1. (4 marks) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $B = \{n \in \mathbb{N} \mid n < 13\}$, and $C = \{e \in \mathbb{N} \mid e \text{ is even}\}$.

In (i)–(iii), write down the sets (so list the elements in the particular sets in curly braces); in (iv), use set notation (as in $B$ and $C$ above) to describe the set clearly.

(i) $A \cap B$
(ii) $B \cap C$
(iii) $B - A$
(iv) $A \cup C$

Solution:
(i) $A \cap B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, or $A \cap B = A$.
(ii) $B \cap C = \{0, 2, 4, 6, 8, 10, 12\}$. [Note that $0 \in \mathbb{N}$ for this course, using the text Epp!]
(iii) $B - A = \{0, 11, 12\}$. [Since $0 \in \mathbb{N}$!]
(iv) $A \cup C = \{n \in \mathbb{N} \mid 1 \leq n \leq 10, \text{ or } n = 2k, k \in \mathbb{N}\}$, or \{w \in \mathbb{N} \mid w = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \text{ or } w \text{ is even}\}.

2. (7 marks) Let $X = \emptyset$, $Y = \{X\}$ and $Z = \{\emptyset, Y\}$.

(i) How many elements are in the set $X$, and in $Y$, and in $Z$? That is, what are $|X|$, $|Y|$ and $|Z|$ equal to? (It may help to write out the elements in these sets explicitly.)

(ii) Write down, by listing the elements inside curly braces, the following sets:

(a) $X \cup Y$
(b) $X \cup Z$
(c) $P(Y)$, the power set of $Y$
(d) $P(Z)$
(e) $X \cup Y \cup Z$
(f) $Z - X$

(iii) State whether the following are true or false.

(a) $X \subseteq Y$
(b) $Y \subseteq Z$
(c) $X \in Y$
(d) $X \in Z$
(e) $Y \in Z$
(f) $X \cap Y = \emptyset$
(g) $X \cap Z = Y$

Solution: Note: Be aware of the difference between the emptyset, $\emptyset$, (a set of size zero, with no elements in it), and $\{\emptyset\}$, which is a set with one element in it. So for instance in (f) below, you have the set $Z$, with two elements in it, and you're removing any elements in $Z$ which are also in $X$. But $X$ doesn't have any elements—it's empty! So the set $Z$ is left unchanged when you take $Z - X = Z - \emptyset$. Even though one of the elements in $Z$ is the emptyset itself, this is in $Z$ as one of its elements; the emptyset isn't an element of $X$! Write $X = \{\} \text{ if it's easier to understand.}$

(i) $X$ contains 0 elements; we write $|X| = 0$. Also $|Y| = 1$, and $|Z| = 2$.

(ii)

(a) $X \cup Y = \{X\} = Y$.
(b) $X \cup Z = \{\emptyset, Y\} = Z$.
(c) $P(Y) = \{\emptyset, \{X\}\} \text{ or } \{\emptyset, Y\}$.
(d) $P(Z) = \{\emptyset, \{\emptyset\}, \{Y\}, \{\emptyset, Y\}\} \text{ (or } \{\emptyset, \{\emptyset\}, \{Y\}, Z\})$.
(e) $X \cup Y \cup Z = \{X, \emptyset, Y\}$.
(f) $Z - X = \{\emptyset, Y\} = Z$.

(iii) (a), (b), (c), (d), (e), (f) are all true; only (g) is false.

No reasons required for marks, but here are some to help you understand!

$X$ is the emptyset (of size zero!); the emptyset is a subset of every set; so (a) is true.

Subsets of $Z$ are: $\emptyset$, then two of size 1: $\{\emptyset\}$ and $\{Y\}$ (we take the two listed elements of $Z$ and put braces round them to get the subsets of $Z$ of size 1), and $\{\emptyset, Y\}$, that is, $Z$ itself. And since $Y = \{X\} = \{\emptyset\}$, part (b) is true; $Y$ is indeed a subset of $Z$.

Note that $Y = \{X\} = \{\emptyset\}$, and $Y \neq \{Y\}$. Here (d) is true because $X = \emptyset$ and $\emptyset$ is listed as an element of $Z$. 

over...
3. (4 marks) Let sets \( X \) and \( Y \) be given by \( X = \{a, b\} \), \( Y = \{1, b\} \).

(i) Write out the set \( X \times Y \).
(ii) Write out the set \( Y - X \).
(iii) Write out the set \( X \times (Y - X) \).
(iv) Write out the power set of \( X \times (Y - X) \), that is, \( \mathcal{P}(X \times (Y - X)) \).

**Solution:**
(i) \( X \times Y = \{(a, 1), (a, b), (b, 1), (b, b)\} \).
(ii) \( Y - X = \{1\} \).
(iii) \( X \times (Y - X) = \{(a, 1), (b, 1)\} \).
(iv) \( \mathcal{P}(X \times (Y - X)) = \{\emptyset, \{(a, 1)\}, \{(b, 1)\}, \{(a, 1), (b, 1)\}\} \).

4. (5 marks) If \( A, B, C \) are any three sets, prove carefully that
\[(A - B) \cap C = (A \cap C) - B.\]

**Solution:**
First we’ll show that \( (A - B) \cap C \subseteq (A \cap C) - B \)

Let \( x \) be any element in \( (A - B) \cap C \). Then \( x \in (A - B) \) and \( x \in C \); so \( x \in A \) and \( x \notin B \) and \( x \in C \).
Thus \( x \in A \) and \( x \in C \) and \( x \notin B \) (all “and” so we can swap order!)
so \( x \in (A \cap C) \) and \( x \notin B \), that is, \( x \in (A \cap C) - B \).
Hence
\[(A - B) \cap C \subseteq (A \cap C) - B \quad (1)\]
because \( x \) was an arbitrary element of \( (A - B) \cap C \).

Next we’ll show that \( (A \cap C) - B \subseteq (A - B) \cap C \)

Now take any element \( y \in (A \cap C) - B \).
Then \( y \in A \cap C \) and \( y \notin B \), so \( (y \in A \) and \( y \in C \) \) and \( y \notin B \) (all “and”);
thus \( y \in A \) and \( y \notin B \) and \( y \in C \), so \( y \in (A - B) \) and \( y \in C \), that is, \( y \in (A - B) \cap C \).
Hence, since \( y \) was an arbitrary element of \( (A \cap C) - B \), we have
\[(A \cap C) - B \subseteq (A - B) \cap C. \quad (2)\]

Now from (1) and (2) we have equality:
\[(A - B) \cap C = (A \cap C) - B,\]
as required.

(A Venn diagram may aid thinking about this, but it does not constitute a formal proof.)