1. (6 marks) For all integers $a$ and $b$, if $a + b$ is even then $a$ and $b$ are either both even or both odd. Prove this statement (i) directly; (ii) by contradiction.

**Solution:**

Note: If integers $a$ and $b$ are both even or both odd, we say that $a$ and $b$ *have the same parity*.

(i) Direct proof:

Let $a, b$ be any integers, and suppose their sum is even. So $a + b = 2k$, for some integer $k$.

Now $a$ is either even or odd.

If $a$ is even then $a = 2A$ for some integer $A$, and then

$$
    b = 2k - a = 2k - 2A = 2(k - A) = 2 \times \text{integer},
$$

so $b$ is also even.

If on the other hand $a$ is odd, then $a = 2A + 1$ for some integer $A$. In this case,

$$
    b = 2k - a = 2k - (2A + 1) = 2(k - A) + 1 = (2 \times \text{integer}) + 1,
$$

so $b$ is also odd.

Hence if the sum of two integers is even, they are both even integers or else they are both odd integers (that is, they have the same parity).

(ii) Proof by contradiction:

Assume that integers $a$ and $b$ satisfy $a + b$ is even, but that one of $a$, $b$ is even and the other one is odd.

Since $a + b = b + a$, there’s no loss of generality in assuming that $a$ is even and $b$ is odd.

So say $a = 2A$ and $b = 2B + 1$ for some integers $A$ and $B$.

Then $a + b = 2A + (2B + 1) = 2(A + B) + 1$, which is odd. Contradiction!

Hence our assumption is false, so we can’t have one of $a$, $b$ even and the other one odd; that is, $a$ and $b$ must have the same parity (both even or both odd).

2. (4 marks) For all integers $c, d$ and $e$, if $c \mid d$ and $c \not| e$, then $c \not| (d + e)$.

Use *proof by contradiction* to prove this statement.

**Solution:**

Proof by contradiction: Note that the negation says

$$
    \exists c, d, e \in \mathbb{Z} \text{ so that } (c \mid d \text{ and } c \not| e) \text{ and } c \mid (d + e).
$$

Assume the negation holds.

Now $c \mid d$ means that $d = cx$ for some integer $x$.

And $c \mid (d + e)$ means that $d + e = cy$, for some integer $y$.

Hence $d + e = cx + e = cy$, so $e = cy - cx = c(y - x)$, and we have $y - x \in \mathbb{Z}$, because $x$ and $y$ are both integers. Thus $c \mid e$, a contradiction.

Hence the negation isn’t true. So the original statement is true.
3. (4 marks) Prove that there exists a unique prime number of the form $n^2 + 2n - 3$, where $n$ is a positive integer.

Hint: (1) Find some value of $n$ for which the expression is prime. Then you must show uniqueness, so suppose there are two different values of $n$ yielding primes $p$ and $q$; then show that in fact we must have $p = q$.

(2) Think about primes and factors; what can you do with a quadratic expression like $n^2 + 2n - 3$?

**Solution:**
First note that $n^2 + 2n - 3 = (n + 3)(n - 1)$.

When $n = 2$, this equals 5, which is a prime.

So there exists at least one value of $n$ for which $n^2 + 2n - 3$ is prime.

To show uniqueness, suppose there are two values of $n$ which make $n^2 + 2n - 3$ prime. So suppose we have

$$p = n_1^2 + 2n_1 - 3 = (n_1 + 3)(n_1 - 1) \quad \text{and} \quad q = n_2^2 + 2n_2 - 3 = (n_2 + 3)(n_2 - 1).$$

Now the only factors a prime number can have are 1 and the prime itself. And of course $n + 3$ and $n - 1$ are both factors of $n^2 + 2n - 3$. Also $n$ is a positive integer; so $n + 3$ must be 4 or more. Therefore we must have $(n_1 - 1) = 1$ and $(n_2 - 1) = 1$, so that $n_1 = n_2 = 2$, so $p = q$ after all, and they both equal 5.

Hence there is a unique prime number of the form $n^2 + 2n - 3$ where $n \in \mathbb{Z}^+$. 

4. (6 marks) Determine whether or not there exist solutions to the following linear Diophantine equations. If a solution exists, give one, that is, give integer values for $x$ and $y$ which satisfy the given equation.

(a) $14x + 3003y = 7$.

**Solution:** Since 3003 = $7 \times 429$, and 14 = $7 \times 2$, a solution does exist.

For simplicity, cancel 7 throughout, and work with $2x + 429y = 1$.

$429 = 2 \times 214 + 1$.

So $1 = 429 \times 1 + 2(-214) = 2x + 429y$.

Thus $x = -214$ and $y = 1$ are a solution to the equation.

(b) $18x + 1028y = 3$.

**Solution:** Here although $3 \mid 18$, we have $3 \nmid 1028$, so there is no solution.

(c) $221x - 255y = 17$.

**Solution:** Now $221 = 17 \times 13$ and $255 = 17 \times 15$, so a solution exists.

We cancel 17 throughout, and work with the equation $13x - 15y = 1$.

$15 = 13 \times 1 + 2$;

$13 = 2 \times 6 + 1$.

Thus $1 = 13 - 2 \times 6 = 13 \times 1 - (15 - 13)6$, or $1 = 13 \times 7 - 15 \times 6$.

So $x = 7$ and $y = 6$ is a solution to $13x - 15y = 1$, and hence also to $221x - 255y = 17$. 