THE DIFFERENCE OF TWO RENEWAL PROCESSES
LEVEL CROSSING AND THE INFIMUM

by

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ABSTRACT

We consider the difference process $N$ of two independent renewal (counting) processes. Second-order approximations to the distribution function of the level crossing time are given. Direct application of the second-order approximation is complicated by the occurrence of an (in general) unknown term $E\bar{\eta}$, which denotes the expected minimum of the stationary version of $N$. However, this number is obtained for a wide class of processes $N$, using matrix-geometric techniques. Numerical experiments have been carried out, in which the new approximations were compared to simulation, first-order and/or exact results. These results confirm that the second-order approximations are considerably better than the (known) first-order ones.
1. **INTRODUCTION**

Although renewal processes are among the most basic and important processes in queuing and reliability theory, some areas in renewal theory seem to have been not fully explored. In Kroese & Kallenberg [8], for example, the difference process $N = N_1 - N_2$ of two independent renewal processes $N_1$ and $N_2$ was studied. In applications, one is often interested in the distribution of the first time that $N$ crosses some level $n$, $r_n$ say. For example, in a reliability model $r_n$ may indicate the time of a system failure. Only in very special cases it is possible to give a tractable formula for $P(r_n \leq x)$. It is however possible to expand $P(r_n \leq x)$ for large $n$. A first-order approximation to $P(r_n \leq x)$, based on asymptotic normality, can be easily established using Cox [4] p.73. To obtain greater accuracy, a second-order approximation was obtained in [8], which reduces the approximation error from $o(1)$ to $o(n^{-1/2})$, as $n \to \infty$.

The motivation for this paper is two-fold. Firstly, we address the difficulty that direct application of the second-order approximation is complicated by the occurrence of the term $E\tilde{N} - E \inf_{t \geq 0} \tilde{N}(t)$, where $\tilde{N}$ is the stationary version of $N$. This number is in general intractable. However, for a large class of processes $N$ it is possible to calculate $E\tilde{N}$ explicitly or numerically. In Sections 3-5 the distribution of the infimum of $N$ is derived for a number of cases in which $N_1$ and/or $N_2$ has some kind of Markov structure. Moreover, the distribution function of $r_n$ is given for a few cases of interest (Sections 3-4). The main new results on the distribution of the infimum of $N$ are given in Section 5, where $N_1$ is a Phase-type renewal process and $N_2$ a general delayed renewal process. This shows that the second-order approximation may be used for a wide class of $N$'s.

Secondly, we need to verify that the second-order approximations are indeed useful for finite (small) $n$. Numerical experiments have been carried out to compare first- and second-order approximations to simulation or exact results. It appears (cf. Section 6) that the second-order approximations indeed give considerable improvement on
the known first-order results. Numerical results show good agreement with the theoretical error structure as described in [8]. Even for small $n$ quite satisfactory approximations are obtained. It is therefore worthwhile to use the new second-order approximation instead of the known first-order one. In these numerical examples we do not restrict ourselves to stationary renewal processes, although formally the expansion from which the second-order approximation is derived, has yet only been proved for the stationary case (cf. [8]). We adopt here a more pragmatic point of view and state an approximation for arbitrary delayed renewal processes. In the next section we give the definitions and model assumptions for the rest of the paper, and state the second-order approximation to $P(r_n \leq x)$.

2. GLOBAL DEFINITIONS / APPROXIMATION

The definitions and assumptions given in this section hold throughout the paper, unless otherwise specified.

Let $N_1 = (N_1(t))_{t \geq 0}$ and $N_2 = (N_2(t))_{t \geq 0}$ denote two independent delayed renewal processes with renewal sequences $(x^{(1)}_k)_{k \geq 1}$ and $(x^{(2)}_k)_{k \geq 1}$, respectively (cf. [8] for definitions). In other definitions $N_1$ and $N_2$ may be called renewal counting processes. Denote the renewal times of $N_1$ by $S^{(1)}_n = x^{(1)}_1 + \ldots + x^{(1)}_n$, $i=1,2$, $n \in \mathbb{N}$. Let $F_i$ be the distribution function (d.f.) of $X^{(i)}_1$ and let $G_i$ be the d.f. of $X^{(i)}_2$, $i=1,2$. We assume that the expectation $\mu_i$, the variance $\sigma^2_i$ and the third central moment $\mu_3_i$ of $X^{(i)}_2$ finitely exist and that $G_i$ is non-lattice, $i=1,2$. We denote $E X^{(i)}_1$ by $\eta_i$, $i=1,2$, which we assume to be finite as well. The Laplace-Stieltjes transforms of $F_i$ and $G_i$ are denoted by $A_i$ and $B_i$, respectively, $i=1,2$. We assume that $\mu_1 < \mu_2$, in which case $N$ has an upward drift $\alpha^{-1}$.

$$\alpha^{-1} = \mu_1^{-1} - \mu_2^{-1}.$$  

The difference process $N = (N(t))_{t \geq 0}$ of $N_1$ and $N_2$ is defined by $N(t) = N_1(t) - N_2(t)$, $t \geq 0$. Let $r_n$ denote the first time that $N$
crosses level \( n \in \mathbb{N}_+ \), that is \( r_n = \inf \{ t \geq 0 : N(t) \geq n \} \). We denote the random variable \( \inf N(t) \) by \( E \). An overview of the definitions is given in fig. 1.

2.1 REMARK The stationary version \( \bar{N} \) of \( N \) is defined as the difference process of two stationary independent renewal processes \( \bar{N}_1 \) and \( \bar{N}_2 \), with corresponding inter-arrival d.f.'s \( F_1, G_1, F_2 \) and \( G_2 \), such that \( G_i - G_1 \), and (hence) \( \bar{F}_i(x) = \int_0^x (1 - G_i(u)) du / \mu_i \), \( i=1,2 \). If \( N \) is itself stationary (i.e. \( F_i = \bar{F}_i \), \( i=1,2 \)), then \( \nu_i = (\mu_i + \sigma_i^2 / \mu_i) / 2 \), \( i=1,2 \) (and hence \( \bar{e} = 0 \) in the next approximation).

In Kroese [7] the following second-order approximation to the d.f. of \( r_n \) was stated. For a proof of the stationary case see [8]. In Kroese & Kallenberg [6] a similar result on the sum process of \( k \) independent delayed renewal processes was proved.

2.2 APPROXIMATION Let \( N_1 \) and \( N_2 \) be independent delayed renewal processes that satisfy the conditions above. Let \( E \bar{N} = E \inf \bar{N}(t) \) be
the expected infimum of the stationary version of \( N \) (cf. Remark 2.1), then \(|E\bar{N}| < \infty\) and moreover,

\[
P(\tau_n \leq x) = \Phi(y_n) + \frac{\varphi(y_n)}{\sqrt{n}} \left\{ p(1-y_n^2) + q \right\},
\]

where \( \Phi \) denotes the standard normal d.f., and \( \varphi \) its density function, and

\[
y_n = \frac{x - na}{\gamma \alpha^{3/2} \sqrt{n}}, \quad p = \frac{(c_1-d_1)(\gamma a_1)^{-3} - (c_2-d_2)(\gamma a_2)^{-3} + \gamma \sqrt{a}}{2},
\]

\[
q = \frac{1}{2 \gamma \sqrt{\alpha}} - \frac{\gamma \sqrt{a}}{2} - \frac{\bar{e}}{\sqrt{\alpha}} - \frac{1}{\gamma \sqrt{\alpha}} E\bar{N}, \quad \gamma = \sqrt{\sigma_1^2 \mu_1^{-3} + \sigma_2^2 \mu_2^{-3}},
\]

\[
s_i = \mu_1^{3/2} \sigma_1^{-1}, \quad c_i = \frac{1}{6} \mu_3 \sigma_1 \mu_1^{3/2}, \quad d_i = \frac{1}{2} \sigma_1 \mu_1^{-1/2},
\]

\[
e_i = -d_i - \frac{a_i}{2} + \mu_1^{1/2} \sigma_1^{-1} \eta_i, \quad i=1,2 \quad \text{and} \quad \bar{e} = e_1(\gamma a_1)^{-1} - e_2(\gamma a_2)^{-1}.
\]

2.4 REMARK A first-order approximation is obtained from (2.3) by suppressing the second term in the right-hand side of (2.3). The approximation error is in this case of order \( o(1) \), as \( n \to \infty \), whereas the second-order approximation (presumably) yields an error of order \( o(n^{-1/2}) \), as \( n \to \infty \). For a proof in the stationary case, cf. [8].

Note that \( E\bar{N} \) is not specified further because it is in general intractable. This complicates the direct application of the approximation. However, we can derive this quantity for a number of important cases for which the exact d.f. of \( r_n \) is very difficult to compute (or cannot be computed at all) and hence an accurate approximation is welcome. In Sections 3–5 we consider typical cases for which \( E\bar{N} \) can be computed. In each case at least one of the renewal processes has a Markov structure (phase-type or Poisson).
A similar approximation can be found for the difference process of two processes $K_1$ and $K_2$, which themselves are sum processes of a finite number of independent renewal processes, cf. [7].

3. $N_1$ DELAYED, $N_2$ POISSON

In this section $N_1$ is a delayed renewal process and $N_2$ a Poisson process. We first give an explicit expression for the d.f. of $r_n$ in terms of $\mu_2$, $F_1$ and $G_1$. This expression is in general too complicated to be handled numerically. However a simple expression for $EM$, in terms of the first few moments of $F_1$ and $G_1$ is found (along with the generating function of $M$) which makes the second-order approximation very simple to evaluate. A numerical example is given in section 6.2.

3.1 PROPOSITION Let $N_1$ be an ordinary renewal process $(F_1 - G_1)$. Denote the n-fold convolution of an arbitrary function $F$ by $F^\ast_n$. The d.f. of $r_n$ is given by

$$P(r_n \leq t) = H^\ast_n(t), t \geq 0,$$

where $H$ is defined by

$$H(t) = \sum_{k=1}^{\infty} \int_{k-1}^{t} \frac{(u/\mu_2)^{k-1}}{k!} e^{-u/\mu_2} dG_1^\ast(u), t \geq 0.$$

Proof Let $T_i$ be the first entrance time of $N$ into level $i$, starting from level $i-1$, $i=1, \ldots, n$. Note that since $N_2$ is a Poisson process, $T_1, T_2, \ldots, T_n$ are i.i.d. random variables. Moreover $r_n = T_1 + T_2 + \ldots + T_n$. Since we can regard $T_1$ as the busy period of an $M/G/1$ queue, we have by Cohen [3], p.250, that $P(T_1 \leq t) = H(t)$ and hence (3.2) follows.

Next we consider the distribution of $M$ and derive an expression for its expectation. We do not restrict ourselves to ordinary renewal processes $N_1$ this time. It is seen in (3.6) that the expression for $EM$ is easily computed and depends only on the first two moments of $G_1$,.
Fig. 2. Difference of renewal processes. $N_1$ delayed, $N_2$ Poisson.

the first moment of $F_1$ and $\mu_2$. For all $x \in \mathbb{R}$, let $x^- = \min(0, x)$ and $x^+ = \max(0, x)$.

3.4 THEOREM For all $r \in \mathbb{C}$, $|r| \leq 1$ we have

\[
\text{Ex}^M = \frac{A_1((1-r)/\mu_2)(r - 1)[1-\mu_1/\mu_2]}{r - B_1((1-r)/\mu_2)},
\]

and moreover,

\[
EM = - \eta_1/\mu_2 - \frac{(\sigma_1^2 + \mu_1^2)/\mu_2^2}{2(1-\mu_1/\mu_2)}.
\]

Proof First let $N_1$ be an ordinary renewal process. Let $M_n$ be the minimum of $N$ until time $S_n^{(1)}$, that is

\[
M_n = \min( N(S_1^{(1)}), \ldots, N(S_n^{(1)}) ) - 1, \quad n \geq 1.
\]

Define

\[
\tilde{M}_n = \min( N(S_1^{(1)}), \ldots, N(S_n^{(1)}) ) - N(X_1^{(1)}) - 1, \quad n \geq 1.
\]

Note that $M = \inf_{n \geq 1} M_n$. Denote $S_1^{(1)} = X_1^{(1)}$ by $X$. It is easy to see that $M_n$ and $\tilde{M}_n$ have the same distribution. Moreover $\tilde{M}_n$ and $N(X)$ are independent and the following recurrence relation holds (see fig. 2 for an illustration)
There are several ways to obtain the generating function of $M$. Note for example that $M$ is the minimum of the random walk $\{N(S_n^{(1)}) - 1; n \geq 1\}$. We adopt however another point of view here. A short reflection will show that $-M_n$ has the same distribution as the number of customers $U_n$ in an $M/G/1$ system, just after the departure of the $n$th customer. For, if we denote the number of customers that arrive during the service of the $n$th customer by $Z_n$, we have for all $n \geq 1$, $U_n = [U_{n-1} - 1]^* + Z_n$, $U_0 = 0$, $Z_n - N(X) + 1$ and the random variables $U_{n-1}$ and $Z_n$ are independent. Hence by (3.8), $M_n = -U_n$, for all $n \geq 1$. Therefore the generating function of $M$ is by Cohen [3], Ch.4 §2 (4.17) equal to

$$E^{r-M} = \frac{B_1((1-r)/\mu_2)[r-1][1-\mu_1/\mu_2]}{r - B_1((1-r)/\mu_2)}.$$  

(3.9)

Next let $N_1$ be a delayed renewal processes ($F_1$ is not necessarily equal to $G_1$).

Similarly to (3.8) we have

$$M = [M^* + 1]^* + N(X) - 1,$$

(3.10)

where $M^*$ and $X$ are independent random variables. $M^*$ is distributed as the random variable $M$ in (3.9) and, for $|r| \leq 1$, $r \in \mathbb{C}$,

$$E^{r-N(X)+1} = E^{r-M^*+1}E^{(1-r)X/\mu_2} = A_1((1-r)/\mu_2).$$

Hence,

$$E^{r-M} = E^{r-M^*}A_1((1-r)/\mu_2).$$

Moreover,

$$E^{r-[M+1]^*} = (1-r^{-1})P(M^* = 0) + r^{-1}E^{r-M^*}$$

(3.11)

$$= (1-r^{-1})(1-\mu_1/\mu_2) + r^{-1}E^{r-M^*}.$$ 

(3.12)

so that (3.5) follows from (3.9), (3.12) and (3.13). (3.6) follows
directly from (3.5) after some calculations. N.B.: In (3.11) $E_X$ denotes conditional expectation w.r.t. $X$.

3.14 REMARK The d.f. of $\tau_n$ in (3.3) is difficult to evaluate. Matters become even more complicated (numerically) in the case that $N_2$ is an arbitrary phase-type renewal process. However, the interpretation of $\tau_n$ as a busy period of a PH/GI/1-queue (starting with $n$ customers) is still valid, and a generalization of Proposition 3.1 can be given for this case. Suppose that $N_2$ is a phase-type renewal process with $m$ phases. Let $V_\tau$ be the number of transitions involved in crossing level $r\geq 1$ (the crossing time is $\tau_\tau$), and let $J_\tau$ denote the phase of $N_2$ at time $\tau_\tau$. Denote by $P_i$ the probability measure under which $N_2$ starts at phase $i$, $i \in \{1, \ldots, m\}$. By Neuts [10], Chapter 2, we can interpret $\tau_\tau$ as the first passage time from level $r$ to level 0 of a Markov renewal process of M/G/1 type. Hence, if we write $G_{i_j}(k;x)$ for $P_i(\tau_\tau \leq x, J_\tau = j, V_\tau = k)$, $r \geq 1$, $x \geq 0$, $k \geq 0$, $1 \leq i, j \leq m$, then, formally, a generalization of Proposition 3.1 is given by Lemma 2.2.1 and Theorem 2.2.1 of Neuts [10]. It seems not very likely that an easy to compute analogue of equations (3.5) or (3.6) can be obtained for this "Delayed, Phase-type" case.

4. $N_1$ Poisson, $N_2$ Poisson

In this section $N_1$ and $N_2$ are Poisson processes with intensities $\lambda = \mu_1^{-1}$ and $\nu = \mu_2^{-1}$, respectively. Of course this model is a special case of the one considered in the previous section. The reason for including this particular case is that the d.f. of $\tau_n$ can actually be evaluated numerically. This gives us the opportunity to compare the exact d.f. of $\tau_n$ with the first- and second-order approximation. This is done for several values of $\lambda - \nu$ in Section 6.3.

4.1 PROPOSITION Let $I_n$ denote the modified Bessel function of order $n$, then

$$P(\tau_n \leq t) = \int_0^t e^{-(\lambda+\nu)u} \lambda^{n/2} \nu^{-n/2} u^{-1} I_n(2u\sqrt{\lambda\nu}) du.$$
Proof Let $f$ denote the probability density of the distribution of $r_n$ and denote its Laplace transform by $d$. Let $a$ be the Laplace transform of $T_1$, where $T_1$ is defined in the previous section. Define for $0 \leq s \leq 1$ the functions $g_a : \mathbb{C} \rightarrow \mathbb{C}$ by

$$g_a(z) = z - \frac{\lambda}{s + \nu(1-z) + \lambda}, \quad z \in \mathbb{C}.$$

By Cohen [3], p.250, $a(s)$ is the only zero of $g_a$ inside the unit circle, thus

$$a(s) = \frac{\lambda + \nu + s - \sqrt{(\lambda + \nu + s)^2 - 4\nu}}{2\nu}$$

and hence, in view of (3.2),

$$d(s) = \left[ \frac{\lambda + \nu + s - \sqrt{(\lambda + \nu + s)^2 - 4\nu}}{2\nu} \right]^n$$

After some manipulation with inverse Laplace transforms (cf. for example [12]), we find $f(t) = e^{-t(\lambda + \nu)t}n^{\alpha/2} \cdot \nu^{-\alpha/2} \cdot I_v(2t\sqrt{\nu})/t$.

5. N₁ PHASE-TYPE, N₂ DELAYED

In this section $N_1$ is a Phase-type renewal process (PH-renewal process) and $N_2$ a delayed renewal process. This is the most general case (together with the "Delayed, Phase-type" case) for which one could expect to obtain explicit, numerically feasible expressions for the distribution of $H$. Since phase-type distributions are widely used in applications, and since any lifetime distribution $G_1$ of renewal process $N_1$ can be approximated by phase-type distributions, this case is also of practical relevance. In contrast to the two previous sections it is not possible to write $r_n$ as the sum of n i.i.d. random variables here. We can however still derive the distribution of $H$. An essential difference with the model in Section 3 is that there $EM$ only depended on the first two moments of $F_1$ and $G_1$, but here $EM$ possibly depends on all moments of $F_2$ and $G_2$. Numerical results are given in Sections 6.4-6.5.
DIFFERENCE OF TWO RENEWAL PROCESSES

Let \( p \in \mathbb{N}_+ \) and let \( \gamma \) denote a probability distribution on \((1, 2, \ldots, p)\), we use the same letter \( \gamma \) to denote the corresponding row vector. We introduce a family of probability measures \((P^\gamma)\) on our probability space \((\Omega, \mathcal{F})\) such that under \( P^\gamma \) \( N_1 \) is a PH-renewal process with \( p \) phases, characteristics \((\beta, S)\) and with initial phase distribution \( \gamma \) (cf. Neuts [9] for definitions on PH-renewal processes). Moreover, under these probability measures we let \( N_2 \) be a delayed renewal process, independent of \( N_1 \). Let \( P_i \) denote the probability measure under which \( N_1 \) starts at phase \( i \), \( i \in \{1, \ldots, p\} \), in other words \( P_i = P^\delta_i \), where \( \delta_i \) is the Dirac measure at \( i \) on \((1, \ldots, p)\).

Define the \( p \times p \) matrices \( P(n, t), \ n \in \mathbb{N}, \ t \geq 0 \), by \( (P(n, t))_{i,j} = P_i(N_1(t) = n, Z_t = j), \ i, j \in \{1, \ldots, p\} \), where \( Z_t \) denotes the phase of \( N_1 \) at time \( t \). Define the \( p \times p \) matrices \( A(k) \) and \( \bar{A}(k), \ k \in \mathbb{N} \) by

\[
A(k) = \int_0^\infty P(k, t) dG_2(t) \quad \text{and} \quad \bar{A}(k) = \int_0^\infty P(k, t) dF_2(t).
\]

Not surprisingly, these matrices are the same as the ones that frequently arise in PH/G/1- or GI/PH/1-queuing systems, see for example [9], [10] and [11].

In the next theorem we give the distribution of \( M \). Note that \( M \) is the limit of the decreasing sequence of random variables \((M_n)\), \( M_n = \min\{0, N(S^{(2)}_1), \ldots, N(S^{(2)}_n)\}, \ n \geq 0 \). We denote \( X^{(2)}_1 \) and \( N(S^{(2)}_n) \) by \( X \) and \( Y \), respectively. And \( Z \) denotes the phase of \( N_1 \) at time \( X \). Let \( 1 = \{1, \ldots, 1\}^T \).

5.1 THEOREM Let \( G \) be the unique minimal non-negative solution to the matrix equation

\[
X = \sum_{k=0}^\infty A(k)X^k,
\]

then for \( m \in \mathbb{N}_+ \),

\[
P^\gamma(M = -m) = \gamma (I - G)G^{m-1} \mathbf{1},
\]

and hence

\[
P^\gamma(M = 0) = \gamma (I - H) \mathbf{1},
\]

where

\[
H = \sum_{k=0}^\infty \bar{A}(k)G^k.
\]
As a consequence we have,

\[(5.5) \quad E^TM = -\gamma H(I-G)^{-1}1.\]

**Proof** First suppose \(N_2\) is an ordinary renewal process \((F_2 = G_2)\).

Denote the negative ladder epochs of \((N(S_2^{(2)}))\) by \(T_1, T_2, \ldots\) put \(T_0 = 0\) and, for \(n \geq 1\), let \(J_n\) denote the phase of \(N_1\) at time \(T_n\) if \(T_n < +\infty\), and \(J_n = \varnothing\) else, for some cemetery state \(\varnothing\). Let \(G\) be the matrix \((g_{ij})\), where \(g_{ij} = P_1(J_1 = j)\), \(i, j \in \{1, \ldots, p\}\). We first show that if \(G\) is sub-stochastic then

\[(5.6) \quad P(M = m) = (I-G)G^m1, \text{ for all } m \in N,\]

where \(P(M = m) = [P_1(M = m), \ldots, P_p(M = m)]^T\). The proof of (5.6) goes as follows: Obviously \((J_n)\) is a Markov chain taking values in \(\{1, \ldots, p\} \cup \{\varnothing\}\), with transition matrix

\[Q = \begin{bmatrix} G & (I-G)1 \\ 0 & 1 \end{bmatrix}.\]

Because \(M = \min(0, N(S_1^{(2)}), N(S_2^{(2)}), \ldots)\), we have for all \(m \in N_*\),

\[(5.7) \quad P_1(M = m) = P_1(J_{m+1} = \varnothing, J_1, \ldots, J_m = \varnothing)\]

\[= \sum_{j=1}^p P_1(J_{m+1} = \varnothing, J_2, \ldots, J_m = \varnothing | J_1 = j) P_1(J_1 = j) = \sum_{j=1}^p P_j(M = m+1) P_1(J_1 = j),\]

which leads to (5.6), provided that \(G\) is sub-stochastic (which we will show later). Next we show that \(G\) satisfies matrix equation (5.2). This follows immediately from Neuts [10], Chapter 2. Namely, matrix \(G\) is exactly the same as matrix \(G\) defined on Page 81 of [10]. Hence by Theorem 2.2.2 and (2.215) of [10] we deduce that \(G\) is the minimal non-negative solution to (5.2).

It remains to show that \(G\) is sub-stochastic and hence that (5.6) holds. This follows from an argument from PH/G/1-queues. Matrix \(G\) (the rate matrix) is stochastic if and only if the mean service time \(\sigma\) is smaller than or equal to the mean inter-arrival time \(\lambda\), cf. Neuts [9] p. 122. In our case \(\sigma = \mu_2 > \mu_1 = \lambda\) by assumption, which implies that \(G\) is sub-stochastic.
Finally consider a delayed renewal process $N_2$. Let $M = \inf \limits_{t \geq 0} N(t)$ and let $M^*$ denote a random variable with the same distribution as the $M$ if $N_2$ were an ordinary renewal process (in other words, the distribution of $M^*$ is given by (5.7)). Since now $P_1(Y=k-1, Z=j) = \tilde{\Lambda}(k)_{i,j}$, we have for $i \in \{1, \ldots, p\}$ and $m \in \mathbb{N}$,

$$P_1(M=m) = \sum_{k=0}^{\infty} \sum_{j=1}^{p} P_1(M=m \mid Y=k-1, Z=j) P_1(Y=k-1, Z=j)$$

$$= \sum_{k=0}^{\infty} \sum_{j=1}^{p} P_j(M^* = m-k+1) \tilde{\Lambda}(k)_{i,j},$$

which in matrix notation is

$$P(M=m) = \sum_{k=0}^{\infty} \tilde{\Lambda}(k)(I-G)G^{m-k-1}1 = H(I-G)G^{m-1}1, \text{ for } m \in \mathbb{N},$$

so that $P(M=0) = (I-H)1$. This concludes the proof, since $P^j(M=m) = \gamma P(M=m)$, for all $m \in \mathbb{N}$.

5.10 COROLLARY If $N_1$ is a Poisson process with intensity $\lambda = 1/\mu_1$, then the generating function of $M$ is given by

$$E r^{-M} = 1 + \frac{(r-1)A_2(\lambda(1-g))}{1 - rg}, \text{ for all } r \in \mathbb{C}, |r| \leq 1,$$

and hence

$$EM = \frac{A_2(\lambda(1-g))}{l - g},$$

where $g$ is the only zero of $f(y) = y - B_2(\lambda(1-y))$ inside the unit circle. (Remember that $A_2$ and $B_2$ are the Laplace-Stieltjes-transforms of $F_2$ and $G_2$, respectively.)

Proof Let $0 \leq g < 1$ denote the minimal non-negative solution to (5.2), which reduces in this case to

$$x = \sum_{k=0}^{\infty} \int_0^{\infty} x^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} dG_2(t) = B_2(\lambda(1-x)).$$

By the proof of Theorem 5.1, we know that $0 \leq g < 1$. By Rouché's theorem
the function \( f: \mathbb{C} \to \mathbb{C} \), defined by \( f(y) = y - B_2(\lambda(1-y)) \), \( y \in \mathbb{C} \) has only one solution inside the unit circle, which must therefore be equal to \( g \). Let \( h = H \) as defined in (5.4), then similarly to (5.13) we have \( h = A_2(\lambda(1-g)) \), so that (5.11) and (5.12) follow easily from equations (5.3) and (5.5).

5.14 REMARK In Neuts [10] is shown that \( G \) can be computed by means of successive substitutions in (5.2). Direct application of this procedure requires however a lot storage and is numerically not very simple. An efficient algorithm for solving the non-linear matrix equation (5.2) is given in Lucantoni & Ramaswami [11], where it is shown that \( G \) also satisfies the following system of equations

\[
(5.15) \quad X = \sum_{n=0}^{\infty} \gamma_n L_n
\]

\[
(5.16) \quad \left\{ \begin{align*}
L_0 &= I \\
L_{n+1} &= PL_n + P^0 \beta L_n X, \quad n \geq 0
\end{align*} \right.
\]

where \( \theta = \max_{1 \leq j \leq m} -S_{1j} \), \( \gamma_n = \int_0^\infty e^{-\theta t} (\theta t)^n \frac{dG_2(t)}{n!} \), \( P = \theta^{-1} S + I \), \( P^0 = 1 - P_1 \).

Remember that \((\beta, S)\) are the characteristics of \( N_1 \). The matrix \( G \) can be computed by iterating equations (5.15) and (5.16) by starting with the initial iterate \( X(0) = 0 \). Moreover, a short reflection will show that matrix \( H \) is given by

\[
(5.17) \quad H = \sum_{n=0}^{\infty} \tilde{\gamma}_n L_n, \quad \text{where} \quad \tilde{\gamma}_n = \int_0^\infty e^{-\theta t} (\theta t)^n \frac{dF_2(t)}{n!}, \quad n \in \mathbb{N}.
\]

5.18 REMARK Other closely related results on first-passage times and busy periods may be found in Hsu & He [5] and Asmussen [1], Chapter X.

6. Numerical results

To verify how well the asymptotic theory applies for finite \( n \), several experiments were carried out. Exact calculations were performed for
the case of Section 4. FORTRAN routines from the NAG library were used to compute modified Bessel functions and to perform numerical integration. In all other cases the true d.f. of \( r_n \) was estimated through Monte-Carlo simulation of size 10000. This simulation size reduces the error in the estimated d.f. to less than 0.01 with confidence 0.95.

For each example (except 6.3) a table is presented with the (estimated) true d.f. of \( r_n \), the (normal) first-order approximation and the second-order approximation which is given in (2.3). The first-order approximation is obtained from (2.3) by suppressing the second term on the right-hand side. Along with these tables come pictures of the approximation errors for the first- and second-order approximation. In the picture we made use of the unrounded numerical results.

6.1 REMARK Note that in the second-order approximations we need to calculate \( \bar{E}\bar{N} \) through the procedures described in Sections 3-5. Now it is important to remember that although \( N_1 \) and \( N_2 \) can be arbitrary delayed renewal processes characterized by \( F_1, F_2, G_1 \) and \( G_2 \), the number \( \bar{E}\bar{N} \) is the expected infimum of \( \bar{N} \), the stationary version of \( N \) (see Remark 2.1).

6.2 \( N_1 \) WEIBULL, \( N_2 \) POISSON

In our first example \( N_2 \) is a Poisson process with intensity \( \nu \), \( N_1 \) is stationary and \( G_1 \) is a Weibull d.f. with shape parameter \( \beta \) and scale parameter \( c \), i.e.

\[
G_1(x) = 1 - \exp((-cx)^\beta), \quad x \geq 0.
\]

For the parameter values we take \( \beta = 3, c = 0.3, \nu = 1/\mu_2 = 0.2 \) and \( n = 20 \). Note that since \( G_1 \) is a Weibull d.f.,

\[
\mu_1 = \Gamma(1+1/\beta)/c \quad \text{and} \quad \sigma_1^2 = \Gamma(1+2/\beta)/c^2 - \left(\Gamma(1+1/\beta)/c\right)^2
\]

(cf. p.31 of Beichelt & Franken [2]). Here \( \Gamma \) is the Gamma function.

Since \( N_1 \) is stationary, we have \( \eta_1 = (\mu_1 + \sigma_1^2/\mu_1)/2 \), where in the
Table 1. Estimated d.f. of $r_n$ together with first- and second-order approximation. Here $N_1$ is a stationary renewal process such that $G_1$ has a Weibull d.f. with parameters $\beta=3.0$, $\gamma=0.3$. $N_2$ is a Poisson process with intensity $0.2$. Level $n=20$.

<table>
<thead>
<tr>
<th>X</th>
<th>ESTIMATED D.F.</th>
<th>NORMAL APPR.</th>
<th>2ND. ORDER APPR.</th>
</tr>
</thead>
<tbody>
<tr>
<td>58.90</td>
<td>0.00</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>67.73</td>
<td>0.00</td>
<td>0.04</td>
<td>0.01</td>
</tr>
<tr>
<td>76.55</td>
<td>0.02</td>
<td>0.05</td>
<td>0.03</td>
</tr>
<tr>
<td>85.37</td>
<td>0.05</td>
<td>0.08</td>
<td>0.07</td>
</tr>
<tr>
<td>94.19</td>
<td>0.09</td>
<td>0.12</td>
<td>0.11</td>
</tr>
<tr>
<td>103.01</td>
<td>0.16</td>
<td>0.16</td>
<td>0.18</td>
</tr>
<tr>
<td>111.83</td>
<td>0.25</td>
<td>0.21</td>
<td>0.25</td>
</tr>
<tr>
<td>120.65</td>
<td>0.33</td>
<td>0.27</td>
<td>0.33</td>
</tr>
<tr>
<td>129.47</td>
<td>0.43</td>
<td>0.34</td>
<td>0.42</td>
</tr>
<tr>
<td>138.29</td>
<td>0.52</td>
<td>0.42</td>
<td>0.51</td>
</tr>
<tr>
<td>147.11</td>
<td>0.60</td>
<td>0.50</td>
<td>0.60</td>
</tr>
<tr>
<td>155.93</td>
<td>0.67</td>
<td>0.58</td>
<td>0.67</td>
</tr>
<tr>
<td>164.75</td>
<td>0.74</td>
<td>0.66</td>
<td>0.73</td>
</tr>
<tr>
<td>173.57</td>
<td>0.79</td>
<td>0.73</td>
<td>0.79</td>
</tr>
<tr>
<td>182.39</td>
<td>0.83</td>
<td>0.79</td>
<td>0.83</td>
</tr>
<tr>
<td>191.21</td>
<td>0.86</td>
<td>0.84</td>
<td>0.86</td>
</tr>
<tr>
<td>200.03</td>
<td>0.89</td>
<td>0.88</td>
<td>0.88</td>
</tr>
<tr>
<td>208.85</td>
<td>0.92</td>
<td>0.92</td>
<td>0.91</td>
</tr>
<tr>
<td>217.67</td>
<td>0.94</td>
<td>0.95</td>
<td>0.92</td>
</tr>
<tr>
<td>226.49</td>
<td>0.95</td>
<td>0.96</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Fig. 3. Error structure for the first- and second-order approximation of table 1.
Fig. 4. Error structure for second-order approximation to the d.f. of \( \tau \_n \) in the case that \( N_1 \) and \( N_2 \) are Poisson processes with \( \lambda = 1 \). Here \( n = 10 \) and \( \nu \) takes the values 0.0, 0.1, 0.3, 0.5 and 0.7.

In the present case we have \( \mu_1 = 2.976598342 \) and \( \sigma_1 = 1.0818341148 \), so that \( \eta_1 = 1.6848935579 \). Hence, by (3.6) we find \( EM = E \bar{M} = -0.83270345 \). Note that, since we deal here with stationary renewal processes, \( \bar{e} = 0 \) in Approximation 2.2 (cf. Remark 2.1).

6.3 \( N_1 \) POISSON, \( N_2 \) POISSON

We can expect poor results of approximation 2.2 when the drift term \( \alpha^{-1} \) is close to 0. In order to study the influence of \( \alpha^{-1} \) on the approximation we compare the second-order approximation for the doubly Poisson case of Section 4 with the exact d.f. of \( \tau_n \), given in Proposition 4.1. This is done for several values of \( \alpha^{-1} \). Specifically, we fix the intensity \( \lambda \) of \( N_1 \) to 1 and let the intensity \( \nu \) of \( N_2 \) vary from \( \nu = 0, 0.1, 0.3, 0.5 \) to 0.7 and we let \( n = 10 \).
6.4 $N_1$ Poisson, $N_2$ Hyper-Exponential

In this example $N_1$ is a Poisson process and $N_2$ a ordinary renewal process with hyper-exponential d.f. $G_2$, that is

$$G_2(x) = p(1-\exp(-\nu_1 x)) + (1-p)(1-\exp(-\nu_2 x)),$$

for some $0 \leq p \leq 1$ and $\nu_i \geq 0$, $i=1,2$. Here we take $\nu_1 = 1$, $\nu_2 = 2$, $p = 1/3$ and $\lambda = 2$. Moreover, we take $n = 15$. It is easy to see that

$$B_2(s) = p \frac{\nu_1}{\nu_1 + s} + (1-p) \frac{\nu_2}{\nu_2 + s}.$$

We need to calculate $E\tilde{W}$, which is obtained by applying Corollary 5.10 to the stationary version $\tilde{N}$ of $N$. Let $\tilde{F}_2$ denote the d.f. of the first inter-arrival epoch of $\tilde{N}_2$, as in Remark 2.1, and let $\tilde{A}_2$ denote the corresponding Laplace-Stieltjes-transform. Then it is not difficult to see that

$$\tilde{A}_2(s) = q \frac{\nu_1}{\nu_1 + s} + (1-q) \frac{\nu_2}{\nu_2 + s},$$

where

$$q = \frac{p \nu_2}{p \nu_2 + (1-p) \nu_1}.$$

In order to apply (5.12), we need to find $g$, the only zero inside the unit circle of function $f$, given by

$$f(y) = y - \frac{1/3}{1+2(1-y)} - \frac{4/3}{2+2(1-y)}, \quad |y|<1,$$

which is

$$g = (15 - \sqrt{33})/12 = 0.771286446.$$

And with $\tilde{A}_2(\lambda(1-g)) = 3/4$, we find $E\tilde{W} = -3.279210992$.

6.5 $N_1$ Erlang2, $N_2$ Erlang2

In the last numerical example we apply the theory developed in Section 5. We use the same notation as in Section 5. Let $N_1$ be an ordinary
Table 2. Estimated d.f. of $r_n$ together with first- and second-order approximation. Here $N_1$ is a Poisson process with intensity 2 and $N_2$ is an ordinary renewal process such that $G_2$ has a hyper-exponential d.f. with parameters $\nu_1 = 1$, $\nu_2 = 2$, $p = 1/3$. Level n=15.

<table>
<thead>
<tr>
<th>X</th>
<th>ESTIMATED D.F.</th>
<th>NORMAL 2ND.ORDER APPR.</th>
<th>2ND.ORDER APPR.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.08</td>
<td>0.02</td>
</tr>
<tr>
<td>3.66</td>
<td>0.00</td>
<td>0.11</td>
<td>0.07</td>
</tr>
<tr>
<td>7.31</td>
<td>0.03</td>
<td>0.15</td>
<td>0.13</td>
</tr>
<tr>
<td>10.97</td>
<td>0.11</td>
<td>0.19</td>
<td>0.20</td>
</tr>
<tr>
<td>14.63</td>
<td>0.22</td>
<td>0.24</td>
<td>0.28</td>
</tr>
<tr>
<td>18.28</td>
<td>0.33</td>
<td>0.29</td>
<td>0.36</td>
</tr>
<tr>
<td>21.94</td>
<td>0.45</td>
<td>0.35</td>
<td>0.45</td>
</tr>
<tr>
<td>25.60</td>
<td>0.54</td>
<td>0.42</td>
<td>0.53</td>
</tr>
<tr>
<td>29.25</td>
<td>0.62</td>
<td>0.49</td>
<td>0.61</td>
</tr>
<tr>
<td>32.91</td>
<td>0.68</td>
<td>0.55</td>
<td>0.67</td>
</tr>
<tr>
<td>36.57</td>
<td>0.73</td>
<td>0.62</td>
<td>0.72</td>
</tr>
<tr>
<td>40.22</td>
<td>0.78</td>
<td>0.68</td>
<td>0.76</td>
</tr>
<tr>
<td>43.88</td>
<td>0.81</td>
<td>0.74</td>
<td>0.79</td>
</tr>
<tr>
<td>47.54</td>
<td>0.84</td>
<td>0.79</td>
<td>0.81</td>
</tr>
<tr>
<td>51.19</td>
<td>0.87</td>
<td>0.84</td>
<td>0.83</td>
</tr>
<tr>
<td>54.85</td>
<td>0.89</td>
<td>0.88</td>
<td>0.84</td>
</tr>
<tr>
<td>58.51</td>
<td>0.91</td>
<td>0.91</td>
<td>0.85</td>
</tr>
<tr>
<td>62.16</td>
<td>0.92</td>
<td>0.93</td>
<td>0.87</td>
</tr>
<tr>
<td>65.82</td>
<td>0.93</td>
<td>0.95</td>
<td>0.88</td>
</tr>
<tr>
<td>69.48</td>
<td>0.94</td>
<td>0.97</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Fig. 5. Error structure for the first- and second-order approximation of table 2.
Table 3. Estimated d.f. of $r_n$ together with first- and second-order approximation. Here $N_1$ and $N_2$ are ordinary Erlang2-renewal processes and level $n=10$.

<table>
<thead>
<tr>
<th>X</th>
<th>ESTIMATED DIST. FNC</th>
<th>NORMAL APPR.</th>
<th>2ND. ORDER APPR.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
<td>-0.01</td>
</tr>
<tr>
<td>3.03</td>
<td>0.00</td>
<td>0.04</td>
<td>0.00</td>
</tr>
<tr>
<td>6.07</td>
<td>0.00</td>
<td>0.06</td>
<td>0.02</td>
</tr>
<tr>
<td>9.10</td>
<td>0.02</td>
<td>0.09</td>
<td>0.06</td>
</tr>
<tr>
<td>12.13</td>
<td>0.06</td>
<td>0.12</td>
<td>0.11</td>
</tr>
<tr>
<td>15.16</td>
<td>0.14</td>
<td>0.17</td>
<td>0.18</td>
</tr>
<tr>
<td>18.20</td>
<td>0.24</td>
<td>0.22</td>
<td>0.27</td>
</tr>
<tr>
<td>21.23</td>
<td>0.35</td>
<td>0.28</td>
<td>0.36</td>
</tr>
<tr>
<td>24.26</td>
<td>0.45</td>
<td>0.35</td>
<td>0.45</td>
</tr>
<tr>
<td>27.30</td>
<td>0.55</td>
<td>0.43</td>
<td>0.55</td>
</tr>
<tr>
<td>30.33</td>
<td>0.63</td>
<td>0.51</td>
<td>0.63</td>
</tr>
<tr>
<td>33.36</td>
<td>0.71</td>
<td>0.59</td>
<td>0.70</td>
</tr>
<tr>
<td>36.40</td>
<td>0.76</td>
<td>0.66</td>
<td>0.76</td>
</tr>
<tr>
<td>39.43</td>
<td>0.81</td>
<td>0.73</td>
<td>0.80</td>
</tr>
<tr>
<td>42.46</td>
<td>0.85</td>
<td>0.79</td>
<td>0.83</td>
</tr>
<tr>
<td>45.49</td>
<td>0.88</td>
<td>0.84</td>
<td>0.85</td>
</tr>
<tr>
<td>48.53</td>
<td>0.90</td>
<td>0.89</td>
<td>0.87</td>
</tr>
<tr>
<td>51.56</td>
<td>0.92</td>
<td>0.92</td>
<td>0.89</td>
</tr>
<tr>
<td>54.59</td>
<td>0.94</td>
<td>0.95</td>
<td>0.91</td>
</tr>
<tr>
<td>57.63</td>
<td>0.95</td>
<td>0.96</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Fig. 6. Error structure for the first- and second-order approximation of table 3.
PH-process with characteristics \((\beta, S)\) (under \(P^\beta\)), given by

\[ S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \] and \(\beta = [1, 0]\),

and let \(N_2\) be an ordinary renewal process with (generalized) Erlang2 inter-arrival d.f. \(G_2\), given by

\[ G_2(x) = 1 + \frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \left( \frac{e^{-\nu_2 x}}{\nu_2} - \frac{e^{-\nu_1 x}}{\nu_1} \right), \quad x \geq 0, \]

where we take \(\nu_1 = 1\) and \(\nu_2 = 2\). Note that \(G_1\) is an Erlang2 d.f. as well. Moreover, \(N_1\) is an ordinary renewal process only under \(P^\gamma\). Under \(P^\gamma\), with \(\gamma = [1/2, 1/2]\), \(N_1\) is stationary. We take \(n = 10\).

\(E\tilde{N}\) is obtained by application of Theorem 5.1 to the stationary version \(\tilde{N}\) of \(N\). Using the iteration scheme of Remark 5.14, we find that the matrices \(G\) and \(H\) are equal to

\[ G = \begin{bmatrix} 0.2312992 & 0.3382979 \\ 0.078248025 & 0.3457446 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0.333333 & 0.333333 \\ 0.077099734 & 0.4460993 \end{bmatrix}. \]

Application of (5.5) with \(\gamma = [1/2, 1/2]\), yields \(E\tilde{N} = -1.12027\).

### 6.6 Conclusions

Several other experiments have been carried out, all giving similar results. The second-order approximations indeed give considerable improvement on the first-order approximations, even for quite small \(n\). The results show good agreement with the theoretical error structure as described in [8]. Compared to the first-order approximation, the new one has additional corrections for systematic shift and skewness, corresponding to the terms \(q \varphi(y_n)n^{-1/2}\) and \(p(1-y_n^{-1/2})\varphi(y_n)n^{-1/2}\), respectively, in (2.3). In [6] similar results for the sum process of \(k\) independent renewal processes were found. Approximation 2.2 tends to give the best results for \(t\) near the expectation of \(\tau_n\). In the various examples we see that for values of \(t\) that are relatively far away from
this expectation the second-order approximation is only slightly better than the first-order one. This is partly due to the relative smallness of $q$ in these cases, which means that there is only an additional correction for skewness in (2.3), when compared to the first-order approximation. But perhaps these kind of approximations are not very suited for such small or large $t$, and one should be looking for other approximations in this range. When $\alpha^{-1}$ is close to zero, application of approximation 2.2 is not very appropriate, as indicated by figure 4. In the case that $\alpha^{-1} = 0$, $\tau_n$ has, when properly normalized, for large $n$ approximately a stable distribution of order 1/2, and not a normal distribution.

REFERENCES


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