right| \leq \overline{F} \sqrt{t_2 - t_1} \, \|\Delta \psi\|_{L^2(\mathbb{T}^d)} \, \|\varrho^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^d))}$$

for every $\psi \in H^2(\mathbb{T}^d)$. This uniform Hoelder continuity in time implies then, by the Arzela-Ascoli Theorem, the convergence of ϱ^ε to ϱ in $C(0, T; H^2(\mathbb{T}^d)')$, where the superscript $'$ refers to the respective dual space. Then, due to the assumption $\mathcal{W} \in H^2(\mathbb{T}^d)$, we can take $\psi(\cdot) := \mathcal{W}(\mathbf{x} - \cdot)$ for every $\mathbf{x} \in \mathbb{T}^d$ and conclude the pointwise in $\mathbf{x}$, uniform in t convergence of $\mathcal{W} * \varrho^\varepsilon$,

$$\sup_{t \in [0,T]} |\mathcal{W} * \varrho^\varepsilon(t, \mathbf{x}) - \mathcal{W} * \varrho(t, \mathbf{x})| \to 0 \,, \qquad \text{as } \varepsilon \to 0 \,, \quad \text{for all } \mathbf{x} \in \mathbb{T}^d \,.$$

Combined with the pointwise convergence of F^ε to F, given by (1.2), we readily get

$$F^\varepsilon(\mathcal{W} * \varrho^\varepsilon) \to \mathcal{F} \qquad \text{pointwise in } (0, T) \times \mathbb{T}^d,$$

for a function $\mathcal{F}$ satisfying (2.5). The pointwise convergence together with uniform boundedness implies then, by the Lebesgue theorem,

$$F^\varepsilon(\mathcal{W} * \varrho^\varepsilon) \to \mathcal{F} \qquad \text{strongly in } L^2(0, T; L^2(\mathbb{T}^d)).$$

Finally, based on the above facts, we may pass to the limit $\varepsilon \to 0$ in (2.4) to conclude that ϱ verifies the distributional formulation (2.6).

■

3 Strong formulation of the free boundary problem

This and the following Section will be devoted to formal results. In this section we derive the strong formulation of the limiting problem (2.6), under the additional assumption that the $(d + 1)$-dimensional Lebesgue measure of the level set $\{(t, \mathbf{x}); \varrho * \mathcal{W}(t, \mathbf{x}) = \mathcal{C}\}$ vanishes and the set is a d-rectifiable curve. Clearly, this is related to smoothness properties of the level set function $\varrho * \mathcal{W}$, and according to Theorem 2.5(i) of [1], Lipschitz continuity of $\varrho * \mathcal{W}$ is sufficient for this assumption to be verified for a.e. $C > 0$. Lipschitz continuity with respect to the $\mathbf{x}$-variables is guaranteed due to the uniform bound on ϱ in $L^\infty(0, T; L^1(\mathbb{T}^d))$ given by mass conservation, as long as $\mathcal{W}$ is Lipschitz continuous. However, Lipschitz continuity of $\varrho * \mathcal{W}$ with respect to time requires appropriate time regularity of ϱ, which we are not able to establish.

Under the above assumption, we have a partition of $(0, T) \times \mathbb{T}^d$ into open, connected sets S_i, $i \in \mathcal{I}$ and D_j, $j \in \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ are index sets, such that

$$\varrho * \mathcal{W} < \mathcal{C} \text{ on } \bigcup_{i \in \mathcal{I}} S_i \,, \qquad \varrho * \mathcal{W} > \mathcal{C} \text{ on } \bigcup_{j \in \mathcal{J}} D_j \,. \tag{3.1}$$

Consequently, the sets S_i will be called "sparse plateaus" and the sets D_j "dense plateaus" in the sequel. At the interfaces between the plateau it holds that $\varrho * \mathcal{W} = \mathcal{C}$. The classical solution of the free boundary problem has classical derivatives on every plateau S_i and D_j, but it might have jumps at the interfaces. When we evaluate ϱ at any sparse plateau, we will occasionally denote it by ϱ_S, as opposed to the symbol ϱ_D which we occasionally use when evaluating ϱ on any set D_j. Moreover, note that we have $\mathcal{F} \equiv \overline{F}$ on any S_i and $\mathcal{F} \equiv \underline{F}$ on any D_j.

The weak formulation of the limit problem, (2.6), if satisfied by a classical solution ϱ, reads

$$\begin{aligned} \sum_{i \in \mathcal{I}} \iint_{S_i} \left(\frac{\partial \varphi}{\partial t} + \overline{F} \Delta \varphi \right) \varrho_S \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \sum_{j \in \mathcal{J}} \iint_{D_j} \left(\frac{\partial \varphi}{\partial t} + \underline{F} \Delta \varphi \right) \varrho_D \,\mathrm{d}\mathbf{x}\,\mathrm{d}t& \\ + \int_{\mathbb{T}^d} \varphi(t=0) \varrho_I \,\mathrm{d}\mathbf{x} = 0& \end{aligned} \tag{3.2}$$

for any test function $\varphi \in C_c^\infty([0, T) \times \mathbb{T}^d)$. To derive the strong formulation we define

$$\vec{a} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

and formulate the differential operators as

$$\partial_t \varphi(t, \mathbf{x}) = \vec{a} \cdot \nabla_{t,\mathbf{x}} \varphi(t, \mathbf{x}) = \nabla_{t,\mathbf{x}} \cdot (\vec{a} \varphi(t, \mathbf{x})) \quad \text{and} \quad \Delta_{\mathbf{x}} \varphi(t, \mathbf{x}) = \nabla_{t,\mathbf{x}} \cdot (A \, \nabla_{t,\mathbf{x}} \varphi(t, \mathbf{x})) \,.$$

Hence, for any dense plateau D_j we have

$$\begin{aligned}
\iint_{D_j} \left(\frac{\partial\varphi}{\partial t} + \underline{F}\,\Delta\varphi\right) \varrho_D \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \iint_{D_j} \nabla_{t,\mathbf{x}} \cdot (\vec{a}\,\varphi + \underline{F} A \nabla_{t,\mathbf{x}}\varphi)\, \varrho_D \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \iint_{D_j} \left[-\vec{a}\cdot\nabla_{t,\mathbf{x}}\varrho_D + \underline{F}\,\nabla_{t,\mathbf{x}}\cdot(A\nabla_{t,\mathbf{x}}\varrho_D)\right]\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + \int_{\partial D_j} \left[\varrho_D\,\varphi\,\vec{a} + \underline{F}\,\varrho_D\, A\nabla_{t,\mathbf{x}}\varphi - \underline{F}\,\varphi A \nabla_{t,\mathbf{x}}\varrho_D\right]\cdot \vec{n}_j \,\mathrm{d}S(\mathbf{x},t) \\
= \iint_{D_j} \left[-\partial_t\varrho_D + \underline{F}\,\Delta_{\mathbf{x}}\varrho_D\right]\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &+ \int_{\partial D_j} \left[\begin{pmatrix} \varrho_D \\ -\underline{F}\,\nabla_{\mathbf{x}}\varrho_D \end{pmatrix}\cdot \vec{n}_j\,\varphi + \begin{pmatrix} 0 \\ \underline{F}\nabla_{\mathbf{x}}\varphi \end{pmatrix} \cdot \vec{n}_j\,\varrho_D\right] \mathrm{d}S(\mathbf{x},t)\,,
\end{aligned} \tag{3.3}$$

where $\vec{n}_j$ is the outer unit normal vector to ∂D_j. Note that the above integration by parts can be carried out due to the assumption of the rectifiability of the level set $\{(t,\mathbf{x}); \varrho*\mathcal{W}(t,\mathbf{x}) = \mathcal{C}\}$. An analogous computation for any sparse plateau S_i gives

$$\begin{aligned}
\iint_{S_i} \left(\frac{\partial\varphi}{\partial t} + \overline{F}\,\Delta\varphi\right) \varrho_S \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \iint_{S_i} \left[-\partial_t\varrho_S + \overline{F}\,\Delta_{\mathbf{x}}\varrho_S\right]\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_{\partial S_i} \left[\begin{pmatrix} \varrho_S \\ -\overline{F}\,\nabla_{\mathbf{x}}\varrho_S \end{pmatrix}\cdot (-\vec{n}_j)\varphi + \begin{pmatrix} 0 \\ \overline{F}\nabla_{\mathbf{x}}\varphi \end{pmatrix} \cdot (-\vec{n}_j)\,\varrho_S\right] \mathrm{d}S(\mathbf{x},t)\,,
\end{aligned} \tag{3.4}$$

where the outer normal vector to ∂S_i is written as $(-\vec{n}_j)$, with $\vec{n}_j$ the respective outer normal vector to the boundary of the neighboring dense plateau ∂D_j. Due to (3.2), the sum of (3.3) over $j \in \mathcal{J}$ plus the sum of (3.4) over $i \in \mathcal{I}$ vanishes for every test function φ compactly supported in $\{t > 0\} \times \mathbb{T}^d$. This restriction is motivated by the fact that we define the plateaus as open sets without intersection with $\{t = 0\}$. By choosing test functions compactly supported in single plateaus we obtain the system

$$\begin{cases} \partial_t\varrho_D = \underline{F}\Delta\varrho_D & \text{on every } D_j,\ j \in \mathcal{J}\ , \\ \partial_t\varrho_S = \overline{F}\Delta\varrho_S & \text{on every } S_i,\ i \in \mathcal{I}\ . \end{cases} \tag{3.5}$$

Using test functions supported in the union of two adjacent plateaus we obtain the interface conditions on ∂D_j,

$$\begin{cases} \overline{F}\varrho_S = \underline{F}\varrho_D\ , \\ \vec{n}_j \cdot \begin{pmatrix} \varrho_D \\ -\underline{F}\nabla_{\mathbf{x}}\varrho_D \end{pmatrix} = \vec{n}_j \cdot \begin{pmatrix} \varrho_S \\ -\overline{F}\nabla_{\mathbf{x}}\varrho_S \end{pmatrix}\ , \\ \varrho * \mathcal{W} = \mathcal{C}\ , \end{cases} \tag{3.6}$$

where ϱ_S and $\nabla_{\mathbf{x}}\varrho_S$ denote the traces of the respective quantities in the sparse plateau adjacent to ∂D_j. Note that we obtain the first equation by collecting the coefficients of $(0, \nabla_{\mathbf{x}}\varphi)^t \cdot \vec{n}_j$

and the second equation collecting the coefficients of φ. The last equation is the free boundary condition defining the interfaces ∂S_i and ∂D_j.

The fact that $\varrho * \mathcal{W}$ is a level set function defining the interfaces between dense and sparse plateaus implies that, by the smoothness assumptions on $\mathcal{W}$, the interface is smooth and the normal vector $\vec{n}$ is defined everywhere. Moreover, it suggests a way to derive the velocity of these interfaces in the normal direction. We start with the regularized solution ϱ^ε constructed in Theorem 1. As the unit normal vector of such an interface in the spatial domain is given by $-\nabla_{\mathbf{x}}\varrho^\varepsilon * W/|\nabla_{\mathbf{x}}\varrho^\varepsilon * W|$, the normal velocity $v^\varepsilon(t,\mathbf{x})$ at point $\mathbf{x}$ of such an interface is calculated as

$$v^\varepsilon(t,\mathbf{x}) = \frac{\partial_t \varrho^\varepsilon * \mathcal{W}(\mathbf{x})}{|\nabla_{\mathbf{x}}\varrho^\varepsilon * \mathcal{W}(\mathbf{x})|} = \frac{\Delta_{\mathbf{x}}(F^\varepsilon(\varrho^\varepsilon * \mathcal{W})\varrho^\varepsilon) * \mathcal{W}(\mathbf{x})}{|\nabla_{\mathbf{x}}\varrho^\varepsilon * \mathcal{W}(\mathbf{x})|} = \frac{(F^\varepsilon(\varrho^\varepsilon * \mathcal{W})\varrho^\varepsilon) * \Delta_{\mathbf{x}}\mathcal{W}(\mathbf{x})}{|\varrho^\varepsilon * \nabla_{\mathbf{x}}\mathcal{W}(\mathbf{x})|}\,.$$

Since the last expression on the right-hand side does not contain any derivatives of ϱ^ε, we may let formally $\varepsilon \to 0$ as we did to derive (3.2), and obtain

$$v(t,\mathbf{x}) = \frac{(F(\varrho * \mathcal{W})\varrho) * \Delta_{\mathbf{x}}\mathcal{W}(\mathbf{x})}{|\varrho * \nabla_{\mathbf{x}}\mathcal{W}(\mathbf{x})|}\,, \tag{3.7}$$

with F given by (1.2). Note that, unfortunately, it is not possible to write a closed level set equation for $\psi := \varrho * \mathcal{W}$, since

$$\frac{\partial \psi}{\partial t} = (F(\psi)\varrho) * \Delta_{\mathbf{x}}\mathcal{W}$$

and the mapping $\varrho \mapsto \psi$ is in general not one-to-one.

3.1 Strong formulation in the spatially 1D setting

Some of the expressions used in the strong formulation of the free boundary problem become more amenable in the special case where the spatial domain is the one dimensional torus, i.e. the unit interval where we identify the endpoints with each other. We assume that there is only a finite number of dense and sparse plateaus, and that the plateaus are separated by ordered points $x_1(t) < \cdots < x_N(t) \in [0,1)$ for some $N \in \mathbb{N}$. The outer normal vectors to these plateaus are parallel to the vector $(-\dot{x}_i, 1)^t$. Hence the second equation in (3.6) evaluated at x_i, reads

$$\dot{x}_i(t)[\varrho_S(t,x_i(t)) - \varrho_D(t,x_i(t))] = \underline{F}\partial_x\varrho_D(t,x_i(t)) - \overline{F}\partial_x\varrho_S(t,x_i(t))\,. \tag{3.8}$$

A direct computation of the total derivative of the identity $(\varrho^\varepsilon(t,.) * W)(x_1^\varepsilon(t)) \equiv \mathcal{C}$ and the formal passage to the limit as $\varepsilon \to 0$ leads to

$$\dot{x}_i(t) = -\frac{[F(\mathcal{W} * \varrho)\varrho] * \partial^2_{xx}\mathcal{W}(x_i(t))}{\varrho * \partial_x\mathcal{W}(x_i(t))}\,, \tag{3.9}$$

which corresponds to (3.7), where $v(t,\mathbf{x})$ represents the velocity in the direction of the outer unit normal to the dense plateau.

Without loss of generality, let the interval $(x_{i-1}(t), x_i(t))$ correspond to the cross-section of a dense plateau and $(x_i(t), x_{i+1}(t))$ to the one of a sparse plateau. We furthermore assume that the support of $W(x(t) - .)$ is contained in the union of these two intervals. We perform one integration by parts in the numerator of (3.9) where the boundary terms at $x_i(t)$ cancels due to the first equation in (4.2) and we obtain

$$\dot{x}_i = -\frac{\underline{F} \int_{x_{i-1}}^{x_i} \partial_x \varrho_D(t,y) \partial_x W(x_i - y)\, dy + \overline{F} \int_{x_i}^{x_{i+1}} \partial_x \varrho_S(t,y) \partial_x W(x_i - y)\, dy}{\int_{x_{i-1}}^{x_i} \varrho_D(t,y) \partial_x W(x_i - y)\, dy + \int_{x_i(t)}^{x_{i+1}} \varrho_S(t,y) \partial_x W(x_i - y)\, dy} .$$

Another integration by parts in the numerator as well as in the denominator gives additional boundary terms. However these cancel due to (3.8) and we obtain

$$\dot{x}_i = -\frac{\underline{F} \int_{x_{i-1}}^{x_i} \partial_{xx} \varrho_D(t,y) W(x_i - y)\, dy + \overline{F} \int_{x_i}^{x_{i+1}} \partial_{xx} \varrho_S(t,y) W(x_i - y)\, dy}{\int_{x_{i-1}}^{x_i} \partial_x \varrho_D(t,y) W(x_i - y)\, dy + \int_{x_i}^{x_{i+1}} \partial_x \varrho_S(t,y) W(x_i - y)\, dy} . \tag{3.10}$$

Remark 1 *The expression* (3.10) *suggests a limit model in the case where the sensing kernel is scaled,* $\frac{1}{\delta} W(\frac{x}{\delta})$ *and becomes a point distribution in the limit as* $\delta \to 0$. *For example in the case where* W *is symmetric, the formal limit of* (3.10) *as* $\delta \to 0$ *suggests*

$$\dot{x}_i = -\frac{\underline{F} \partial_{xx} \varrho_D(t, x_i) + \overline{F} \partial_{xx} \varrho_S(t, x_i)}{\partial_x \varrho_D(t, x_i) + \partial_x \varrho_S(t, x_i)} .$$

Non-symmetric weights to compute averages of data at the boundary of the two plateaus can be derived based on non-symmetric sensing kernels.

4 Stationary classical solutions

In this section we characterize stationary classical solutions under the additional assumption that the partition of $\mathbb{T}^d$ into sparse and dense plateaus is finite. Note that the assumed rectifiability of the level set $\{\mathcal{W} * \varrho = \mathcal{C}\}$ from Section 3 implies piecewise C^1 boundaries of the plateaus.

To formulate the stationary version of (3.5), (3.6) we use the same notation as for the time dependent model, i.e. we consider a partition of $\mathbb{T}^d$ into open, connected sets S_i, $i \in \mathcal{I}$ and D_j, $j \in \mathcal{J}$ (sparse and dense plateaus) such that (3.1) holds. The stationary version of (3.5), (3.6) then reads

$$\begin{cases} 0 = \Delta \varrho_D & \text{on every } D_j,\ j \in \mathcal{J} , \\ 0 = \Delta \varrho_S & \text{on every } S_i,\ i \in \mathcal{I} , \end{cases} \tag{4.1}$$

coupled to the interface conditions on ∂D_j,

$$\begin{cases} \overline{F} \varrho_S = \underline{F} \varrho_D , \\ \overline{F} \vec{n}_{\mathbf{x},j} \cdot \nabla_{\mathbf{x}} \varrho_S = \underline{F} \vec{n}_{\mathbf{x},j} \cdot \nabla_{\mathbf{x}} \varrho_D , \\ \varrho * \mathcal{W} = \mathcal{C} , \end{cases} \tag{4.2}$$

where ϱ_S and $\nabla_{\mathbf{x}}\varrho_S$ denote the respective traces in the sparse plateau S_i adjacent to ∂D_j. The second equation in (4.2) is derived by noticing that the first component of the normal vector $\vec{n}_j$ in (3.6) vanishes, $\vec{n}_j = (0, \vec{n}_{\mathbf{x},j})^T \in \mathbb{R}^{d+1}$, where $\vec{n}_{\mathbf{x},j}$ is the spatial unit normal vector to the respective dense plateau.

We distinguish the cases of one and more spatial dimensions. Although the statements of Propositions 1 (for the 1D case) and 2 (for the general case) are the same, we prove them separately. This is for the sake of the reader, since the easily intelligible proof in the 1D case helps to understand the multidimensional case. Moreover, in Section 4.2 we study the geometry of the stationary plateaus.

4.1 Stationary solutions in the spatially 1D setting

Proposition 1 *In the 1-dimensional case $d = 1$ with the assumption of finite number of plateaus, solutions to the system (4.1), (4.2), i.e., stationary classical solutions, are piecewise constant. Either they are globally constant or there exist constants $\lambda \in (0,1)$ and*

$$\overline{\varrho}_S = \frac{\mathcal{C}\underline{F}}{\lambda\overline{F} + (1-\lambda)\underline{F}}\,, \qquad \overline{\varrho}_D = \frac{\mathcal{C}\overline{F}}{\lambda\overline{F} + (1-\lambda)\underline{F}}\,. \tag{4.3}$$

satisfying $\mathcal{C}\underline{F}\,/\,\overline{F} < \overline{\varrho}_S < M < \overline{\varrho}_D < \mathcal{C}\overline{F}/\underline{F}$, where $M = \int_0^1 \varrho(x)\,dx$, such that

$$\varrho \equiv \overline{\varrho}_D \text{ in } D_j,\ j \in \mathcal{J}\,, \quad \text{and} \quad \varrho \equiv \overline{\varrho}_S \text{ in } S_i,\ i \in \mathcal{I}\,.$$

Proof: Let us assume that the solution is not globally constant and observe that the first two interface conditions listed in (4.2) read in the 1D setting

$$\overline{F}\varrho_S = \underline{F}\varrho_D\,, \quad \overline{F}\partial_x\varrho_S = \underline{F}\partial_x\varrho_D\,. \tag{4.4}$$

Due to (4.1) the solution ϱ is affine on every plateau. Let us assume that in one given plateau the slope is strictly positive. The second interface condition in (4.4) implies that it will be positive on any plateau. Thus, starting from a specific point in a specific plateau, we may follow a trajectory along which the value of ϱ constantly increases due to the positive slope and which, due to the periodic boundary finally returns to the point of origin. Due to the first interface condition in (4.4), every time we cross from a sparse plateau to a dense one the value of ϱ increases by the factor $\overline{F}/\underline{F}$ and every time we cross from a dense plateau to a sparse one by the inverse factor $\underline{F}/\overline{F}$ which thus cancels the effect of the previous transition. Ordering the plateaus from left to right like $S_i, D_i, S_{i+1}, D_{i+1}, \ldots$ and denoting by $\varrho_S^{i,\text{left}}$ the value of ϱ_S at the left endpoint of S_i, etc., the value of ϱ along this trajectory follows the following chain of inequalities starting at a point in the interior of S_1,

$$\varrho_S^{1,\text{start}} < \varrho_S^{1,\text{right}} = \varrho_D^{1,\text{left}}\underline{F}\,/\,\overline{F} < \varrho_D^{1,\text{right}}\underline{F}\,/\,\overline{F} = \varrho_S^{2,\text{left}} < \varrho_S^{2,\text{right}} = \varrho_D^{2,\text{left}}\underline{F}\,/\,\overline{F} <$$
$$< \varrho_D^{2,\text{right}}\underline{F}\,/\,\overline{F} = \varrho_S^{3,\text{left}} < \ldots = \varrho_S^{1,\text{left}} < \varrho_S^{1,\text{end}}\,.$$

As a consequence the value of ϱ at the endpoint of this trajectory should be larger than at the beginning. But since we returned to the same point in the same plateau, this is a contradiction. Hence only zero slopes are possible.

Therefore, in the case of a finite number of plateaus the solution is constant on each plateau, and, moreover, it only assumes two values, $\overline{\varrho}_D,\ \overline{\varrho}_S \in \mathbb{R}^+$,

$$\varrho \equiv \overline{\varrho}_D \text{ in } D_j,\ j \in \mathcal{J}\ , \quad \text{and} \quad \varrho \equiv \overline{\varrho}_S \text{ in } S_i,\ i \in \mathcal{I}\ .$$

Then, using the normalization $\int_0^1 \mathcal{W}(x)\,\mathrm{d}x = 1$, the third interface condition implies that there exists a $\lambda \in (0,1)$ such that for every interface point x_k,

$$\varrho * \mathcal{W}(x_k) = \lambda\overline{\varrho}_D + (1-\lambda)\overline{\varrho}_S = \mathcal{C}\ .$$

Combined with the first condition of (4.4), this gives

$$\overline{\varrho}_S = \frac{\mathcal{C}\underline{F}}{\lambda\overline{F} + (1-\lambda)\underline{F}}\ , \qquad \overline{\varrho}_D = \frac{\mathcal{C}\overline{F}}{\lambda\overline{F} + (1-\lambda)\underline{F}}\ . \tag{4.5}$$

Finally, with $M = \int_0^1 \varrho(x)\,\mathrm{d}x$ the total mass of the solution, we have the chain of inequalities

$$\mathcal{C}\underline{F}\,/\,\overline{F} < \overline{\varrho}_S < M < \overline{\varrho}_D < \mathcal{C}\overline{F}/\underline{F}. \tag{4.6}$$

■

Remark 2 *Observe that* (4.6) *gives the necessary condition* $\mathcal{C}\underline{F}\,/\,\overline{F} < M < \mathcal{C}\overline{F}/\underline{F}$ *for existence of stationary plateaus.*

Remark 3 *Observe that the constant* λ *in the above proof is calculated as* $\lambda = \int_{\bigcup\{D_j,j\in\mathcal{J}\}} \mathcal{W}(x_k - y)\,\mathrm{d}y$ *for an arbitrary interface point* x_k*. Obviously, assuming a symmetric kernel* $\mathcal{W}$*, there are two particular cases when* $\lambda = 1/2$*:*

- *all plateaus have the same width ("periodic solution"),*
- *the width of any plateau is wider then half of the support of the sensing kernel* $\mathcal{W}$*.*

In fact, supported by our numerical observations (see Section 5), we hypothesize that only these two cases are possible.

4.2 Stationary solutions in the spatially multidimensional setting

A similar argument as in the 1D case can be formulated to argue that the stationary densities within every plateau are constant if the total number of plateaus is finite.

Proposition 2 *Under the assumption of finite number of plateaus, solutions to the system* (4.1)*,* (4.2)*, i.e., stationary classical solutions, are piecewise constant. Either they are globally constant, or there exist constants* $\lambda \in (0,1)$ *and*

$$\overline{\varrho}_S = \frac{\mathcal{C}\underline{F}}{\lambda\overline{F} + (1-\lambda)\underline{F}}, \qquad \overline{\varrho}_D = \frac{\mathcal{C}\overline{F}}{\lambda\overline{F} + (1-\lambda)\underline{F}}. \tag{4.7}$$

satisfying $\mathcal{C}\underline{F}\,/\,\overline{F} < \overline{\varrho}_S < M < \overline{\varrho}_D < \mathcal{C}\overline{F}/\underline{F}$ *such that*

$$\varrho \equiv \overline{\varrho}_D \text{ in } D_j,\ j \in \mathcal{J}, \quad \text{and} \quad \varrho \equiv \overline{\varrho}_S \text{ in } S_i,\ i \in \mathcal{I}.$$

Proof: Due to (4.1) the solution ϱ satisfies a Laplace equation on every plateau. Hence the minimal and maximal values in every plateau are assumed at its boundary. Moreover, due to Hopf's Lemma [5], the normal derivative at a maximum point on the boundary is non-negative (and strictly positive if the solution on the plateau is not constant). Due to the second condition in (4.2) the normal derivative on the adjacent plateau at the same point is therefore negative and hence the density on the adjacent plateau does not assume its maximum at the respective boundary point.

We start in a sparse plateau S_1 within which the solution is not constant. More specifically we start at a boundary point where the maximum value $\varrho_S^{1,\max}$ within that plateau is assumed and we construct a trajectory by the following strategy: We always leave the plateau at a boundary point where the maximum value of ϱ within the plateau is assumed. This way, from entering a plateau to leaving it, we always increase the value of ϱ and, as in the 1-D case, the effect of the first equation in (4.2) after subsequent transitions from a sparse into a dense and into another sparse plateau cancels. As the present value of ϱ is always higher then the previous values along the trajectory we can not return to a plateau which we already have visited. This gives, for k plateaus, the chain of inequalities

$$\varrho_S^{1,\max} = \varrho_D^{1,\mathrm{adj}}\underline{F}\,/\,\overline{F} < \varrho_D^{1,\max}\underline{F}\,/\,\overline{F} = \varrho_S^{2,\mathrm{adj}} < \varrho_S^{2,\max} = \varrho_D^{2,\mathrm{adj}}\underline{F}\,/\,\overline{F} <$$
$$< \varrho_D^{2,\max}\underline{F}\,/\,\overline{F} = \varrho_S^{2,\mathrm{adj}} < \ldots < \varrho_D^{k,\max}\underline{F}\,/\,\overline{F} = \varrho_S^{k,\mathrm{adj}},$$

where indices are ordered along the trajectory and where $\varrho_D^{i,\max}$ denotes the maximum value of ϱ_D at the boundary of D_i accoring to the maximum principle of the Laplace equation, and analogously for $\varrho_S^{i,\max}$. Furthermore $\varrho_D^{i,\mathrm{adj}}$ denotes the value of ϱ_D adjacent to the point where ϱ_S assumes $\varrho_S^{i,\max}$, etc. Clearly, as the total number of plateaus k is finite, in the $\overline{k}$-th step, where $\overline{k} \le k$, we reach the boundary of the starting sparse plateau S_1 with the value $\varrho_S^{\overline{k},\mathrm{adj}}$. The above chain of inequalities implies that $\varrho_S^{\overline{k},\mathrm{adj}} > \varrho_S^{1,\max}$, which is a contradiction. Thus, we conclude that ϱ has to be constant on every plateau, only assuming two values $\overline{\varrho}_S$ and $\overline{\varrho}_D$ given by (4.7).

■

In the special case of *well separated plateaus*, i.e., when the plateaus are such that the support of the sensing kernel $\mathcal{W}(\mathbf{x}-\cdot)$ at any interface point overlaps with only one dense and one sparse plateau, we are able to describe the geometry of the stationary solutions:

Proposition 3 *Under the assumption of well separated plateaus, the interfaces between plateaus are either all straight or all spherical with the same radius. Hence we have the following alternatives,*

- *either all dense plateaus are balls (with the same radius), and the sparse plateau is their complement,*
- *or all sparse plateaus are balls (with the same radius), and the dense plateau is their complement,*
- *or interfaces are flat and plateaus are (generalized) stripes.*

Proof: Let us recall that, as a generic assumption, we work with rotationally symmetric kernels $\mathcal{W}$. The interface condition $\varrho * \mathcal{W} \equiv \mathcal{C}$ implies that for any interface point $\mathbf{x}$,

$$\varrho * \mathcal{W}(\mathbf{x}) = \lambda \overline{\varrho}_D + (1-\lambda)\overline{\varrho}_S = \mathcal{C},$$

where, due to the assumption of well separated plateaus,

$$\lambda = \int_{D_j} \mathcal{W}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{4.8}$$

for the (unique) plateau D_j such that $\mathbf{x} \in \partial D_j$. Note that $\lambda \in (0,1)$ is the same for all interface points $\mathbf{x}$.

Let us now fix an interface point $\mathbf{x} \in \partial D_j \cap \partial S_i$ for some $j \in \mathcal{J}$ and $i \in \mathcal{I}$. Due to the assumption of well separated plateaus, we have $\operatorname{supp} \mathcal{W}(\mathbf{x} - \cdot) \subset \overline{D}_j \cup \overline{S}_i$. Since $\varrho * \mathcal{W}$ is the level set function, the vector $\nabla_{\mathbf{x}}(\varrho * \mathcal{W})(\mathbf{x})$ is parallel to $\vec{n}(\mathbf{x})$, the outer unit normal vector to ∂D_j at $\mathbf{x}$. Denote $\mathcal{B} := \partial D_j \cap \operatorname{supp} \mathcal{W}(\mathbf{x} - \cdot) = \partial S_i \cap \operatorname{supp} \mathcal{W}(\mathbf{x} - \cdot)$. Then, with integration by parts, we obtain

$$\begin{aligned}
\nabla(\varrho * \mathcal{W})(\mathbf{x}) &= \overline{\varrho}_D \int_{D_j} \nabla \mathcal{W}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} + \overline{\varrho}_S \int_{S_i} \nabla \mathcal{W}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= -\overline{\varrho}_D \int_{\mathcal{B}} \mathcal{W}(\mathbf{x}-\mathbf{y})\,\vec{n}(\mathbf{y})\,\mathrm{d}\mathbf{y} - \overline{\varrho}_S \int_{\mathcal{B}} \mathcal{W}(\mathbf{x}-\mathbf{y})\,(-\vec{n}(\mathbf{y}))\,\mathrm{d}\mathbf{y} \\
&= (\overline{\varrho}_S - \overline{\varrho}_D) \int_{\mathcal{B}} \mathcal{W}(\mathbf{x}-\mathbf{y})\,\vec{n}(\mathbf{y})\,\mathrm{d}\mathbf{y}.
\end{aligned}$$

Hence it holds that

$$\frac{\int_{\mathcal{B}} \mathcal{W}(\mathbf{x}-\mathbf{y})\,\vec{n}(\mathbf{y})\,\mathrm{d}\mathbf{y}}{\left|\int_{\mathcal{B}} \mathcal{W}(\mathbf{x}-\mathbf{y})\,\vec{n}_(\mathbf{y})\,\mathrm{d}\mathbf{y}\right|} = \vec{n}(\mathbf{x})$$

for all interface points $\mathbf{x} \in \partial D_j \cap \partial S_i$, for any $j \in \mathcal{J}$ and $i \in \mathcal{I}$. As the left hand side represents a weighted average of normal vectors on a segment of the interface, we conclude that all the interfaces $\partial D_j \cap \partial S_i$ have to be either circular or straight.

Clearly, due to the rotational symmetry of $\mathcal{W}$, λ given by (4.8) is a function of the radius of the dense plateaus: we have $\lambda \in (0, 1/2)$ if the dense plateau is a circle, $\lambda \in (1/2, 1)$ if the plateau is a complement of a circle, and $\lambda = 1/2$ if the interface is straight. Recalling that the value of λ is the same for all points $\mathbf{x}$ of all interfaces, we conclude that all the interfaces must have the same curvature, which gives the three alternatives given in the statement of the Lemma.

■

We refer to Section 5 for numerical examples illustrating the statement of Proposition 3 in 2D. Let us note that although the Proposition is formulated under the assumption of well separated plateaus, we believe that its validity can be extended to more general situations. However, do not make any claims in this direction.

5 Numerical examples

In this section we present numerical examples in 1D and 2D for the limiting mean-field model (1.1) where we use (1.2) with $\mathcal{C} = 1$ and $\overline{F} = 2$, $\underline{F} = 1$. Moreover, we present simulations of the corresponding discrete agent-based model (1.6)–(1.7) with the same parameters. In the 1D setting, we obtain a good quantitative match between a Monte Carlo simulation of the discrete model and the numerical solution of the mean-field model. In 2D, we demonstrate that the geometric constraints on the stationary solution derived in Section 4 apply also to the discrete model with sufficiently high number of agents.

Note that although for our analysis we assumed that the kernel $\mathcal{W}$ be sufficiently smooth, in the numerical simulations we set $\mathcal{W}$ to be (properly scaled) characteristic functions of intervals in 1D or balls in 2D. This is for the mere sake of simplicity and implementational convenience.

5.1 The discrete (1.6)–(1.7) and mean-field (1.1) models in 1D

We simulate the limiting model (1.1) in the 1D periodic domain $[0, 1)$, using first order central finite difference discretization for the space variable and first-order semi-implicit Euler method for the time variable, see, e.g., [10]. In particular, the nonlinear term of (1.1) is discretized in time as

$$\frac{\varrho^{k+1} - \varrho^k}{\Delta t} = \Delta(F(\mathcal{W} * \varrho^k)\varrho^{k+1}),$$

where ϱ^k is the solution at time $t_k = k\Delta t$. The time step is $\Delta t = 10^{-4}$, the space grid consists of 500 equidistant points. We choose $\mathcal{W}(x)$ to be characteristic function of the interval $[-0.1, 0.1]$ scaled to unit mass.

For our first simulation we use the initial datum

$$\varrho_I(s) = \begin{cases} 0.5 & \text{for } 0 < s \leq 0.6, \\ 2.0 & \text{for } 0.6 < s \leq 1. \end{cases}$$

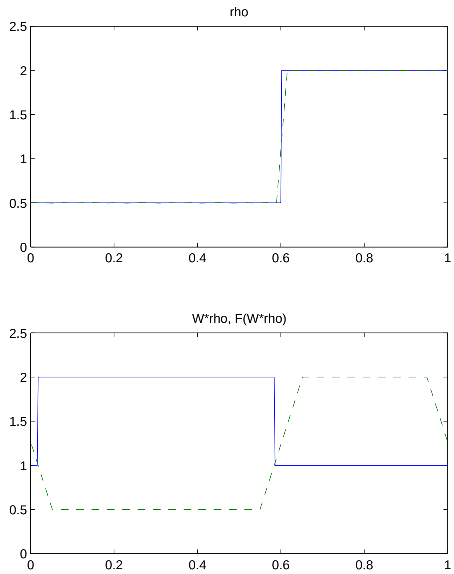
Snapshots of the evolution are shown in Fig. 1. We observe that the discontinuity in the initial datum is smoothed out, but another discontinuity (of smaller height) is created at $x \simeq 0.58$. This moves to the right until it reaches its final position at $x \simeq 0.48$ and $t \simeq 0.23$, when a steady state is reached with piecewise constant ϱ. Note that in the steady state there is only one sparse and one dense plateau, so the formula (4.5) of Section 4 applies with $\lambda = 1/2$ and gives $\overline{\varrho}_S = 2/3$, $\overline{\varrho}_D = 4/3$.

We also solve the stochastic individual based model (1.6)–(1.7) with $G = \sqrt{2F}$ and the same kernel $\mathcal{W}$ as above, using $N = 10^4$ particles in the interval $(0, 1)$ with periodic boundary conditions. The initial positions of particles are chosen to approximate the initial datum ϱ_I and equidistant in the subintervals $(0, 0.6)$ and $(0.6, 1)$. The system of stochastic differential equations (1.6)–(1.7) is integrated in time using the Euler-Maruyama scheme [7] with time-step length $\Delta t = 10^{-5}$. We performed a Monte Carlo simulation with 5 realizations of the stochastic simulation (dashed line in the upper panels of Fig. 1) and we observe a good quantitative match with the numerical solution of the mean-field model (solid line). Of course, as can be expected, the Monte Carlo method has difficulties in capturing the discontinuities of the solution (interfaces between the plateaus).

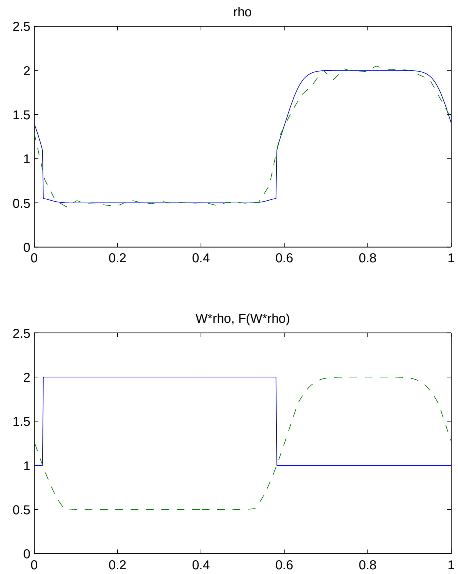
We also perform another simulation of the mean-field model (1.1) where we impose a random initial condition for ϱ, generated such that for every grid point a random number from the uniform distribution in $[0, 2]$ is drawn. Snapshots of the evolution are shown in Fig. 2. Note that in the steady state there are three dense plateaus separated by three sparse ones, but all the plateaus are wider then half of the support of the kernel $\mathcal{W}$, which is 0.1. This supports the conjecture posed in Section 4 about the structure of the stationary solution.

5.2 The discrete (1.6)–(1.7) and mean-field (1.1) models in 2D

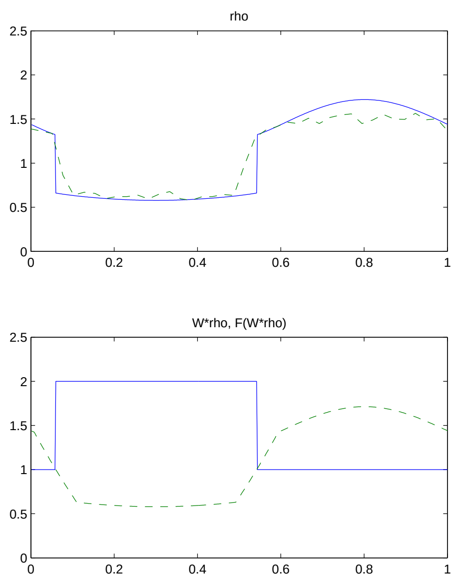
We simulate the mean-field model (1.1) in the 2D periodic domain $(0, 1)^2$, using the same type of discretization as in the 1D case. The space grid consists of 100×100 equidistant points, the time step is 10^{-5}. We choose $\mathcal{W}(\mathbf{x})$ to be the (properly scaled) characteristic function of the ball with radius 0.1 centered at the origin. We impose a random initial condition for ϱ, generated such that for every grid point a random number from the uniform distribution in $[0, 2]$ is drawn. We ran two simulations with two different realizations of the random initial condition. The snapshots of the evolution of the density are given in Figures 3 and 4. In Fig. 3, the solution converges to the steady state with one sparse plateau with $\overline{\varrho}_S \equiv 2/3$ of circular shape on dense background with $\overline{\varrho}_D \equiv 4/3$. On the other hand, in Fig. 4 the plateaus in the asymptotic steady state are stripes. This nicely illustrates the analytic conclusions made in Section 4, where we found that the boundaries of the stationary plateaus are either circular or straight.



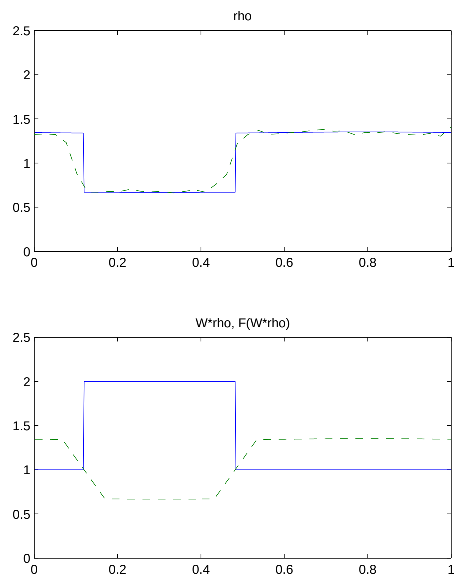
t=0.000



t=0.001



t=0.030



t=0.230

Figure 1: The 1D mean field model (1.1) and Monte Carlo simulation of the discrete model (1.6)–(1.7) in the interval $(0, 1)$ with periodic boundary conditions. Upper panels: the density ϱ as a solution of the mean-field model (solid line) and Monte Carlo simulation (dashed line). Lower panels: the convolution $\mathcal{W} * \varrho$ (dashed line) and the diffusivity $F(\mathcal{W} * \varrho)$ (solid line) calculated from the mean-field solution.

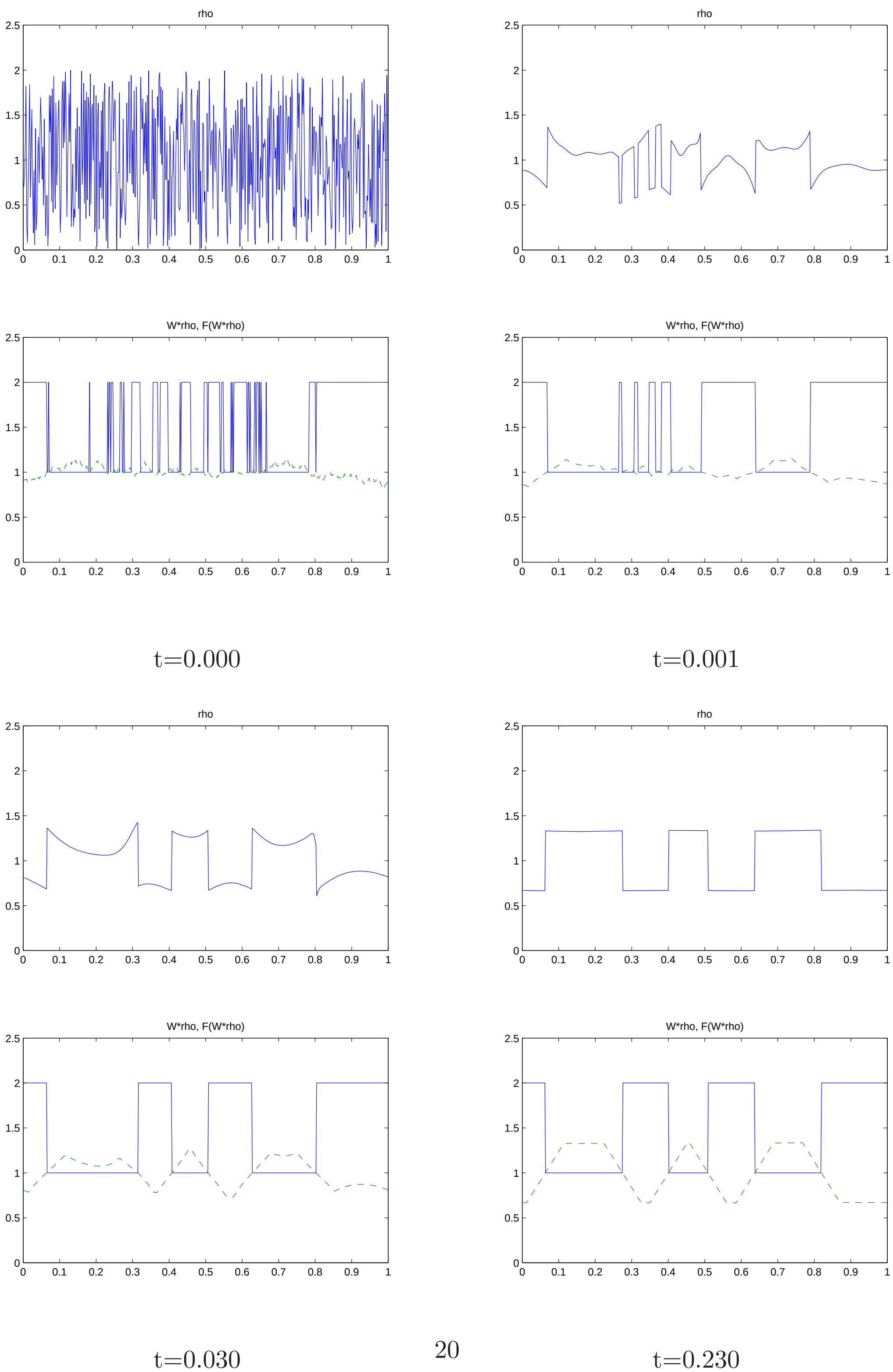
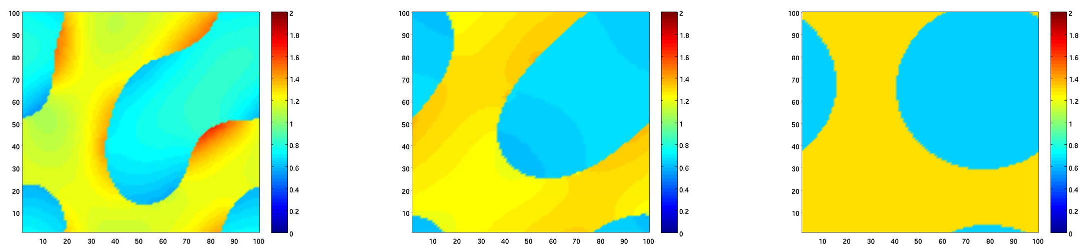
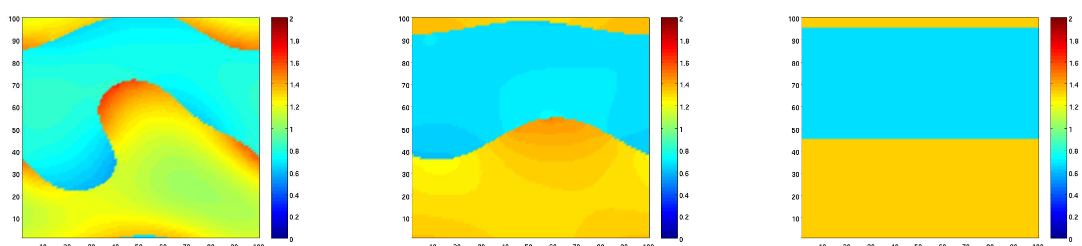


Figure 2: The 1D mean field model (1.1) in the interval $[0, 1)$ with periodic boundary conditions. Upper panels: the density ϱ, lower panels: the convolution $\mathcal{W} * \varrho$ (dashed line) and the diffusivity $F(\mathcal{W} * \varrho)$ (solid line).



t=0.01 t=0.07 t=0.30

Figure 3: The 2D mean field model (1.1) in a periodic 2D setting with random initial condition (not shown).



t=0.01 t=0.07 t=0.30

Figure 4: The 2D mean field model (1.1) in a periodic 2D setting with random initial condition (not shown).

Finally, we present 2D simulations of the stochastic individual based model (1.6)–(1.7) with $G = \sqrt{2F}$ and $F(s) = 10$ for $0 \le s \le 1$ and $F(s) = 1$ for $s > 1$. We increased the ratio $\overline{F}/\underline{F}$ to 10 in order to obtain sharp boundaries between the dense and sparse plateaus, which otherwise would be smudged by stochastic effects. The kernel $\mathcal{W}$ in (1.7) is again chosen as the characteristic function of the ball with radius 0.1 centered at the origin, scaled such that $\int \mathcal{W}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$. We consider a system consisting of $N = 2000$ individuals in a 2D domain $\Omega = (0,1) \times (0,1)$ with periodic boundary conditions. The initial positions are generated randomly and independently for each individual from the uniform distribution on Ω. The system of stochastic differential equations (1.6) is integrated in time using the Euler-Maruyama scheme [7] with time-step length $\Delta t = 10^{-5}$. In Fig. 5 we present four different patterns that typically emerge from different realizations of the stochastic system (1.6) with the above parameters, namely, (a) circular dense plateau, (b) circular sparse plateau, (c) broad stripe and (d) narrow stripe. Once these patterns have been formed, they typically remain stable, in the sense that the stochastic motion of the individual particles does not change the global pattern.

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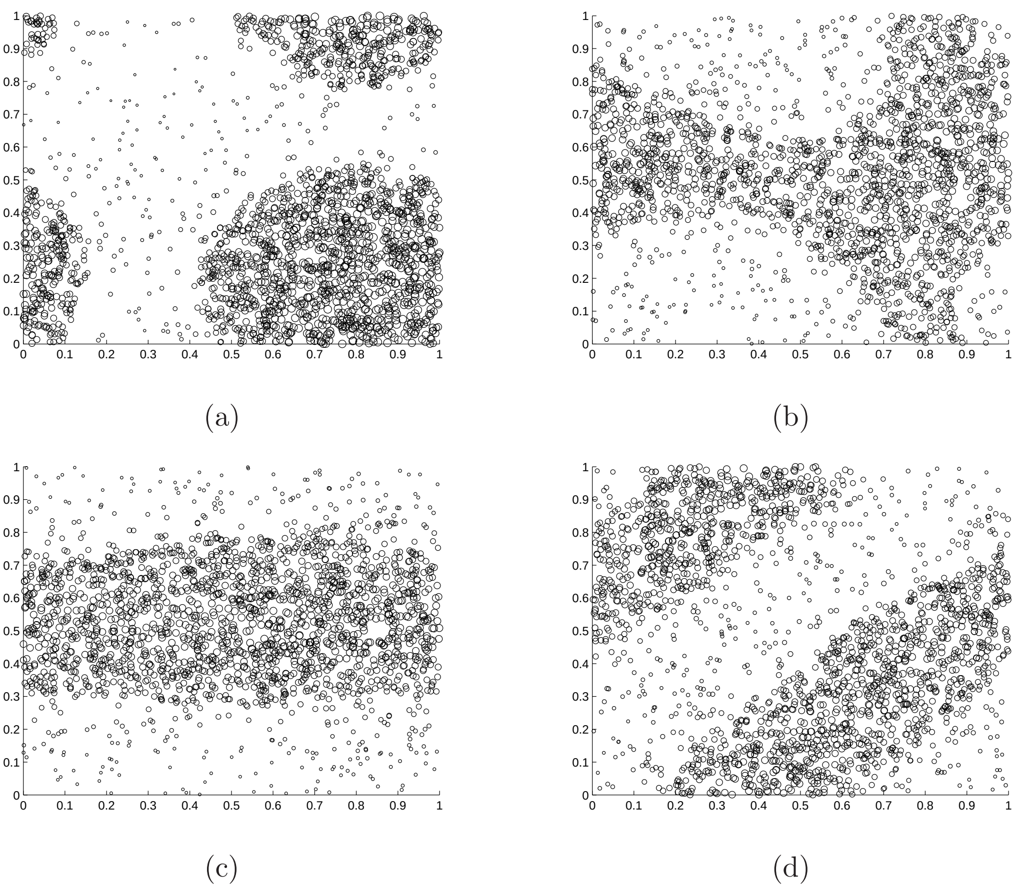

(a) (b) (c) (d)

Figure 5: Typical patterns formed in different realizations of the stochastic individual based model (1.6)–(1.7) with $N = 2000$ agents in the 2D domain $\Omega = [0, 1) \times [0, 1)$ with periodic boundary conditions, subject to a random initial condition. The size of the markers is proportional to the locally sensed density ϑ_i for each individual, given by (1.7).

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