1. (4 marks)
   (i) Find the unique factorization (up to order of factors) of the integer 6840.
   (ii) Find integers \( q \) and \( r \), with \( 0 \leq r < d \), so that \( n = dq + r \) in the case that \( n = -78 \) and \( d = 11 \).
   (iii) Evaluate \( 38 \text{ div } 11 \) and \( 38 \text{ mod } 11 \).

   (i) \[ 6840 = 10 \times 4 \times 171 = 5 \times 8 \times 3 \times 57 = 5 \times 8 \times 3 \times 3 \times 19 = 2^3 \times 3^2 \times 5^1 \times 19^1. \]

   (ii) \((-78) = 11q + r\). We want \( r \geq 0 \) (and \( r < 11 \)).
   So take \(-78 = 11(-8) + 10\).
   So \( q = -8 \) and \( r = 10 \).

   (iii) \( 38 \text{ div } 11 = 3 \) (because \( 11 \times 3 = 33 \) and \( 38 = 33 + 5 \)).
   \( 38 \text{ mod } 11 = 5 \) (because \( 38 = 11 \times 3 + 5 \)).

2. (4 marks)
   (i) An integer \( n \) divided by 7 leaves the remainder 4. What is the remainder (between 0 and 6) when \( 3n \) is divided by 7?
   (ii) When the integer \( m \) is divided by 11, it leaves the remainder 9. What is the remainder when \( 4m \) is divided by 22?

   (i) We have \( n = 7t + 4 \) where \( t \in \mathbb{Z} \).
   So \( 3n = 7 \cdot 3t + 12 = 7(3t + 1) + 5 \).
   So the remainder now, when \( 3n \) is divided by 7, is 5.

   (ii) Now \( m = 11s + 9 \) for some \( s \in \mathbb{Z} \).
   
   \[ 4m = 44s + 36 \]
   \[ = 22(2s + 1) + 14. \]

   So when \( 4m \) is divided by 22, it leaves remainder 14.

3. (8 marks) Prove carefully each of the following statements.

   (i) The quotient \( r/s \) of any two rational numbers, with \( s \neq 0 \), is a rational number.

   SOLUTION:
   We have \( r \) and \( s \) are rational and \( s \neq 0 \). So let
   \[ r = \frac{a}{b}, \quad s = \frac{c}{d}, \]
   where \( b, d \neq 0 \) and \( c \neq 0 \), and \( a, b, c, d \in \mathbb{Z} \).
   Then \( \frac{r}{s} = \frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \), and \( bc \neq 0 \) because \( b \neq 0 \) and \( c \neq 0 \).
   Here \( ad \in \mathbb{Z} \), and \( bc \in \mathbb{Z} \) and \( bc \neq 0 \).
   Hence \( \frac{r}{s} = \frac{ad}{bc} \in \mathbb{Q} \), the rationals.
(ii) The difference between the cube of two consecutive integers leaves the remainder 1 when divided by 6.
Solution:
Take the consecutive integers to be \( n \) and \( n+1 \). Then the difference between their cubes is
\[
(n+1)^3 - n^3 = (n^3 + 3n^2 + 3n + 1) - n^3 = 3n^2 + 3n + 1 = 3n(n+1) + 1.
\]
Now \( n(n+1) \) is the product of two consecutive integers, and so ONE of \( n \) or \( n+1 \) must be even; so \( n(n+1) \) is even. Thus \( 3n(n+1) \) is \( 3 \times \) even number, so is divisible by 6. Hence \( 3n(n+1) + 1 \) leaves remainder 1 when divided by 6.

(iii) For any odd integer \( n \), the number \( n^2 - 1 \) is divisible by 8.
Solution:
Here \( n \) is odd, so let \( n = 2k + 1 \). Then
\[
n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k + 1).
\]
Now \( k(k + 1) \) is even (because ONE of \( k \) and \( k + 1 \) must be even, so their product is). Hence \( 4k(k + 1) \) is \( 4 \times \) even number; say \( k(k + 1) = 2w \) for some \( w \in \mathbb{Z} \). That is, \( 4k(k + 1) = 4 \times 2w = 8w \), with \( w \in \mathbb{Z} \). Hence when \( n \) is odd, \( n^2 - 1 \) is a multiple of 8, that is, \( n^2 - 1 \) is divisible by 8.

(iv) If the integers \( k, m \) and \( n \) satisfy \( k \mid m \) and \( k \mid n \), then it follows that \( k \mid (m - 2n) \).
Solution:
Given \( k \mid m \) and \( k \mid n \). Must prove that \( k \mid (m - 2n) \).
Now \( k \mid m \) means that \( m = kr \) for some \( r \in \mathbb{Z} \).
And \( k \mid n \) means that \( n = ks \) for some \( s \in \mathbb{Z} \).
So \( m - 2n = kr - 2ks = k(r - 2s) \), where \( r - 2s \in \mathbb{Z} \).
Hence \( m - 2n = k \times \) integer; thus \( k \mid (m - 2n) \), as required.

4. (4 marks) For each of the following statements about the positive integers \( m, n, q \) and prime \( p \), if the statement is true, prove it carefully, and if the statement is false, give a counter-example.

(i) If \( m \mid n \) and \( n \mid q \), then \( m^2 \mid nq \).
(ii) If \( \gcd(m, n) = d \) and \( \gcd(n, q) = 1 \), then \( \gcd(m, q) = 1 \).

Solution:
(i) This is true: \( m \mid n \) means that there exists some \( r \in \mathbb{Z} \) with \( n = rm \).
And \( n \mid q \) means that there exists some \( s \in \mathbb{Z} \) with \( q = ns \).
So \( nq = rnm = rms.n = rms.rm = m^2.r^2s \), where \( r^2s \in \mathbb{Z} \).
So \( nq = m^2 \times \) integer, and so \( m^2 \mid nq \).

(ii) This is false; one counter-example suffices.
For example take \( m = 6 \), \( n = 2 \), \( q = 3 \). Then \( \gcd(m, n) = d = 2 \), and \( \gcd(n, q) = \gcd(2, 3) = 1 \).
But \( \gcd(m, q) = \gcd(6, 3) = 3 \), NOT 1.