A fluid queue driven by a fluid queue

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Abstract Consider a fluid system consisting of two reservoirs. The first reservoir receives and releases fluid at rates which depend on the state of a two-state Markov process. The second reservoir, which is called a credit buffer, is driven by the first one, in such a way that it accumulates credit at a constant rate when the first buffer is empty, and releases credit at a constant rate otherwise. The first reservoir models an overflow buffer at the entrance of an ATM-telecommunication network, while the amount of credit in the second buffer is an indication for the load of the connection.

We obtain the stationary joint distribution of both fluid reservoirs via a spectral analysis of an approximating fluid model, in which the second reservoir is regulated by a countably infinite Markov process.

Keywords: Fluid flow model, Markov modulated rate process, infinite state space, spectral analysis, joint distribution.

AMS Subject Classifications (1991): Primary 60K25; Secondary 90B22.

1 Introduction

Markov modulated fluid flow models have turned out to be useful to capture the basic characteristics of a variety of systems arising in modern telecommunication networks, especially those based on the Asynchronous Transfer Mode (ATM). The basic ingredients of such a model are a fluid reservoir and a regulating Markov process (X_t). The rate of change of the content of the fluid reservoir (or the *net input rate*) is determined by the current state of the Markov process.

In case (X_t) has a finite state space, it is well-known that the joint distribution of the Markov process and the content of the fluid reservoir can be determined analytically, at least in principle, see, e.g., [3], [11], [4] and [7]. Also for the case in which (X_t) is a birth-death process on an infinite state space, several analytical results are known, see [14], [2], [5] and [6]. In this report, we consider a particular case in which (X_t) has a non-denumerable state space. Specifically, the net input rate depends on the content of another fluid reservoir, which in turn is being regulated by a two-state Markov process.

The way in which the first reservoir modulates the input rate of the second one is such that the second reservoir (or *credit buffer*) is being filled when the first reservoir (or *data buffer*) is empty. Furthermore, the credit buffer is being depleted as long as the data buffer is non-empty. The interpretation is the following. The data buffer can be viewed as a model for an overflow buffer at the edge of a network, which receives data only during on periods of an on-off data source, and which releases data as long as it is non-empty at a constant rate (the capacity of the outgoing link). The amount of credit in the credit buffer is an indication for the load of the connection.

Our primary focus will not be the practical relevance of the model. Rather, this report may be viewed as a contribution to the theory of fluid queues, since it is, at least to our knowledge, the first Markov modulated fluid model for which the stationary distribution can be obtained when the modulating Markov process has a non-denumerable state space. Moreover, it is a first step towards the analysis of more complicated fluid models which do have relevance for the performance analysis of ATM-based communication networks. Two such models, that are currently being investigated, will be described elsewhere, namely a model for a two-level traffic shaper in [10] (a generalisation of the model in [1]) and a model for a simple tandem queue in [9].

The structure of the report is as follows. First, we will give a more precise description of the model at hand and some preliminaries in Section 2. Then, in Section 3 we consider an approximative model, in which the credit buffer is being regulated by a Markov process with a countably infinite state space. Although this process has no birth-death structure, we will use similar techniques as in Sections 3 and 4 of [5], to indicate how this model can be solved, without working out the details of the analysis. The main result of the paper is presented in Section 4, where the stationary joint distribution of the on-off source, data buffer and credit buffer is obtained, using the results of Section 3. Finally, a rigourous proof of the main result is given in the Appendix.

2 The model and some preliminaries

Consider a fluid system consisting of two reservoirs, which we will call *data buffer* and *credit buffer*. The content of the data buffer is regulated by an on-off fluid source with exponentially distributed on-times and off-times, with parameters μ and λ, respectively. Thus, the state of the source is given by a continous-time Markov process (M_t) with state space {0,1} (where 0 and 1 represent the off-state and on-state respectively) and generator

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}. \tag{1}$$

The content of the data buffer is regulated by (M_t) in the following way. The net input rate into the buffer at time t is $d_+ > 0$ if $M_t = 1$ and $-d_- < 0$ if $M_t = 0$, provided that the data buffer is non-empty.

The content of the second reservoir, the credit buffer, is regulated by the content of the data buffer in the following way. The net input rate into the credit buffer is $c_+ > 0$ if the data buffer is empty, and $-c_- < 0$ otherwise, provided that the credit buffer is non-empty.

We let D_t and C_t denote the content of data buffer and credit buffer at time t, respectively. A schematic overview of the interaction between the three subsystems is given in Figure 1. Note that the rate parameters d_+, d_-, c_+, c_- are positive numbers. Their meaning is reflected in the notation.

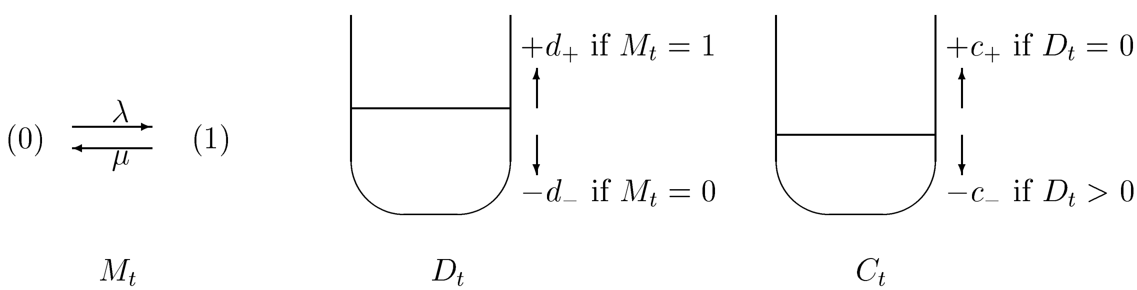

Figure 1: Interaction between the subsystems of the total fluid flow system.

Clearly, (M_t, D_t, C_t) is a Markov process with state space $\{0,1\} \times \mathbb{R}_+ \times \mathbb{R}_+$. It will be our goal to find the stationary distribution of this process. In the next subsection the stability conditions under which this distribution exists are given, and in Subsection 2.2 the system of partial differential equations will be derived, which must be satisfied by the solution we are looking for.

2.1 Stability

To have stability of the content of the data buffer, it is necessary and sufficient that the average net rate of data flowing into the buffer, given enough data in the buffer, is negative.

Thus, we must have $d_+ \Pr[M = 1] - d_- \Pr[M = 0] < 0$, or equivalently

$$\lambda/d_- < \mu/d_+. \tag{2}$$

Assuming that this condition is satisfied, it is possible to determine the condition under which stability of the credit buffer is guaranteed. To do so, we first observe that the regulating process (M_t, D_t) corresponds to a finite-state fluid queue. Assuming that (M_t, D_t) is in equilibrium, we have (see, e.g., [3]),

$$\begin{aligned}
\Pr[M_t = 0, D_t \leq x] &= \frac{\mu}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\frac{d_+}{d_-}\, e^{-\alpha x}, \qquad x \geq 0, \\
\Pr[M_t = 1, D_t \leq x] &= \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\, e^{-\alpha x}, \qquad x \geq 0,
\end{aligned} \tag{3}$$

where

$$\alpha = \frac{\mu}{d_+} - \frac{\lambda}{d_-}. \tag{4}$$

¿From (3), it is immediate that $\Pr[D_t = 0] = d_+\alpha/(\lambda+\mu)$, so that the stability condition for the credit buffer, $c_+ \Pr[D_t = 0] - c_- \Pr[D_t > 0] < 0$, reduces to

$$\frac{\lambda}{c_+} > \frac{\mu d_-}{c_- d_- + c_- d_+ + c_+ d_+}. \tag{5}$$

We will assume both (2) and (5) to be satisfied in what follows.

2.2 The differential equations

Our goal is to find the stationary distribution of the process (M_t, D_t, C_t). To this end, we derive a system of differential equations from the Kolmogorov forward equations.

First, we define the time-dependent probability distribution of the Markov process (M_t, D_t, C_t) as

$$F_i(t, x, y) \equiv \Pr[M_t = i,\ D_t \leq x,\ C_t \leq y], \quad t, x, y \geq 0,\ i \in \{0, 1\}.$$

In the derivation of the Kolmogorov forward equations, we consider three different cases separately, namely the case where both x and y are strictly positive, the case $x = 0$ and the case $y = 0$. As an example, the Kolmogorov forward equations for the case $x, y > 0$ are (for $t \geq 0$) given by

$$\begin{aligned}
\frac{\partial F_0(t,x,y)}{\partial t} - d_- \frac{\partial F_0(t,x,y)}{\partial x} - c_- \frac{\partial F_0(t,x,y)}{\partial y} &= -\lambda F_0(t,x,y) + \mu F_1(t,x,y)\ , \\
\frac{\partial F_1(t,x,y)}{\partial t} + d_+ \frac{\partial F_1(t,x,y)}{\partial x} - c_- \frac{\partial F_1(t,x,y)}{\partial y} &= +\lambda F_0(t,x,y) - \mu F_1(t,x,y)\ .
\end{aligned}$$

Since we are looking for a stationary distribution, we assume that (M_t, D_t, C_t) is in equilibrium. Hence, we set $(\partial/\partial t)F_i(t,x,y) \equiv 0$ and replace $F_i(t,x,y)$ by $F_i(x,y)$ where

$$F_i(x,y) \equiv \Pr[M_t = i,\ D_t \le x,\ C_t \le y] \qquad x,y \ge 0,\ i \in \{0,1\}. \tag{6}$$

Thus, we obtain for $x, y > 0$,

$$\begin{aligned} -\lambda F_0(x,y) + \mu F_1(x,y) &= -d_- \frac{\partial F_0(x,y)}{\partial x} - c_- \frac{\partial F_0(x,y)}{\partial y} \\ +\lambda F_0(x,y) - \mu F_1(x,y) &= +d_+ \frac{\partial F_1(x,y)}{\partial x} - c_- \frac{\partial F_1(x,y)}{\partial y} \ . \end{aligned}$$

Similar equations can be found for the cases $x = 0$ and $y = 0$.

Denoting the stationary distribution $\mathbf{F}$ by

$$\mathbf{F}(x,y) \equiv (F_0(x,y), F_1(x,y))^T, \qquad x,y \ge 0, \tag{7}$$

it follows that $\mathbf{F}$ must be a solution of the system of partial differential equations

$$Q^T \mathbf{F}(x,y) = \begin{cases} (\mathcal{A}_0 \mathbf{F})(x,y), & x > 0,\ y \ge 0 \\ (\mathcal{A}_1 \mathbf{F})(0,y), & y > 0, \end{cases} \tag{8}$$

where Q is given in (1),

$$\mathcal{A}_0 = \begin{pmatrix} -d_- \frac{\partial}{\partial x} - c_- \frac{\partial}{\partial y} & 0 \\ 0 & d_+ \frac{\partial}{\partial x} - c_- \frac{\partial}{\partial y} \end{pmatrix}, \tag{9}$$

and

$$\mathcal{A}_1 = \begin{pmatrix} -d_- \frac{\partial}{\partial x} + c_+ \frac{\partial}{\partial y} & 0 \\ 0 & d_+ \frac{\partial}{\partial x} - c_- \frac{\partial}{\partial y} \end{pmatrix}. \tag{10}$$

We want to find a solution to (8), subject to the normalization condition

$$F_0(\infty,\infty) + F_1(\infty,\infty) = 1. \tag{11}$$

Note that, since the credit buffer cannot be empty while the data buffer is empty, we must have

$$F_0(0,0) = 0. \tag{12}$$

Also, the solution should be in agreement with (3) as we let y grow to infinity; in particular we have $F_1(0,y) = 0$ for $y \ge 0$.

There are several ways to solve (8). We will apply a spectral-analysis method in Section 4, using the results for an approximating model, which will be considered in Section 3.

3 The approximating model

This section is devoted to an approximation of the model in Section 2, for which the state space of the modulating Markov process is denumerable. We prove a lemma about a system of difference equations and indicate how this can be used to solve the (approximating) model. More important however is that in Section 4, we can show that, by taking appropriate limits, the system of difference equations leads to a system of differential equations, which is used to find the solution to the original model.

Specifically, we approximate the process (M_t, D_t) by another Markov process $(M_t, \tilde{D}_t)$, in which $\tilde{D}_t$ now represents the number of *data blocks* in the data buffer. Instead of a continuous inflow of data during on-periods, we suppose that data blocks of size $1/n$ arrive according to a Poisson process with rate nd_+. In order to avoid the possibility of an empty data buffer during on-periods, we will suppose that a block is added to the buffer as soon as an on-period starts. Also, we suppose that blocks are removed from the buffer during off periods according to a Poisson process with rate nd_-. Thus, the transition diagram of $(M_t, \tilde{D}_t)$ is given in Figure 2, and the corresponding (infinite-dimensional) generator has the following block-tridiagonal structure.

$$T \equiv \begin{pmatrix} H_0 & D_+ & & & \\ D_- & H & D_+ & & \\ & D_- & H & D_+ & \\ & & \cdots & \cdots & \cdots \end{pmatrix}, \tag{13}$$

where

$$H = \begin{pmatrix} -\lambda - nd_- & 0 \\ \mu & -\mu - nd_+ \end{pmatrix}, \qquad H_0 = \begin{pmatrix} -\lambda & 0 \\ \mu & -\mu - nd_+ \end{pmatrix},$$

$$D_- = \begin{pmatrix} nd_- & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D_+ = \begin{pmatrix} 0 & \lambda \\ 0 & nd_+ \end{pmatrix}$$

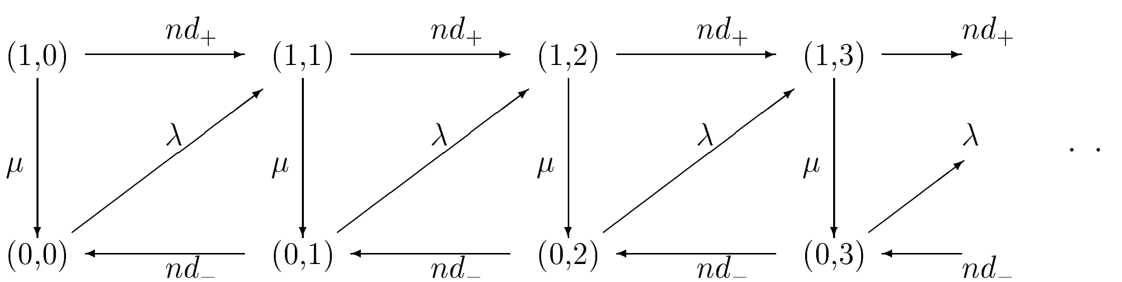

Figure 2: Transition diagram of the Markov process $(M_t, \tilde{D}_t, C_t)$.

We assume that the Markov process $(M_t, \tilde{D}_t, C_t)$ is ergodic, and we will write $\mathbb{G}$ for its stationary distribution, with

$$\mathbb{G}(y) \equiv (\mathbf{G}_0(y), \mathbf{G}_1(y), \mathbf{G}_2(y), \ldots)^T, \qquad y \geq 0, \tag{14}$$

where

$$\mathbf{G}_j(y) = (G_{0,j}(y), G_{1,j}(y)), \quad j \in \mathbb{N},\ y \geq 0,$$

with

$$G_{i,j}(y) \equiv \Pr[M_t = i, \tilde{D}_t = j, C_t \leq y], \quad i \in \{0,1\},\ j \in \mathbb{N},\ y \geq 0.$$

Notation. Throughout, blackboard bold characters (such as $\mathbb{G}$, $\mathbb{P}$, etc.) will indicate vectors that can typically be partitioned into two-dimensional vectors (corresponding to the two possible values for M_t), which will be written in bold-faced type.

Note that the Markov process $(M_t, \tilde{D}_t)$ has been constructed such that state (1,0) is transient. We could therefore eliminate it from the state space as we are interested in stationary behaviour only. However, to maintain our notational convention we prefer not to do so, but rather let $G_{1,0}(y) = 0$ for all $y \geq 0$.

It can be shown, by writing down the Kolmogorov forward equations for the system and then assuming equilibrium, see, e.g. [3], that $\mathbb{G}$ must be a solution of the differential equation

$$\frac{d}{dy}\mathbb{G}(y) = R^{-1}T^T\mathbb{G}(y). \tag{15}$$

Here, T is the generator in (13), while the diagonal matrix R, containing the net input and output rates of credit in the various states of the modulating proces $(M_t, \tilde{D}_t)$, is given by

$$R \equiv \mathrm{diag}(c_+, 1, -c_-, -c_-, \ldots), \tag{16}$$

Note that the choice of the second diagonal element is not relevant. Since the credit buffer cannot be empty while the data buffer is empty, the solution must satisfy the boundary condition

$$G_{0,0}(0) = 0. \tag{17}$$

Moreover, we must impose

$$\lim_{y\to\infty} \sum_{j=0}^{\infty} (G_{0,j}(y) + G_{1,j}(y)) = 1. \tag{18}$$

To analyse (15), we apply a similar procedure as described in [5, Section 4], see also [1]. First, we truncate the state space of the process $(M_t, \tilde{D}_t)$ to $\{0,1\} \times \{0,1,2,\ldots,m\}$, for $m > 0$. Also, we eliminate $(1,0)$ from the state space. The generator is changed to the $(2m+1) \times (2m+1)$-matrix

$$T_m \equiv \begin{pmatrix} -\lambda & \mathbf{d}_+ & & & \\ \mathbf{d}_- & H & D_+ & & \\ & \cdots & \cdots & \cdots & \\ & & D_- & H & D_+ \\ & & & D_- & H_1 \end{pmatrix}, \tag{19}$$

where H, D_+ and D_- are as before, and

$$\begin{aligned} H_1 &= \begin{pmatrix} -nd_- & 0 \\ \mu & -\mu \end{pmatrix}, \\ \mathbf{d}_+ &= (0, \lambda), \\ \mathbf{d}_- &= (nd_-, 0)^T, \end{aligned}$$

and we let

$$R_m \equiv \mathrm{diag}(c_+, \overbrace{-c_-, -c_-, \ldots, -c_-}^{2m}). \tag{20}$$

The solution of the differential system

$$\frac{d}{dy}\mathbb{G}_m(y) = R_m^{-1}T_m^T\mathbb{G}_m(y). \tag{21}$$

is now formally given by

$$\mathbb{G}_m(y) = e^{-R_m^{-1}T_m^T y}\mathbb{G}_m(0).$$

By [13], we know that the eigenvalues ξ_k, $k = 0, 1, \ldots, 2m$, of the matrix $R_m^{-1}T_m^T$, when ordered properly, satisfy $\xi_0 < 0 = \xi_1 < \mathrm{Re}(\xi_2) \le \mathrm{Re}(\xi_3) \le \cdots \le \mathrm{Re}(\xi_{2m})$, where ξ_0 and ξ_1 are real. Since we are looking for a bounded solution, $\mathbb{G}_m$ must be of the form

$$\mathbb{G}_m(y) = c_0\mathbb{v}_0 e^{\xi_0 y} + c_1\mathbb{v}_1, \quad y \ge 0,$$

where $\mathbb{v}_k$ is a suitably normalized eigenvector corresponding to ξ_k and c_k is a constant, $k = 0, 1$. The values for c_0 and c_1 follow from boundary conditions, similar to the ones in (17) and (18). Therefore we may conclude that, *for any* m, the solution of (21) is of the form

$$\mathbb{G}_m(y) = \mathbb{P}_m - \mathbb{v}_m e^{-\beta_m y}, \quad y \ge 0.$$

Here, $\mathbb{P}_m$ denotes the stationary distribution of the truncated modulating Markov process, $-\beta_m$ is the unique negative eigenvalue of $R_m^{-1}T_m^T$, and $\mathbb{v}_m$ is the corresponding, suitably normalized eigenvector. Finally, letting $m \to \infty$, we get the same form for the original process,

$$\mathbb{G}(y) = \mathbb{P} - \mathbb{v}e^{-\beta y}, \quad y \ge 0, \tag{22}$$

where

$$\begin{aligned} \beta &= \lim_{m\to\infty}\beta_m, \\ \mathbb{P} &= (\mathbf{p}_0, \mathbf{p}_1, \ldots)^T, \quad \text{with} \quad \mathbf{p}_j = (p_{0,j}, p_{1,j}), \quad j = 0, 1, \ldots, \\ \mathbb{V} &= (\mathbf{v}_0, \mathbf{v}_1, \ldots)^T, \quad \text{with} \quad \mathbf{v}_j = (v_{0,j}, v_{1,j}), \quad j = 0, 1, \ldots, \end{aligned}$$

respectively. After determining β, $\mathbb{P}$ and $\mathbb{v}$, it can be shown by substitution that (22) indeed is a solution of (15). Note that the components of $\mathbb{P}$ and $\mathbb{v}$ have a probabilistic interpretation, namely

$$\begin{aligned} p_{i,j} &= \Pr[M_t = i, \tilde{D}_t = j], \qquad i \in \{0,1\}, j \in \mathbb{N}, \\ v_{i,j} &= \Pr[M_t = i, \tilde{D}_t = j, C_t > 0], \quad i \in \{0,1\}, \ j \in \mathbb{N}, \end{aligned}$$

so that we immediately have that $p_{1,0} = v_{1,0} = 0$.

To learn more about the relation between β and $\mathbb{v}$, we investigate the eigenvalue problem

$$T^T \mathbb{v} = -\beta R \mathbb{v}. \tag{23}$$

Writing out (23) (where we leave out the equation concerning state $(1,0)$, and set $v_{1,0} = 0$), we obtain

$$\begin{aligned} -\lambda v_{0,0} + nd_- v_{0,1} &= -\beta c_+ v_{0,0} \\ \lambda v_{0,0} - (\mu + nd_+) v_{1,1} &= \beta c_- v_{1,1}, \end{aligned}$$

and, for $j = 1, 2, \ldots,$

$$\begin{aligned} -(\lambda + nd_-) v_{0,j} + \mu v_{1,j} + nd_- v_{0,j+1} &= \beta c_- v_{0,j} \\ \lambda v_{0,j} + nd_+ v_{1,j} - (\mu + nd_+) v_{1,j+1} &= \beta c_- v_{1,j+1}. \end{aligned}$$

The following lemma is now immediate.

Lemma 1 *The components of the eigenvector* $\mathbb{v}$ *corresponding to eigenvalue* $-\beta$ *satisfy the difference equation*

$$\mathbf{v}_{j+1} = A \mathbf{v}_j \quad j = 1, 2, \ldots, \tag{24}$$

where

$$A = \begin{pmatrix} \dfrac{\lambda + nd_- + \beta c_-}{nd_-} & -\dfrac{\mu}{nd_-} \\ \dfrac{\lambda}{\mu + nd_+ + \beta c_-} & \dfrac{nd_+}{\mu + nd_+ + \beta c_-} \end{pmatrix},$$

with initial condition

$$\mathbf{v}_1 = \begin{pmatrix} \dfrac{-\beta c_+ + \lambda}{nd_-} \\ \dfrac{\lambda}{\mu + nd_+ + \beta c_-} \end{pmatrix} v_{0,0} \ . \tag{25}$$

Since det $A > 0$ and det $(A - I_2) < 0$ for $\beta > 0$, it turns out that for the eigenvalues $\zeta_1(\beta)$ and $\zeta_2(\beta)$ of A we have $0 < \zeta_1(\beta) < 1 < \zeta_2(\beta)$. Therefore, in order for a bounded solution $\mathbb{v}$ to exist, $\mathbf{v}_1$ must be in the eigenspace of $\zeta_1(\beta)$, which gives an equation from which we can determine β. Also, since $\mathbb{p}$ is the eigenvector corresponding to eigenvalue 0, we can use Lemma 1 to find an expression for its components. Finally, applying conditions (17) and (18), it is possible to solve the model exactly. We will however proceed by showing how the results of this section can be used to obtain the solution of our original problem.

4 The stationary distribution

In this section we will show how the stationary distribution $\mathbf{F}$ of the process (M_t, D_t, C_t) can be found, using the results for the approximating system of the previous section. We will first derive an analogon to Lemma 1 for our original model. Then, we proceed by carrying out an analysis similar to the one suggested after Lemma 1, which will lead to the main result of this report.

We first consider a sequence of auxiliary systems, as described in the previous section. However, now we are interested in the *amount of data* in the data buffer, rather than the *number of data blocks* (apart from the state of the on-off source and the content of the credit buffer). Concretely, let $M_t^{(n)}$, $D_t^{(n)}$ and $C_t^{(n)}$ denote the state of the on-off source, the amount of data and the amount of credit at time t, respectively, for any $n \in \mathbb{N}\backslash\{0\}$, where $1/n$ is the size of a data block as before. In particular, $D_t^{(n)} = \tilde{D}_t/n$. The stationary distribution of the Markov process $(M_t^{(n)}, D_t^{(n)}, C_t^{(n)})$ is for any n given by

$$F_i^{(n)}(x,y) \equiv \Pr[M_t^{(n)} = i,\ D_t^{(n)} \le x,\ C_t^{(n)} \le y], \qquad x, y \ge 0,\ i \in \{0,1\},$$

or, in vector notation

$$\mathbf{F}^{(n)}(x,y) \equiv (F_0^{(n)}(x,y), F_1^{(n)}(x,y))^T, \qquad x, y \ge 0,$$

and can be expressed easily in terms of the quantities in Section 3, namely

$$\mathbf{F}^{(n)}(x,y) = \sum_{j=0}^{\lfloor xn \rfloor} \mathbf{G}_j^{(n)}(y) = \sum_{j=0}^{\lfloor xn \rfloor} \mathbf{p}_j^{(n)} - \mathbf{v}_j^{(n)} e^{-\beta^{(n)} y} \qquad x, y \ge 0,$$

where we have indicated dependence on n.

Clearly, the stochastic process $(M_t^{(n)}, D_t^{(n)}, C_t^{(n)})$ is a good approximation for the original process (M_t, D_t, C_t) if n is large. It seems therefore plausible that the stationary distribution $\mathbf{F}$ of the latter satisfies

$$\mathbf{F}(x,y) = \lim_{n\to\infty} \mathbf{F}^{(n)}(x,y), \qquad x, y \ge 0.$$

We draw the conclusion that

$$\mathbf{F}(x,y) = \mathbf{P}(x) - \mathbf{V}(x) e^{-\beta y}, \qquad x, y \ge 0, \tag{26}$$

assuming that the following limits exist,

$$\mathbf{P}(x) = \lim_{n\to\infty} \sum_{j=0}^{\lfloor xn \rfloor} \mathbf{p}_j^{(n)}, \qquad x \ge 0, \tag{27}$$

$$\mathbf{V}(x) = \lim_{n\to\infty} \sum_{j=0}^{\lfloor xn \rfloor} \mathbf{v}_j^{(n)}, \qquad x \ge 0, \tag{28}$$

$$\beta = \lim_{n\to\infty} \beta^{(n)}. \tag{29}$$

We note that for $\mathbf{v}(x) \equiv \mathbf{V}'(x)$ we have

$$\mathbf{v}(x) = \lim_{n\to\infty} n\mathbf{v}^{(n)}_{\lfloor xn \rfloor + 1}, \qquad x \geq 0, \tag{30}$$

assuming the right hand side exists, since for $x \geq 0$

$$\begin{aligned}\int_0^x \lim_{n\to\infty} n\mathbf{v}^{(n)}_{\lfloor tn \rfloor + 1} dt &= \lim_{n\to\infty} \int_0^x n\mathbf{v}^{(n)}_{\lfloor tn \rfloor + 1} dt \\ &= \lim_{n\to\infty} \left(\sum_{j=1}^{\lfloor xn \rfloor} \mathbf{v}^{(n)}_j + \mathbf{v}^{(n)}_{\lfloor xn \rfloor + 1} (xn - \lfloor xn \rfloor) \right) = \mathbf{V}(x) - \mathbf{V}(0).\end{aligned}$$

Similarly, we have

$$\mathbf{v}'(x) = \lim_{n\to\infty} n^2 (\mathbf{v}^{(n)}_{\lfloor xn \rfloor + 2} - \mathbf{v}^{(n)}_{\lfloor xn \rfloor + 1}), \qquad x \geq 0. \tag{31}$$

We are now ready to prove the following Lemma.

Lemma 2 *If it exists, the vector function* $\mathbf{v}(x)$ *satisfies the differential equation*

$$\mathbf{v}'(x) = B\mathbf{v}(x) \qquad x \geq 0, \tag{32}$$

where

$$B = \begin{pmatrix} \dfrac{\lambda + \beta c_-}{d_-} & -\dfrac{\mu}{d_-} \\ \dfrac{\lambda}{d_+} & -\dfrac{\mu + \beta c_-}{d_+} \end{pmatrix},$$

with initial condition

$$\mathbf{v}(0) = \begin{pmatrix} \dfrac{-\beta c_+ + \lambda}{d_-} \\ \dfrac{\lambda}{d_+} \end{pmatrix} V_0(0). \tag{33}$$

Proof. By (24), we have

$$\mathbf{v}^{(n)}_{\lfloor xn \rfloor + 2} - \mathbf{v}^{(n)}_{\lfloor xn \rfloor + 1} = (A^{(n)} - I)\mathbf{v}^{(n)}_{\lfloor xn \rfloor + 1}\,, \qquad x \geq 0,$$

where we have indicated the dependence on n of matrix A in Lemma 1. Multiplying both sides of this equation by n^2 and taking the limit for $n \to \infty$ while applying (30) and (31), yields (32), where $B = \lim_{n\to\infty} n(A^{(n)} - I)$. Similarly, (33) follows from multiplying (25) by n and taking the limit for $n \to \infty$. $\Box$

Since for $-\beta < 0$ also $\det B < 0$, we have for the eigenvalues $\zeta_1(\beta)$ and $\zeta_2(\beta)$ of B that $\zeta_1(\beta) < 0 < \zeta_2(\beta)$. For $\mathbf{v}(x)$ to be bounded for $x \geq 0$, we must therefore have that $\mathbf{v}(0)$ is an eigenvector of $\zeta_1(\beta)$, or equivalently, $\mathbf{v}(0)$ must be orthogonal to the left eigenvector of $\zeta_2(\beta)$. After some calculus it follows that this is the case if and only if

$$\beta = \frac{\lambda}{c_+} - \frac{\mu d_-}{c_- d_- + c_- d_+ + c_+ d_+}. \tag{34}$$

Using (34), we now solve (32) – (33) and obtain

$$\mathbf{v}(x) = e^{-\gamma x}\mathbf{v}(0), \qquad x \geq 0,$$

where

$$\gamma = -\zeta_1(\beta) = \frac{\lambda c_-}{c_+ d_+} + \frac{\mu c_-}{c_- d_- + c_- d_+ + c_+ d_+} \tag{35}$$

and

$$\mathbf{v}(0) = \begin{pmatrix} \dfrac{\mu c_+}{c_- d_- + c_- d_+ + c_+ d_+} \\ \dfrac{\lambda}{d_+} \end{pmatrix} V_0(0).$$

Integration yields for $x \geq 0$,

$$\begin{aligned} \mathbf{V}(x) &= \mathbf{V}(0) - \int_0^x \mathbf{v}(t)dt \\ &= \begin{pmatrix} V_0(0) \\ 0 \end{pmatrix} + \frac{V_0(0)}{\gamma} \begin{pmatrix} \dfrac{\mu c_+}{c_- d_- + c_- d_+ + c_+ d_+} \\ \dfrac{\lambda}{d_+} \end{pmatrix} \left(1 - e^{-\gamma x}\right). \end{aligned} \tag{36}$$

We could follow a similar procedure to obtain $\mathbf{P}(x)$ from $\mathbf{p}_j^{(n)}$ based on (27). However, since $\mathbf{P}(x) = \lim_{y\to\infty} \mathbf{F}(x, y)$, it is the solution to the standard fluid flow problem involving just the on-off source and data buffer, which has already been given in (3). Finally, by condition (12), we have that

$$V_0(0) = P_0(0) = \frac{d_+\alpha}{\lambda + \mu}. \tag{37}$$

Combining (26) with (3), (4), (34), (35), (36) and (37), we obtain the following theorem, which is rigourously proved in the Appendix.

Theorem 1 *The unique stationary distribution of the Markov process (M_t, D_t, C_t) is given by*

$$\mathbf{F}(x, y) = \mathbf{P}(x) - \mathbf{V}(x)e^{-\beta y}, \qquad x, y \geq 0,$$

where

$$\mathbf{P}(x) = \begin{pmatrix} \dfrac{\mu}{\lambda + \mu} \\ \dfrac{\lambda}{\lambda + \mu} \end{pmatrix} - \begin{pmatrix} \dfrac{\lambda}{\lambda + \mu}\dfrac{d_+}{d_-} \\ \dfrac{\lambda}{\lambda + \mu} \end{pmatrix} e^{-\alpha x}\ , \qquad x \geq 0,$$

and

$$\mathbf{V}(x) = (1-\rho_d)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1-\rho_d}{\gamma}\begin{pmatrix}\dfrac{\mu c_+}{c_-d_- + c_-d_+ + c_+d_+}\\ \dfrac{\lambda}{d_+}\end{pmatrix}\left(1-e^{-\gamma x}\right), \qquad x \geq 0,$$

with

$$\begin{aligned}
\alpha &= \frac{\mu}{d_+} - \frac{\lambda}{d_-},\\
\beta &= \frac{\lambda}{c_+} - \frac{\mu d_-}{c_-d_- + c_-d_+ + c_+d_+},\\
\gamma &= \frac{\lambda c_-}{c_+d_+} + \frac{\mu c_-}{c_-d_- + c_-d_+ + c_+d_+},\\
1-\rho_d &= \Pr[D_t = 0] = \frac{d_+\alpha}{\lambda+\mu}.
\end{aligned}$$

Corollary 1 *The stationary distribution in Theorem 1, interpreted as a probability measure, can be written as*

$$\begin{aligned}
\mathbf{F}(dx, dy) &= \beta(1-\rho_d)\begin{pmatrix}\dfrac{\mu c_+}{c_-d_- + c_-d_+ + c_+d_+}\\ \dfrac{\lambda}{d_+}\end{pmatrix} e^{-\gamma x - \beta y}\ dx\ dy\\
\mathbf{F}(\{0\}, dy) &= \beta(1-\rho_d)\begin{pmatrix}1\\0\end{pmatrix} e^{-\beta y}\ dy\\
\mathbf{F}(dx, \{0\}) &= (1-\rho_d)\left\{\begin{pmatrix}\dfrac{\lambda}{d_-}\\ \dfrac{\lambda}{d_+}\end{pmatrix} e^{-\alpha x} - \begin{pmatrix}\dfrac{\mu c_+}{c_-d_- + c_-d_+ + c_+d_+}\\ \dfrac{\lambda}{d_+}\end{pmatrix} e^{-\gamma x}\right\} dx.
\end{aligned}$$

Appendix

First we will prove that Theorem 1 indeed presents a stationary distribution for the Markov process (M_t, D_t, C_t). In the following, $\mathbf{F}$ will denote both probability measure and distribution function. We can write down the infinitesimal generator $\mathcal{B}$ of (M_t, D_t, C_t) as a (vector-valued) operator, mapping a function $\mathbf{h} \equiv (h_0, h_1)^T$ to $\mathcal{B}\mathbf{h}$, with

$$(\mathcal{B}\mathbf{h})(x,y) = \lim_{t\downarrow 0} t^{-1}\begin{pmatrix}\mathbf{E}[h_{M_t}(D_t, C_t) - h_0(x,y)|M_0 = 0,\ D_0 = x,\ C_0 = y]\\ \mathbf{E}[h_{M_t}(D_t, C_t) - h_1(x,y)|M_0 = 1,\ D_0 = x,\ C_0 = y]\end{pmatrix} \qquad x, y \geq 0.$$

Some algebra shows that

$$(\mathcal{B}\mathbf{h})(x,y) = Q\mathbf{h}(x,y) + \begin{cases} (\mathcal{B}_0\mathbf{h})(x,y), & x,y>0, \\ (\mathcal{B}_1\mathbf{h})(0,y), & y>0, \\ (\mathcal{B}_2\mathbf{h})(x,0), & x>0, \end{cases} \tag{38}$$

where Q is given in (1),

$$\mathcal{B}_0 = \begin{pmatrix} -d_-\frac{\partial}{\partial x} - c_-\frac{\partial}{\partial y} & 0 \\ 0 & d_+\frac{\partial}{\partial x} - c_-\frac{\partial}{\partial y} \end{pmatrix}, \tag{39}$$

$$\mathcal{B}_1 = \begin{pmatrix} c_+\frac{\partial}{\partial y} & 0 \\ 0 & d_+\frac{\partial}{\partial x} - c_-\frac{\partial}{\partial y} \end{pmatrix}, \tag{40}$$

and

$$\mathcal{B}_2 = \begin{pmatrix} -d_-\frac{\partial}{\partial x} & 0 \\ 0 & d_+\frac{\partial}{\partial x} \end{pmatrix}. \tag{41}$$

The operator $\mathcal{B}$ can be viewed as a generalization of the Q-matrix corresponding to a continuous-time Markov process with a finite state space. In the latter context we would prove that the probability measure π is stationary by showing that it satisfies $\pi Q = \mathbf{0}$, i.e. $\pi Q\mathbf{v} = 0$ for all vectors $\mathbf{v}$. Likewise, here we should prove that the probability measure $\mathbf{F}$ is stationary by showing that it satisfies $\mathbf{F}\mathcal{B}\mathbf{h} = 0$ for all (vector-valued) functions $\mathbf{h}$, i.e. that

$$\int_0^\infty \int_0^\infty \mathbf{F}^T(dx,dy)\mathcal{A}\mathbf{h} \ = 0 \tag{42}$$

(see, e.g., [8], p. 239). Upon substitution of

$$\begin{aligned} \mathbf{F}(dx,dy) &= \mathbf{F}_{x,y}(x,y)\ dx\ dy, \\ \mathbf{F}(dx,\{0\}) &= \mathbf{F}_x(x,0)dx, \\ \mathbf{F}(\{0\},dy) &= \mathbf{F}_y(0,y)\ dy, \end{aligned}$$

where $\mathbf{F}_{x,y}$, $\mathbf{F}_x$ and $\mathbf{F}_y$ denote partial derivatives of $\mathbf{F}$, and applying partial integration, we obtain the following equations,

$$Q^T\mathbf{F}_{x,y}(x,y) - \mathcal{A}_0\mathbf{F}_{x,y}(x,y) = 0, \quad x,y>0, \tag{43}$$

$$Q^T\mathbf{F}_x(x,0) - \mathcal{A}_0\mathbf{F}_x(x,0) = 0, \quad x>0, \tag{44}$$

$$Q^T\mathbf{F}_y(0,y) - \mathcal{A}_1\mathbf{F}_y(0,y) = 0, \quad y>0, \tag{45}$$

$$d_-\tfrac{\partial}{\partial x}F_0(0,0) - c_+\tfrac{\partial}{\partial y}F_0(0,0) = 0, \tag{46}$$

$$\tfrac{\partial}{\partial x}F_1(0,0) = 0, \tag{47}$$

$$\tfrac{\partial}{\partial y}F_1(0,y) = 0, \quad y>0, \tag{48}$$

where $\mathcal{A}_0$ and $\mathcal{A}_1$ are given in (9) and (10). The validity of (46) – (48) is immediate, while the verification of (43) – (45) can be done either by noting that they are implied by (8), since it is straightforward to check that $\mathbf{F}$ indeed satisfies (8), or directly by using Corollary 1.

The uniqueness of the stationary distribution is proved via a coupling argument, as in [12]. Let $\mathbf{F}'$ be an arbitrary distribution on $\{0,1\} \times \mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbf{F}$ the stationary distribution in Theorem 1. We construct two processes (M_t, D_t, C_t) and (M'_t, D'_t, C'_t) on one sample space in the following way: starting with initial distributions $\mathbf{F}$ and $\mathbf{F}'$, respectively, let (M_t) and (M'_t) evolve independently, until the first time τ_1 such that $M_{\tau_1} = M'_{\tau_1}$. After τ_1 we let the two processes (M_t) and (M'_t) have a common trajectory. It is easy to see that $\Pr[\tau_1 < \infty] = 1$. Notice that the increments of (D_t) and (D'_t) are equal for $t \geq \tau_1$, but $D_{\tau_1} \neq D'_{\tau_1}$ in general. Now let τ_2 be the first time at which both data buffers are empty. Since $\max\{D_{\tau_1}, D'_{\tau_1}\} < \infty$ and both (D_t) and (D'_t) have negative drift when the data buffers are not empty, we know that $\tau_2 < \infty$ with probability 1. It is clear that for $t \geq \tau_2$ we have $D_t = D'_t$ and, moreover, the increments of (C_t) and (C'_t) coincide. Therefore, we can find a time τ_3 at which both credit buffers are empty, which is finite by a similar argument as for τ_2. It follows that the two processes (M_t, D_t, C_t) and(M'_t, D'_t, C'_t) coincide for $t \geq \tau_3$ and that for any Borel set A in $\{0,1\} \times \mathbb{R}_+ \times \mathbb{R}_+$, we have

$$| \Pr[(M'_t, D'_t, C'_t) \in A] - \mathbf{F}(A) \mid \leq \Pr[\tau_3 > t].$$

Since the right-hand side converges to 0, as $t \to \infty$, we have that $\Pr[(M'_t, D'_t, C'_t) \in \cdot]$ converges in total variation to $\mathbf{F}$, irrespective of the initial distribution $\mathbf{F}'$. Hence (M'_t, D'_t, C'_t) converges to (M_t, D_t, C_t) in distribution and $\mathbf{F}$ is the unique stationary distribution of (M_t, D_t, C_t).

References

[1] I.J.B.F. Adan, E.A. van Doorn, J.A.C. Resing and W.R.W. Scheinhardt, A queueing system with two different service speeds. Memorandum No. 1316, University of Twente, Enschede, The Netherlands, 1996.

[2] I.J.B.F. Adan and J.A.C. Resing, Simple analysis of a fluid queue driven by an M/M/1 queue. *Queueing Systems* **22** (1996) 171-174.

[3] D. Anick, D. Mitra and M.M. Sondhi, Stochastic theory of a data-handling system with multiple sources. *Bell System Tech. J.* **61** (1982) 1871-1894.

[4] E.A. van Doorn, A.A. Jagers and J.S.J. de Wit, A fluid reservoir regulated by a birth-death process. *Stochastic Models* **4** (1988) 457-472.

[5] E.A. van Doorn and W.R.W. Scheinhardt, Analysis of birth-death fluid queues. In B.D. Choi, editor, *Proceedings of Applied Mathematics Workshop*, KAIST, Korea, (1996) 13-29.

[6] E.A. van Doorn and W.R.W. Scheinhardt, A fluid queue driven by an infinite-state birth-death process. Memorandum No. 1342, University of Twente, Enschede, The Netherlands, 1996.

[7] A.I. Elwalid and D. Mitra, Analysis and design of rate-based congestion control of high speed networks, Part I: Stochastic fluid models, access regulation. *Queueing Systems* **9** (1991) 29-64.

[8] S.N. Ethier and T. G. Kurtz, *Markov Processes: Characterisation and Convergence.* Wiley, New York, 1986.

[9] D.P. Kroese, Analysis of a tandem fluid queue, preprint.

[10] D.P. Kroese and W.R.W. Scheinhardt, A fluid queue with feedback, preprint.

[11] D. Mitra, Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Adv. Appl. Probab.* **20** (1988) 646-676.

[12] L.C.G. Rogers, Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Ann. Appl. Probab.* **4** (1994) 390-413.

[13] P. Sonneveld, Some properties of the generalized eigenvalue problem $Mx = \lambda(\Gamma - cI)x$, where M is the infinitesimal generator of a Markov process and Γ is a real diagonal matrix, Report, Faculty of Mathematics and Informatics, Delft University of Technology, The Netherlands, 1988.

[14] J. Virtamo and I. Norros, Fluid queue driven by an M/M/1 queue. *Queueing Systems* **16** (1994) 373-386.