APPROXIMATIONS TO THE LIFETIME DISTRIBUTION OF K-OUT OF-N SYSTEMS WITH COLD STANDBY*

D. P. KROESE AND W. C.M. KALLENBERG

University of Twente

We consider several approximations to the lifetime distribution of general k-out-of-n systems with cold standby, when n becomes large. The approximations are derived from limiting results for $N(t)$, the number of failures in $[0, t]$, when $t \to \infty$. The approximations can be easily evaluated even though the derivation of the limiting result is rather technical, due to the discrete character of $N(t)$. Numerical work shows excellent agreement with the theoretical error structure of the approximations.

1. The model. A general k-out-of-n system consists of k component positions and n identical components. Each position has to be occupied by exactly one component in order that the entire system works. The lifetime distribution of a component at position i is given by a common distribution function (d.f.) F. (The more complicated case of different d.f.'s F_i, $i = 1, 2, \ldots, k$, can be handled by the same method, cf. Remark 7.1.) The components that have not been placed into a component position are held in storage. The moment a component at position i fails, it is replaced by a component from the storage room. The entire system fails when the $n - k + 1$th component fails.

In Figure 1 the system is shown when p components have failed, k components are working and hence $n - k - p$ components are still in storage.

2. Main results. Let M_n be the lifetime of a k-out-of-n system and let X_i be the time between the $(i - 1)$th and ith component failure, $i = 1, \ldots, n - k + 1$. The exact system lifetime d.f., i.e. $P(M_n \le x)$ for fixed k, is in general intractable. We therefore apply an asymptotic approach with fixed k and $n \to \infty$. A natural approach to obtain asymptotic results would be to consider M_n as the sum of $n - k + 1$ random variables (r.v.'s) X_i. This however gives rise to severe difficulties, that arise from the fact that the r.v.'s X_i are neither identically distributed, nor independent. These problems can be circumvented in the following way.

Consider k independent ordinary renewal processes $\{ N_i(t), t \ge 0 \}$, $i = 1, \ldots, k$, in operation simultaneously, all with the same d.f. F of failure time. Define $N(t) = N_1(t) + \cdots + N_k(t)$, $t \ge 0$. For $t \le M_n$, $N_i(t)$ denotes the number of component failures during $[0, t]$ at position i, $i = 1, \ldots, k$ and $N(t)$ denotes the total number of failures during $[0, t]$. The following relationship between $N(t)$ and M_n is obvious

$$(2.1) \qquad \{ M_n \le t \} \Leftrightarrow \{ N(t) \ge n - k + 1 \}.$$

In view of (2.1) we investigate the behaviour of $N(t)$ for large values of t. Basic normal approximations for $P(M_n \le t)$ as $n \to \infty$ follow from the central limit theorem and
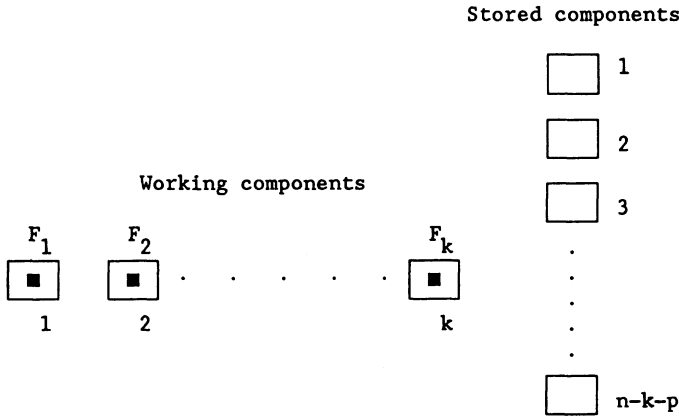
Stored components

Working components

FIGURE 1. k-out-of-n system with cold standby.

(2.1), cf. Cox (1962, p. 73). Here we consider higher order expansions to obtain better approximations. The expansion of $P(M_n \leq t)$ is obtained in three steps. In §3 we start with the expansion of the lifetime distribution of a 1-out-of-n system. M_n is then simply the sum of n i.i.d. r.v.'s, and much work has been done in this field. From this first expansion we derive an expansion of the discrete d.f. of $N_1(t)$, the number of renewals in $[0, t]$, as $t \to \infty$ (Theorem 3.1). For the general case of a k-out-of-n system, $N(t)$ is the sum of k independent r.v.'s $N_i(t)$, each distributed as $N_1(t)$. The expansion of $P(N(t) \leq x)$ is derived in §4 (Theorem 4.1) from the expansion of $P(N_i(t) \leq x)$. The final step then will be the construction of the expansion of $P(M_n \leq t)$ through that of $P(N(t) \leq x)$. This inversion step is well known for central limit theorems. We refer for general results in this area to Iglehart and Whitt (1971), Whitt (1980) and Glynn and Whitt (1988). Here a similar argument is used in the context of expansions.

Henceforth we will denote the standard normal distribution function by Φ and its density by φ. A $\mathbb{N}(\mu, \sigma^2)$-distribution means a normal distribution with expectation μ and variance σ^2.

For the lifetime distribution of the components F we put
Condition C. F is nonlattice, $F(0) = 0$ and $E_F X_i^3 < \infty$.
Further we define $\mu = E_F X_i$, $\sigma^2 = \text{var}_F X_i$.
Our main result is given in the next theorem.

THEOREM 2.1.   *Assume Condition C, then*

(2.2)

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{M_n - (n - k + 1)\mu k^{-1}}{\sqrt{n - k + 1}\, \sigma k^{-1}} \leq x \right) - \Phi(x) \right.$$

$$\left. - \frac{\varphi(x)}{\sqrt{n - k + 1}} \left\{ (1 - x^2)\frac{\mu_3}{6\sigma^3} + \frac{1}{2}(k - 1)\left(\frac{\sigma}{\mu} - \frac{\mu}{\sigma} \right) \right\} \right| = o(n^{-1/2}) \quad \text{as } n \to \infty.$$

The proof of Theorem 2.1 is given in §5.
We infer from Theorem 2.1 the well-known result (cf. Cox 1962, p. 73) that

(2.3)
$$\frac{M_n - (n - k + 1)\mu k^{-1}}{\sqrt{n - k + 1}\, \sigma k^{-1}} \xrightarrow{\mathscr{D}} \mathbb{N}(0, 1).$$

Hence we take as approximation

$$(2.4) \qquad P(M_n \le x) \approx \Phi\left(\frac{x - (n - k + 1)\mu k^{-1}}{\sqrt{n - k + 1}\,\sigma k^{-1}} \right).$$

Thus M_n is approximated by a $\mathbb{N}((n - k + 1)\mu k^{-1}, (n - k + 1)\sigma^2 k^{-2})$-distribution. The error term is easily obtained from (2.2) and is given by

$$(2.5) \quad \frac{\varphi(y_n)}{\sqrt{n - k + 1}}\left\{ \left(1 - y_n^2\right)\frac{\mu_3}{6\sigma^3} + \frac{1}{2}(k - 1)\left(\frac{\sigma}{\mu} - \frac{\mu}{\sigma}\right) \right\} + o(n^{-1/2}), \quad \text{where}$$

$$(2.6) \qquad y_n = \frac{x - (n - k + 1)\mu k^{-1}}{\sqrt{n - k + 1}\,\sigma k^{-1}}.$$

REMARK 2.1.   From a mathematical point of view (2.3) may also be written as

$$\frac{M_n - n\mu k^{-1}}{\sqrt{n}\,\sigma k^{-1}} \xrightarrow{\mathscr{D}} \mathbb{N}(0, 1),$$

since k is fixed. The induced approximation would be

$$P(M_n \le x) \approx \Phi\left(\frac{x - n\mu k^{-1}}{\sqrt{n}\,\sigma k^{-1}} \right).$$

It is easily seen that

$$\Phi\left(\frac{x - (n - k + 1)\mu k^{-1}}{\sqrt{n - k + 1}\,\sigma k^{-1}} \right) - \Phi\left(\frac{x - n\mu k^{-1}}{\sqrt{n}\,\sigma k^{-1}} \right)$$

is of exact order $n^{-1/2}$. Therefore to give a mathematical justification for the factor $(n - k + 1)$ instead of n in (2.4) one has to go beyond first order approximation.

REMARK 2.2.   Let X_i be the lifetime of the ith component in a k-out-of-n system. Construct a 1-out-of-$(n - k + 1)$ system by contracting the k component positions to only one position which generates independent component lifetimes X_i/k, $i = 1, \ldots, n - k + 1$. Approximation (2.4) states that at first order we may replace the k-out-of-n system by this new system. In that case the lifetime distribution immediately follows from the central limit theorem, since M_n is then simply a sum of i.i.d. r.v.'s. However, as is seen in Theorem 2.1 and the following approximations, this sloppy argument is inadequate for more accurate approximations.

Neglecting the $o(n^{-1/2})$ term, the error term (2.5) has a systematic component $\varphi(y_n)(n - k + 1)^{-1/2}\frac{1}{2}(k - 1)(\sigma\mu^{-1} - \mu\sigma^{-1}) = S(y_n)$, say. Denoting the remainder term by $R(y_n)$, this may be argued as follows. Firstly, one would like the expectation of M_n to be equal to the expectation of its approximation. Since $\int x\, d(\varphi(x)(1 - x^2)) = 0$ and $\int x\, d\varphi(x) \ne 0$, the contribution of R to the expectation is indeed equal to 0, but that of S is unequal to 0. This argument is made more rigorous in Remark 7.4. Secondly, the systematic error in location can also be seen by considering the integrated error. Since $\int \varphi(x)(1 - x^2)\, dx = 0$ and $\int \varphi(x)\, dx \ne 0$, the contribution to the integrated error is indeed equal to 0 for R, but unequal to 0 for S. The systematic

component may be canceled out, leading to our second approximation

$$(2.7) \quad P(M_n \leq x) \approx \Phi\left(\frac{x - (n - k + 1)\mu k^{-1}}{\sqrt{n - k + 1}\,\sigma k^{-1}} + \frac{(k - 1)}{2\sqrt{n - k + 1}}\left(\frac{\sigma}{\mu} - \frac{\mu}{\sigma} \right) \right).$$

Here M_n is approximated by a $N((n - k + 1)\mu k^{-1} - \frac{1}{2}(k - 1)k^{-1}\sigma(\sigma\mu^{-1} - \mu\sigma^{-1})$, $(n - k + 1)k^{-2}\sigma^2)$-distribution. In this case the error term equals

$$(2.8) \qquad \frac{\varphi(y_n)}{\sqrt{n - k + 1}}\left(1 - y_n^2 \right)\frac{\mu_3}{6\sigma^3} + o(n^{-1/2}) \quad \text{as } n \to \infty,$$

where y_n is given by (2.6).

REMARK 2.3.   Note that these approximations are easy to evaluate and that only the first and second moment are involved. Theorem 2.1 justifies the heuristic argument used by Cox (1962, p. 75) to take $(n - k + 1)\mu k^{-1} - \frac{1}{2}(k - 1)k^{-1}\sigma(\sigma\mu^{-1} - \mu\sigma^{-1})$ as a normalizing constant instead of $(n - k + 1)\mu k^{-1}$ (cf. Remark 7.4). We can however improve on this result by taking the third central moment into account, stating

$$(2.9) \quad P(M_n \leq x) \approx \Phi\left(y_n + \frac{(1 - y_n^2)}{\sqrt{n - k + 1}}\frac{\mu_3}{6\sigma^3} + \frac{(k - 1)}{2\sqrt{n - k + 1}}\left(\frac{\sigma}{\mu} - \frac{\mu}{\sigma} \right) \right),$$

with y_n as in (2.6).

The error term now becomes $o(n^{-1/2})$ as $n \to \infty$. This approximation is not a normal approximation in the usual sense, since the expectation and variance vary with x!

A second approach that also involves the third central moment is to use the Edgeworth approximation suggested by (2.2)

$$(2.10) \quad P(M_n \leq x)$$

$$\approx \Phi(y_n) + \frac{\varphi(y_n)}{\sqrt{n - k + 1}}\left\{ (1 - y_n^2)\frac{\mu_3}{6\sigma^3} + \frac{1}{2}(k - 1)\left(\frac{\sigma}{\mu} - \frac{\mu}{\sigma} \right) \right\},$$

where y_n is given by (2.6). The error term of (2.10) is also $o(n^{-1/2})$ as $n \to \infty$.

One may expect that the approximations given by (2.9) and (2.10) are closest to the true d.f., since they have the smallest error term of the four suggested approximations. One may also expect that, on the whole, (2.7) is better than (2.5). The simulation results (§6) are in excellent agreement with the predicted error structure, even for very small n, cf. also Remark 7.3.

3.   **The 1-out-of-n case.** The most simple nontrivial example of a k-out-of-n system with cold standby is the 1-out-of-n system with cold standby. In this case there is only one component position that has to be occupied by a working component. Suppose component i has a lifetime X_i, $i = 1, \ldots, n$, then the lifetime of the entire system is simply the sum of the n component lifetimes, i.e. $M_n = X_1 + \cdots + X_n$. The investigation of sums of independent r.v.'s has a rich history. Here we use Theorem XVI.4.1 in Feller (1971, p. 539).

Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with d.f. F, and write $M_n = \sum_{i=1}^n X_i$. Suppose that X_i has a nonlattice distribution and that the third central moment $\mu_3 = E(X_i - EX_i)^3$ finitely exists.

Then

$$(3.1) \quad \sup_{x \in \mathbb{R}} \left| P\left(\frac{M_n - n\mu}{\sigma\sqrt{n}} \le x \right) - \Phi(x) - \frac{\varphi(x)}{\sqrt{n}} \frac{\mu_3}{6\sigma^3} (1 - x^2) \right| = o(n^{-1/2})$$

as $n \to \infty$, where $\mu = EX_i$ and $\sigma^2 = \operatorname{var} X_i$.

As a consequence we have the following result for $N_1(t)$, the number of renewals at position 1 in $[0, t]$.

THEOREM 3.1. *Assume Condition C, then*

$$(3.2) \quad \sup_{x \in \mathbb{R}} \left| P\left(\frac{N_1(t) - t/\mu}{\sigma\mu^{-3/2}\sqrt{t}} \le x \right) - \Phi(x) \right.$$

$$\left. - \frac{\varphi(x)}{\sqrt{t}} \left\{ (c - d)(x^2 - 1) - d + a\theta\left(\frac{t}{\mu} + x\sigma\mu^{-3/2}\sqrt{t} \right) \right\} \right|$$

$$= o(t^{-1/2})$$

as $t \to \infty$.

Here $c = \frac{1}{6}\mu_3\sigma^{-3}\mu^{1/2}$, $d = \frac{1}{2}\sigma\mu^{-1/2}$, $a = \mu^{3/2}\sigma^{-1}$ and $\theta(y) = [y] + 1 - y$, where $[y]$ denotes the largest integer less than or equal to y.

PROOF. By definition we have $\{ N_1(t) \le n - 1 \} \Leftrightarrow \{ M_n > t \}$. Let $|x_t| \le \log t$. As $t \to \infty$ we obtain

$$P\left(\frac{N_1(t) - t/\mu}{\sigma\mu^{-3/2}\sqrt{t}} \le x_t \right)$$

$$= P\left(N_1(t) \le \frac{t}{\mu} + x_t\sigma\mu^{-3/2}\sqrt{t} \right)$$

$$= P\left(N_1(t) \le m_t - \theta\left(\frac{t}{\mu} + x_t\sigma\mu^{-3/2}\sqrt{t} \right) \right) = P(M_{m_t} > t)$$

$$= 1 - \Phi(y_t) - \frac{\mu_3}{6\sigma^3} m_t^{-1/2}(1 - y_t^2)\varphi(y_t) + o(m_t^{-1/2}), \quad \text{where}$$

$$m_t = \left[\frac{t}{\mu} + x_t\sigma\mu^{-3/2}\sqrt{t} \right] + 1 \quad \text{and}$$

$$y_t = \frac{t - m_t\mu}{\sigma\sqrt{m_t}}$$

$$= -x_t + \frac{1}{2} x_t^2 \sigma\mu^{-1/2} t^{-1/2} - \mu^{3/2}\sigma^{-1}t^{-1/2}\theta\left(\frac{t}{\mu} + x_t\sigma\mu^{-3/2}\sqrt{t} \right)$$

$$+ O\left((\log^3 t) t^{-1} \right).$$

Therefore

$$(3.3) \quad P\left(\frac{N_1(t) - t/\mu}{\sigma\mu^{-3/2}\sqrt{t}} \le x_t\right)$$

$$= \Phi(x_t) + \frac{\varphi(x_t)}{\sqrt{t}}\left\{(c-d)(x_t^2 - 1) - d + a\theta\left(\frac{t}{\mu} + x_t\sigma\mu^{-3/2}\sqrt{t}\right)\right\}$$

$$+ o(t^{-1/2}) \quad \text{as } t \to \infty.$$

In particular we have

$$P\left(\frac{N_1(t) - t/\mu}{\sigma\mu^{-3/2}\sqrt{t}} \le -\log t\right) = o(t^{-1/2}) \quad \text{and}$$

$$P\left(\frac{N_1(t) - t/\mu}{\sigma\mu^{-3/2}\sqrt{t}} \le \log t\right) = 1 - o(t^{-1/2}).$$

By monotonicity of d.f.'s it now follows that (3.3) also holds if $|x_t| > \log t$. This completes the proof. ∎

REMARK 3.1. Of course $N_1(t)$ has a discrete distribution. This is reflected in the expansion by the function θ, which gives the expansion also a discrete character.

4. Expansions for $N(t)$. In this section we derive an expansion for $N(t)$. Note that $N(t)$ is the sum of k independent r.v.'s $N_i(t)$, each distributed as $N_1(t)$. For each $N_i(t)$ we derived an expansion in §3. We will show that the convolution of these k expansions constitutes the expansion for $N(t)$. One of the main problems is to handle the discrete part of the expansions for $N(t)$ and $N_i(t)$. The function θ plays an essential role for that matter. Therefore we will state two technical lemmas on the function θ first.

LEMMA 4.1. Let $\theta(y) = [y] + 1 - y$, $y \in \mathbb{R}$. Let g be a differentiable function with derivative g', such that

$$(4.1) \qquad\qquad \int |f(x-u)g'(u)|\,du < \infty.$$

For $t > 0$, $c_t \in \mathbb{R}$, $m \in \mathbb{Z}$ define $x_m = (m - c_t)t^{-1/2}$.
  Assume that for $x_{m-1} < u \le x_m$

$$(4.2) \qquad |f(x-u)g'(u) - f(x-x_m)g'(x_m)| \le h_x(u)a_t \quad \text{with}$$

$$\lim_{t \to \infty} a_t = 0 \quad \text{and} \quad \int h_x(u)\,du < \infty.$$

Then as $t \to \infty$

$$(4.3) \quad \int \theta\big(c_t + (x - u)t^{1/2}\big) f(x - u)\, dg(u) = \tfrac{1}{2} \int f(x - u)\, dg(u) + o(1).$$

PROOF.   We have, writing $\chi_t(u) = \theta(c_t + (x - u)t^{1/2}) - \tfrac{1}{2}$,

$$\int \chi_t(u) f(x - u)\, dg(u) = \sum_{m = -\infty}^{\infty} \int_{(x_{m-1},\, x_m]} \chi_t(u) f(x - x_m) g'(x_m)\, du$$

$$+ \sum_{m = -\infty}^{\infty} \int_{(x_{m-1},\, x_m]} \chi_t(u) \{ f(x - u) g'(u)$$

$$- f(x - x_m) g'(x_m) \}\, du.$$

Since $\int_{(x_{m-1},\, x_m]} \chi_t(u)\, du = 0$ and by (4.2)

$$\left| \sum_{m = -\infty}^{\infty} \int_{(x_{m-1},\, x_m]} \chi_t(u) \{ f(x - u) g'(u) - f(x - x_m) g'(x_m) \}\, du \right|$$

$$\leq \sum_{m = -\infty}^{\infty} \int_{(x_{m-1},\, x_m]} h_x(u) a_t\, du = a_t \int h_x(u)\, du,$$

(4.3) easily follows.   ∎

REMARK 4.1.   If $f(x) = P_1(x)\varphi(ax)$ and $g(x) = \Phi(bx)$ or $g(x) = P_2(x)\varphi(cx)$ with P_1 and P_2 polynomials, then (4.2) is satisfied by application of the mean value theorem, taking $a_t = t^{-1/2}$.

REMARK 4.2.   Note that (4.3) holds for every choice of c_t, such that (4.2) holds true. Usually this does not imply any restriction on c_t.

REMARK 4.3.   If $\sup_{x \in \mathbb{R}} \int h_x(u)\, du < \infty$, then (4.3) holds uniformly in x. This condition is satisfied for the choices of f, g in Remark 4.1.

LEMMA 4.2.   *Assume that the conditions in Lemma 4.1 hold with (4.2) replaced by*

$$(4.4) \qquad |f(x - u) g(u) - f(x - x_m) g(x_m)| \leq h_x(u) a_t \quad \text{with}$$

$$\lim_{t \to \infty} a_t = 0 \quad \text{and} \quad \int h_x(u)\, du < \infty.$$

Let $b_t \in \mathbb{R}$. Then as $t \to \infty$

$$(4.5) \quad t^{-1/2} \int \left\{ \theta\big(b_t + (x - u)t^{1/2}\big) - \tfrac{1}{2} \right\} f(x - u)\, d\left(\left\{ \theta\big(c_t + ut^{1/2}\big) - \tfrac{1}{2} \right\} g(u) \right)$$

$$= \left\{ \theta\big(b_t + c_t + xt^{1/2}\big) - \tfrac{1}{2} \right\} \int f(x - u) g(u)\, du + o(1).$$

PROOF. By definition of θ we have $\theta(b_t + (x - x_m)t^{1/2}) = \theta(b_t + c_t + xt^{1/2}) = \zeta_t$, say. Writing $\chi_t(u) = \theta(b_t + (x - u)t^{1/2}) - \frac{1}{2}$, we have

$$t^{-1/2} \int \left\{ \theta\left(b_t + (x - u)t^{1/2}\right) - \tfrac{1}{2} \right\} f(x - u) \, d\left(\left\{ \theta\left(c_t + ut^{1/2}\right) - \tfrac{1}{2} \right\} g(u) \right)$$

$$= t^{-1/2}\left(\zeta_t - \tfrac{1}{2}\right) \sum_{m=-\infty}^{\infty} f(x - x_m)g(x_m)$$

$$+ t^{-1/2} \sum_{m=-\infty}^{\infty} \int_{(x_{m-1},\, x_m]} \chi_t(u) f(x - u) \, d\left(\left\{ \tfrac{1}{2} - (u - x_{m-1})t^{1/2} \right\} g(u) \right)$$

$$= \left(\zeta_t - \tfrac{1}{2}\right) \sum_{m=-\infty}^{\infty} \int_{(x_{m-1},\, x_m]} \left\{ f(x - x_m)g(x_m) - f(x - u)g(u) \right\} du$$

$$+ \left(\zeta_t - \tfrac{1}{2}\right) \int f(x - u)g(u) \, du$$

$$- \sum_{m=-\infty}^{\infty} \int_{(x_{m-1},\, x_m]} \chi_t(u) \left\{ f(x - u)g(u) - f(x - x_m)g(x_m) \right\} du$$

$$- \sum_{m=-\infty}^{\infty} \int_{(x_{m-1},\, x_m]} \chi_t(u) f(x - x_m)g(x_m) \, du$$

$$+ \sum_{m=-\infty}^{\infty} t^{-1/2} \int_{(x_{m-1},\, x_m]} \chi_t(u) f(x - u) g'(u) \left\{ \tfrac{1}{2} - (u - x_{m-1})t^{1/2} \right\} du.$$

Application of (4.4), $\int_{(x_{m-1},\, x_m]} \chi_t(u) \, du = 0$ and the inequalities

$$|\zeta_t - \tfrac{1}{2}| \leq \tfrac{1}{2}, \quad |\chi_t(u)| \leq \tfrac{1}{2}, \quad |\tfrac{1}{2} - (u - x_{m-1})t^{1/2}| \leq \tfrac{1}{2} \quad \text{for } x_{m-1} < u \leq x_m$$

and (4.1) yield the result. ∎

REMARK 4.4. If $f(x) = P_1(x)\varphi(ax)$ and $g(x) = P_2(x)\varphi(bx)$ with P_1 and P_2 polynomials, then (4.4) is satisfied.

REMARK 4.5. The only (very weak) restriction on c_t is that (4.4) holds for $x_{m-1} < u \leq x_m$.

REMARK 4.6. If $\sup_{x \in \mathbf{R}} \int h_x(u) \, du < \infty$ and $\sup_{x \in \mathbf{R}} \int |f(x - u)g'(u)| \, du < \infty$, then (4.5) holds uniformly in x. This condition is satisfied for the choices of f and g in Remark 4.4.

The preceding lemmas enable us to derive an expansion for $N(t)$. The proof of the next theorem shows that such an expansion can indeed be derived by convoluting the expansions for $N_i(t)$, $i = 1, \ldots, k$.

THEOREM 4.1. Let $N_1(t), \ldots, N_k(t)$ be i.i.d. r.v.'s with

$$(4.6) \qquad \sup_{x \in \mathbf{R}} \left| P\left(\frac{N_i(t) - t/\mu}{\sigma\mu^{-3/2}\sqrt{t}} \leq x \right) - \Phi(x) \right.$$

$$\left. - \frac{\varphi(x)}{\sqrt{t}} \left\{ (c - d)(x^2 - 1) - d + a\theta\left(\frac{t}{\mu} + x\sigma\mu^{-3/2}\sqrt{t} \right) \right\} \right|$$

$$= o\left(t^{-1/2}\right)$$

as $t \to \infty$, where

$$c = \tfrac{1}{6}\mu_3\sigma^{-3}\mu^{1/2}, \quad d = \tfrac{1}{2}\sigma\mu^{-1/2}, \quad a = \mu^{3/2}\sigma^{-1}, \quad \theta(y) = [y] + 1 - y.$$

Then we have for the d.f. of $N(t) = N_1(t) + \cdots + N_k(t)$ the following expansion

(4.7)
$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{N(t) - tk\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{kt}} \le x \right) - \Phi(x) \right.$$

$$- \frac{\varphi(x)}{\sqrt{kt}} \left\{ (c - d)(x^2 - 1) + \left(\frac{a}{2} - d \right)k \right.$$

$$\left. \left. + a\left[\theta\left(k\frac{t}{\mu} + x\sqrt{kt}\, a^{-1} \right) - \frac{1}{2} \right] \right\} \right|$$

$$= o\!\left(t^{-1/2} \right) \quad \text{as } t \to \infty.$$

PROOF (By induction).   For $k = 1$ the result is true. Assume that (4.7) holds for $1, \ldots, k - 1$. Write G_k for the d.f. of $(N(t) - tk\mu^{-1})/(\sigma\mu^{-3/2}t^{1/2})$ and $\tilde{G}_k$ for *its* approximation in (4.7).

We now have

$$G_k(y) = \int G_{k-1}(y - u)\, dG_1(u)$$

$$= \int \{ G_{k-1}(y - u) - \tilde{G}_{k-1}(y - u) \}\, dG_1(u) + \int \tilde{G}_{k-1}(y - u)\, dG_1(u)$$

$$= \int \{ G_{k-1}(y - u) - \tilde{G}_{k-1}(y - u) \}\, dG_1(u) + \int G_1(y - u)\, d\tilde{G}_{k-1}(u)$$

$$= \int \{ G_{k-1}(y - u) - \tilde{G}_{k-1}(y - u) \}\, dG_1(u)$$

$$+ \int \{ G_1(y - u) - \tilde{G}_1(y - u) \}\, d\tilde{G}_{k-1}(u)$$

$$+ \int \tilde{G}_1(y - u)\, d\tilde{G}_{k-1}(u),$$

and therefore

$$\sup_{y \in \mathbb{R}} \left| G_k(y) - \int \tilde{G}_1(y - u)\, d\tilde{G}_{k-1}(u) \right| = o\!\left(t^{-1/2} \right).$$

Hence

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{N(t) - tk\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{kt}} \le x \right) - \int \tilde{G}_1(xk^{1/2} - u)\, d\tilde{G}_{k-1}(u) \right| = o\!\left(t^{-1/2} \right)$$

as $t \to \infty$.

Next consider the following terms, which together form $\int \tilde{G}_1(xk^{1/2} - u)\, d\tilde{G}_{k-1}(u)$:

(4.8)
$$\int \Phi(xk^{1/2} - u)\, d\Phi\left(u(k-1)^{-1/2}\right) = \Phi(x),$$

(4.9)
$$\int \varphi(xk^{1/2} - u)t^{-1/2}(c-d)\left\{(xk^{1/2} - u)^2 - 1\right\} d\Phi\left(u(k-1)^{-1/2}\right)$$

$$= \frac{t^{-1/2}(c-d)\varphi(x)(x^2 - 1)}{k^{3/2}},$$

(4.10)
$$\left(\frac{a}{2} - d\right)\int \varphi(xk^{1/2} - u)t^{-1/2}\, d\Phi\left(u(k-1)^{-1/2}\right)$$

$$= \left(\frac{a}{2} - d\right)t^{-1/2}k^{-1/2}\varphi(x),$$

(4.11)
$$at^{-1/2}\int\left[\theta\left(\frac{t}{\mu} + (xk^{1/2} - u)t^{1/2}a^{-1}\right) - \frac{1}{2}\right]$$

$$\times \varphi(xk^{1/2} - u)\, d\Phi\left(u(k-1)^{-1/2}\right)$$

$$= o\left(t^{-1/2}\right)$$

as $t \to \infty$, uniformly in x by Lemma 4.1 and Remark 4.3,

(4.12)
$$\Phi(xk^{1/2} - u)d\left\{(k-1)^{-1/2}t^{-1/2}\varphi\left(u(k-1)^{-1/2}\right)\right.$$

$$\times\left[(c-d)\left(\frac{u^2}{k-1} - 1\right) + \left(\frac{a}{2} - d\right)(k-1)\right.$$

$$\left.\left. + a\left\{\theta\left((k-1)\frac{t}{\mu} + ut^{1/2}a^{-1}\right) - \frac{1}{2}\right\}\right]\right\}$$

$$= k^{-1/2}t^{-1/2}\varphi(x)(k-1)$$

$$\times\left\{\frac{(c-d)}{k}(x^2 - 1) + \frac{a}{2} - d\right\} + o\left(t^{-1/2}\right)$$

as $t \to \infty$, uniformly in x by Lemma 4.1 and Remark 4.3,

(4.13)

$$\int \varphi(xk^{1/2} - u)t^{-1/2}\left[(c-d)\left\{(xk^{1/2} - u)^2 - 1\right\}\right.$$

$$\left. + \frac{a}{2} - d + a\left\{\theta\left(\frac{t}{\mu} + (xk^{1/2} - u)t^{1/2}a^{-1}\right) - \frac{1}{2}\right\}\right]$$

$$\times d\left[(k-1)^{-1/2}t^{-1/2}\varphi\left(u(k-1)^{-1/2}\right)\left\{(c-d)\left(\frac{u^2}{k-1} - 1\right) + \left(\frac{a}{2} - d\right)(k-1)\right\}\right]$$

$$= O\left(t^{-1}\right) = o\left(t^{-1/2}\right)$$

as $t \to \infty$, uniformly in x by Lemma 4.1 and Remark 4.3,

$$(4.14) \quad \int \varphi(xk^{1/2} - u)t^{-1/2}\left[(c - d)\left\{(xk^{1/2} - u)^2 - 1\right\} + \frac{a}{2} - d\right]$$

$$\times d\left[(k - 1)^{-1/2}t^{-1/2}\varphi\left(u(k - 1)^{-1/2}\right)a\left\{\theta\left((k - 1)\frac{t}{\mu} + ut^{1/2}a^{-1}\right) - \frac{1}{2}\right\}\right]$$

$$= o\left(t^{-1}\right) = o\left(t^{-1/2}\right)$$

as $t \to \infty$, uniformly in x by Lemma 4.1 and Remark 4.3,

$$(4.15) \quad \int \varphi(xk^{1/2} - u)t^{-1/2}a\left\{\theta\left(\frac{t}{\mu} + (xk^{1/2} - u)t^{1/2}a^{-1}\right) - \frac{1}{2}\right\}$$

$$\times d\left[(k - 1)^{-1/2}t^{-1/2}\varphi\left(u(k - 1)^{-1/2}\right)a\left\{\theta\left((k - 1)\frac{t}{\mu} + ut^{1/2}a^{-1}\right) - \frac{1}{2}\right\}\right]$$

$$= at^{-1/2}k^{-1/2}\varphi(x)\left\{\theta\left(k\frac{t}{\mu} + xk^{1/2}t^{1/2}a^{-1}\right) - \frac{1}{2}\right\} + o\left(t^{-1/2}\right)$$

as $t \to \infty$, uniformly in x by Lemma 4.2 and Remark 4.6.
   Combination of (4.8)–(4.15) yields the result.   ∎

5. Proof of Theorem 2.1.   In this section we will prove Theorem 2.1. The essential steps already have been taken in §4.

PROOF.   Writing $t_n = (n - k + 1)\mu k^{-1} + x_n(n - k + 1)^{1/2}\sigma k^{-1}$ we have by the definition of M_n

$$p_n(x_n) = P\left(\frac{M_n - (n - k + 1)\mu k^{-1}}{\sqrt{n - k + 1}\,\sigma k^{-1}} \leq x_n\right) = P(M_n \leq t_n)$$

$$= P(N(t_n) > n - k) = 1 - P(N(t_n) \leq n - k)$$

$$= 1 - P\left(\frac{N(t_n) - t_n k\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{kt_n}} \leq \frac{n - k - t_n k\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{kt_n}}\right).$$

Let $x_n = O(\log n)$. Then

$$\frac{n - k - t_n k\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{kt_n}} = -x_n + \frac{1}{2}x_n^2(n - k + 1)^{-1/2}\mu^{-1}\sigma$$

$$- \frac{\mu}{\sigma}(n - k + 1)^{-1/2} + O(n^{-1}\log^3 n).$$

Hence

$$p_n(x_n) = \Phi\left(x_n - \frac{1}{2}x_n^2(n - k + 1)^{-1/2}\mu^{-1}\sigma + \frac{\mu}{\sigma}(n - k + 1)^{-1/2}\right)$$

$$-k^{-1/2}\varphi(x_n)\mu^{-1/2}(n - k + 1)^{-1/2}k^{1/2}$$

$$\times\left\{(c - d)(x_n^2 - 1) + \left(\frac{a}{2} - d\right)k + \frac{a}{2}\right\} + o(n^{-1/2})$$

$$= \Phi(x_n) + \varphi(x_n)(n - k + 1)^{-1/2}$$

$$\times\left\{\frac{1}{6}\mu_3\sigma^{-3}(1 - x_n^2) + \frac{1}{2}(k - 1)\left(\frac{\sigma}{\mu} - \frac{\mu}{\sigma}\right)\right\}$$

$$+o(n^{-1/2}) \quad \text{as } n \to \infty.$$

In particular this implies that $p_n(-\log n) = o(n^{-1/2})$ and $p_n(\log n) = 1 - o(n^{-1/2})$, and the result easily follows. ∎

6. Numerical results. To verify how well the asymptotic theory applies for finite n, several experiments were carried out to compare the true lifetime d.f. of the system with the approximations to it as developed in §2. Only in some special cases the true lifetime d.f. of the system can be evaluated. In the other cases it has been estimated by means of 10000 Monte-Carlo experiments, reducing the simulation error to less than 0.01 with confidence 0.95. Here we give some typical examples.

We have the following presentation. For each example a table is presented with the (estimated) true lifetime d.f. of the system and the various approximations. In each case a picture is made of the errors of the approximations with respect to the (estimated) true d.f. In the pictures we made use of the unrounded numerical results. In several examples approximation (2.9) and (2.10) almost coincide. In the pictures we then only present the error of approximation (2.9).
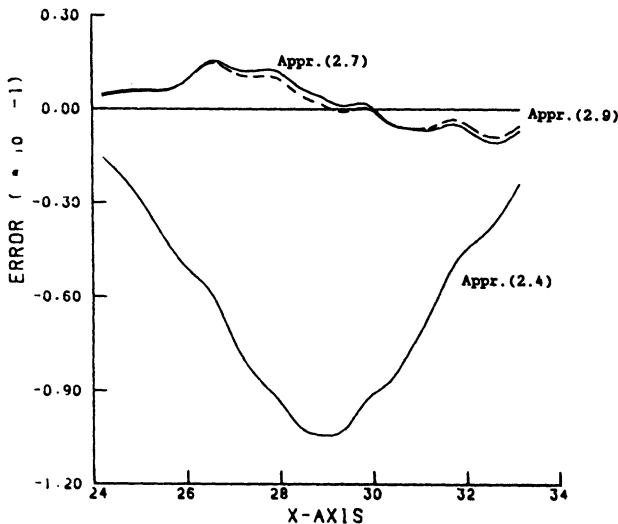

FIGURE 2. Approximation errors for a 2-out-of-20 system with a Weibull component distribution ($\alpha = 3$, $\lambda = 0.3$).

TABLE 1

2-out-of-20 System with a Weibull Component Distribution ($\alpha = 3$, $\lambda = 0.3$)

x	Estimated D.F.	Appr. (2.4)	Appr. (2.7)	Appr. (2.9)	Appr. (2.10)
24.21	0.03	0.04	0.02	0.02	0.02
24.68	0.04	0.06	0.04	0.04	0.03
25.15	0.06	0.09	0.05	0.05	0.05
25.62	0.09	0.13	0.08	0.08	0.07
26.09	0.13	0.18	0.12	0.12	0.11
26.57	0.17	0.23	0.16	0.16	0.15
27.04	0.22	0.30	0.21	0.21	0.21
27.51	0.29	0.37	0.27	0.28	0.27
27.98	0.36	0.45	0.34	0.35	0.34
28.45	0.43	0.53	0.42	0.42	0.42
28.92	0.50	0.61	0.50	0.50	0.50
29.40	0.58	0.68	0.58	0.58	0.59
29.87	0.66	0.75	0.66	0.66	0.66
30.34	0.72	0.81	0.73	0.73	0.73
30.81	0.78	0.86	0.79	0.79	0.80
31.28	0.83	0.90	0.84	0.84	0.85
31.75	0.88	0.93	0.88	0.88	0.89
32.22	0.91	0.95	0.92	0.92	0.92
32.70	0.93	0.97	0.95	0.94	0.95
33.17	0.96	0.98	0.96	0.96	0.97

6.1. *2-out-of-20 system. Weibull-distribution.*   In our first example we consider the lifetime distribution of a 2-out-of-20 system of which the component lifetime distribution is a Weibull-distribution with shape parameter α and scale parameter λ, i.e. $F(x) = 1 - \exp((-\lambda x)^\alpha)$, $x \geq 0$. Here we take $\alpha = 3$ and $\lambda = 0.3$. (See Table 1 and Figure 2.)

Notice that approximation (2.7) gives an excellent correction for the relatively large systematic error in approximation (2.4). In this case approximation (2.9) is only slightly better than approximation (2.7), due to the fact that $\mu_3/6\sigma^3 \approx 0.028$, which is very small. The maximal absolute error of approximations (2.7) and (2.9) is 0.015.

6.2. *2-out-of-7 system. Erlang^2-distribution.*   To see how the approximations apply for a very small number of components n, we take in this example n equal to 7. (See Table 2 and Figure 3.) There are two component positions, both generating component lifetimes with an Erlang^2 distribution, i.e. $F(x) = 1 - e^{-\lambda x}(1 + \lambda x)$, $x \geq 0$. Here we take $\lambda = 0.2$. Note that F can be interpreted as the d.f. of the sum of two independent exponentially distributed r.v.'s. It can be shown that in this special case the system lifetime d.f. G is given by

$$G(x) = 1 - \left[\frac{1}{2} \frac{(0.4x)^{12}}{12!} + \sum_{j=1}^{12} \frac{(0.4x)^{12-j}}{(12-j)!} \right] e^{-0.4x}.$$

Approximation (2.9) and (2.10) are surprisingly close to the estimated d.f. The maximal absolute error is 0.012. In contrast to the previous example here approximation (2.7) is worse than (2.9) and (2.10). On the whole (2.7) is better than (2.4), which again has a systematic error. However, in the middle part (2.7) gives an overcorrection.

6.3. *3-out-of-n. Hyperexponential distribution.*   Next we compare two 3-out-of-n systems with the same component lifetime distributions but with different n. Here we

TABLE 2

2-out-of-7 System with an Erlang² Component Distribution (λ = 0.2)

x	Estimated D.F.	Appr. (2.4)	Appr. (2.7)	Appr. (2.9)	Appr. (2.10)
10.00	0.001	0.010	0.007	0.002	− 0.005
12.50	0.004	0.022	0.015	0.007	− 0.001
15.00	0.014	0.042	0.030	0.019	0.012
17.50	0.040	0.074	0.056	0.045	0.039
20.00	0.088	0.124	0.097	0.092	0.088
22.50	0.161	0.193	0.156	0.162	0.160
25.00	0.256	0.282	0.235	0.255	0.255
27.50	0.366	0.386	0.333	0.365	0.365
30.00	0.481	0.500	0.443	0.481	0.481
32.50	0.592	0.614	0.557	0.592	0.592
35.00	0.691	0.718	0.667	0.690	0.691
37.50	0.774	0.807	0.765	0.772	0.774
40.00	0.840	0.876	0.844	0.836	0.840
42.50	0.890	0.926	0.903	0.884	0.891
45.00	0.927	0.958	0.944	0.919	0.928
47.50	0.952	0.978	0.970	0.943	0.956
50.00	0.970	0.990	0.985	0.960	0.974
52.50	0.981	0.995	0.993	0.971	0.986
55.00	0.989	0.998	0.997	0.979	0.993
57.50	0.993	0.999	0.999	0.984	0.997

use a Hyperexponential distribution, $F(x) = \rho(1 - e^{-\lambda_1 x}) + (1 - \rho)(1 - e^{\lambda_2 x})$, with $\lambda_1 = 1$, $\lambda_2 = 4$ and $\rho = 0.2$. The theory indicates that the error in approximation (2.4) and (2.7) should be reduced by a factor $\sqrt{5} \approx 2.2$, while the reduction factor for (2.9) and (2.10) should be 5, cf. Remark 7.4. Indeed generally those reductions come true. (See Tables 3 and 4 and Figures 4 and 5.)

6.4. *Conclusion.* The Monte-Carlo results are in excellent agreement with the error structure presented in §2. Note the important role of the correction term
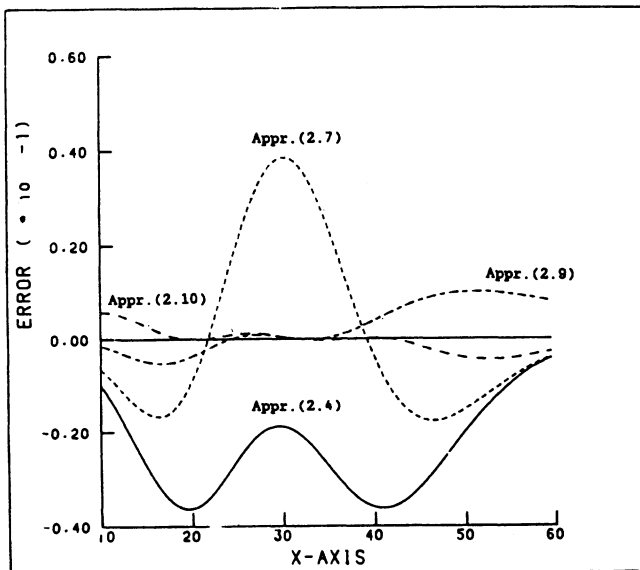

FIGURE 3.   Approximation errors for a 2-out-of-7 system with an Erlang² component distribution ($\lambda = 0.2$).

TABLE 3

3-out-of-20 System with a Hyperexponential Component Distribution ($\lambda_1 = 1$, $\lambda_2 = 4$, $\rho = 0.2$)

x	Estimated D.F.	Appr. (2.4)	Appr. (2.7)	Appr. (2.9)	Appr. (2.10)
0.60	0.00	0.01	0.02	0.00	0.00
0.77	0.00	0.02	0.04	0.01	0.01
0.93	0.00	0.04	0.05	0.03	0.02
1.10	0.02	0.06	0.08	0.05	0.05
1.26	0.05	0.08	0.12	0.09	0.09
1.43	0.10	0.12	0.16	0.14	0.14
1.59	0.18	0.16	0.21	0.21	0.21
1.76	0.27	0.22	0.27	0.30	0.29
1.92	0.37	0.28	0.34	0.38	0.38
2.09	0.47	0.35	0.42	0.47	0.47
2.25	0.56	0.43	0.50	0.56	0.56
2.41	0.64	0.51	0.58	0.64	0.64
2.58	0.72	0.59	0.66	0.71	0.72
2.74	0.78	0.66	0.73	0.77	0.78
2.91	0.83	0.73	0.79	0.82	0.82
3.07	0.87	0.79	0.84	0.85	0.86
3.24	0.90	0.85	0.88	0.88	0.89
3.40	0.92	0.89	0.92	0.91	0.91
3.57	0.94	0.92	0.95	0.92	0.92
3.73	0.96	0.95	0.96	0.94	0.94

$\frac{1}{2}(k - 1)(\sigma\mu^{-1} - \mu\sigma^{-1})$. Without this term the approximated lifetime distribution (2.4) is indeed shifted to the right or to the left with respect to the true distribution. An intuitive explanation is given in remark 7.4. It is also remarkable that the approximations (2.9) and (2.10) are already quite good for very small values of n, which is very convenient for real case applications. In some examples (2.9) and (2.10) differed significantly in the tail. In those cases the latter one yielded slightly better approxima-

TABLE 4

3-out-of-100 System with a Hyperexponential Component Distribution ($\lambda_1 = 1$, $\lambda_2 = 4$, $\rho = 0.2$)

| x	Estimated D.F.	Appr. (2.4)	Appr. (2.7)	Appr. (2.9)	Appr. (2.10)
9.07	0.01	0.02	0.02	0.01	0.01
9.45	0.02	0.03	0.04	0.02	0.02
9.84	0.03	0.05	0.05	0.04	0.04
10.22	0.06	0.07	0.08	0.07	0.07
10.61	0.10	0.10	0.12	0.11	0.11
10.99	0.15	0.14	0.16	0.16	0.16
11.38	0.21	0.19	0.21	0.22	0.22
11.76	0.28	0.25	0.27	0.29	0.29
12.15	0.36	0.32	0.34	0.36	0.36
12.53	0.44	0.39	0.42	0.45	0.44
12.92	0.52	0.47	0.50	0.53	0.53
13.30	0.61	0.55	0.58	0.61	0.61
13.69	0.68	0.63	0.66	0.68	0.68
14.07	0.75	0.70	0.73	0.74	0.74
14.46	0.81	0.76	0.79	0.80	0.80
14.84	0.85	0.82	0.84	0.84	0.84
15.23	0.89	0.87	0.88	0.88	0.88
15.61	0.92	0.91	0.92	0.91	0.91
16.00	0.94	0.94	0.95	0.93	0.93
16.38	0.96	0.96	0.96	0.95	0.95
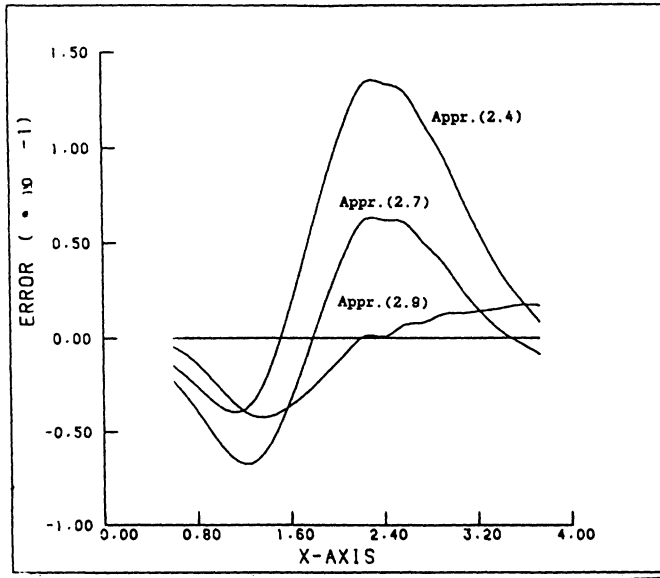
FIGURE 4.   Approximation errors for a 3-out-of-20 systems with a hyperexponential component distribution ($\lambda_1 = 1$, $\lambda_2 = 4$, $\rho = 0.2$).
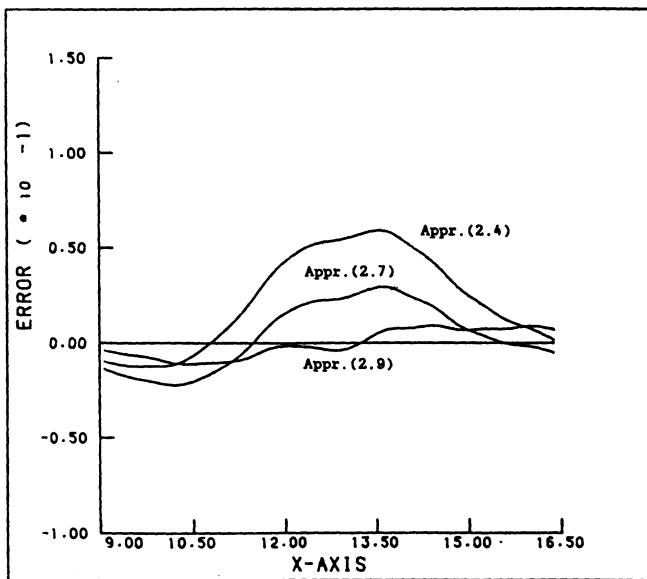

FIGURE 5.   Approximation errors for a 3-out-of-100 system with a hyperexponential component distribution ($\lambda_1 = 1$, $\lambda_2 = 4$, $\rho = 0.2$).

tions. Several other simulations have been performed, all giving the same kind of results.

7.   Recommendations and remarks.   We recommend the use of the approximation (2.7) when only μ and σ^2 are known, since it corrects for the systematic error that occurs in (2.4). If however μ_3 is known too, then approximation (2.7) can be considerably improved by using approximation (2.9) or (2.10). The extra amount of calculations that have to be made is almost negligible. Approximation (2.10) seems to behave better in the tails than (2.9).

REMARK 7.1.   In the preceding sections we have assumed that the lifetime d.f. F_i of a component at position i is the same for all positions $i = 1, \ldots, k$. This assumption is however not really used. It is easy to see that all the preceding theorems and lemmas hold when we only assume that F_i is a nonlattice distribution with $F_i(0) = 0$ and first three moments independent of i. If the first three moments in fact depend on i, straightforward but tedious generalizations of the results can be made, leading to slightly more complicated formulae.

REMARK 7.2.   Although the absolute error of the approximations in the tail of the distribution is almost zero, the relative error can be considerable. To obtain approximations with a small relative error in the tails, we have to replace Feller's theorem in §3 by a moderate or large deviation theorem for sums of i.i.d. r.v.'s, which can be found for instance in Petrov (1975, Chapter VIII). We do not work out the details of such approximations here.

REMARK 7.3.   In this paper we have only assumed the finite existence of the first three moments of the component lifetime distributions. If the fourth moment is also finite, then in Theorem 2.1 the $o(n^{-1/2})$-term is in fact $O(n^{-1})$. This can be easily seen, since in (3.1) we may replace $o(n^{-1/2})$ by $O(n^{-1})$, leading to $O(t_n^{-1})$ in Theorem 3.1. Moreover, the $o(1)$-term in Lemmas 4.1 and 4.2 are in fact $O(a_t)$ and $O(a_t + t^{-1/2})$, respectively. Therefore $o(t_n^{-1/2})$ in Theorem 4.1 may now be replaced by $O(t_n^{-1})$ and with some minor changes in the proof this gives the $O(n^{-1})$-term in Theorem 2.1.

REMARK 7.4.   To explain the correction term in (2.7), we use an argument of Cox (1959). Denote the remaining lifetime of the $k - 1$ working components at time M_n by R_n^i, $i = 1, \ldots, k - 1$. Consider the system at instant M_n. The total time for which all components have been in use is kM_n. But $k - 1$ of the components have still not failed. Thus we have

$$kM_n = X_1 + X_2 + \cdots + X_n - \left(R_n^1 + R_n^2 + \cdots + R_n^{k-1} \right)$$

and hence

$$\lim_{n \to \infty} \left(EM_n - \frac{n\mu}{k} \right) = -\frac{k - 1}{k} \left(\lim_{n \to \infty} ER_n^1 \right) = -\frac{k - 1}{k} \frac{1}{2} \left(\mu + \frac{\sigma^2}{\mu} \right).$$

References

[1]   Cox, D. R. (1959). A Renewal Problem with Bulk Ordering of Components. *J. Roy. Statist. Soc. Ser.* (*B*) **21** 180–189.

[2]   _____. (1962). *Renewal Theory*. Methuen, London.

[3]   Feller, W. (1971). *An Introduction to Probability Theory and Its Applications. II*. Wiley & Sons, New York.

[4]   Glynn, P. W. and Whitt, W. (1988). Ordinary CLT and WLLN versions of $L = \lambda W$. *Math. Oper. Res.* **13** 674–692.

[5]   Iglehart, D. L. and Whitt, W. (1971). The Equivalence of Functional Central Limit Theorems for Counting Processes and Associated Partial Sums. *Ann. Math. Statist.* **42** 1372–1378.

[6]   Petrov, V. V. (1975). *Sums of Independent Random Variables*. Springer-Verlag, New York.

[7]   Whitt, W. (1980). Some Useful Functions for Functional Limit Theorems. *Math. Oper. Res.* **5** 67–85.

FACULTY OF APPLIED MATHEMATICS, UNIVERSITY OF TWENTE, PO BOX 217, 7500 AE ENSCHEDE, THE NETHERLANDS