1. (5 marks) Let $G$, $H$ and $K$ be the graphs below.

(a) Answer the following questions with yes/no, and also give a one line brief explanation for each of your answers.

(i) $G$ is a simple graph.
   NO; $G$ has loops and a parallel (multiple) edge.

(ii) The total degree of $G$ is 18.
   NO; loops count 2 to the total degree. Total degree is $4 + 6 + 4 + 2 + 4 = 20$.

(iii) $H$ is a subgraph of $G$.
   YES; all vertices in $H$ are in $G$, and all edges in $H$ are in $G$.

(iv) $K$ is a subgraph of $G$.
   NO; edge $\{v_1, v_5\}$ is in $K$ but is not in $G$.

(v) $H$ is a tree.
   NO; $H$ contains a simple circuit (cycle), $v_1, v_2, v_3, v_4, v_1$.

(vi) $G$ contains an Euler circuit.
   YES; every vertex has even degree.

(vii) $H$ contains an Euler path.
   YES; every vertex, except two, has even degree, and two vertices have odd degree.

(viii) $H$ is bipartite.
   YES; with vertices partitioned as $X = \{v_1, v_3\}$ and $Y = \{v_2, v_4, v_5\}$, the only edges are from $X$ to $Y$.

(b) Now draw any one subgraph of $G$ which contains all five vertices of $G$ and which is a tree.
Lots of solutions! Your graph must be a simple connected graph using all five vertices and with no circuit. Here is one:

![Tree subgraph](image)

2. (4 marks) For each of the following, state whether or not there exists a simple graph with vertices having degrees as stated. If there is no such simple graph, explain why; if a simple graph does exist with the given degrees, then draw such a graph.

(a) Seven vertices with degrees 6, 5, 4, 3, 2, 2, 1.
   Since $6 + 5 + 4 + 3 + 2 + 2 + 1$ is odd (three odd numbers there!), there is no such graph, because the total degree is twice the number of edges, and so is always even.
(b) Five vertices with degrees 3, 3, 2, 1, 1.
Here is such a graph; its degrees are as marked:

3. (5 marks)

(a) If a tree contains exactly 9 vertices, how many edges does it contain? (Explain your
answer very briefly.)
A tree with 9 vertices must have 8 edges. For any tree with \( n \) vertices has \( n - 1 \) edges.
(b) A tree \( T \) has 9 vertices. The degrees of its vertices are
\[ 1, 1, 1, 1, 2, 2, r, s. \]

If \( 3 \leq r \leq s \), find \( r \) and \( s \).
The sum of all the degrees is twice the number of edges, and from (a) we know that there
are 8 edges. So we have
\[ 1 + 1 + 1 + 1 + 1 + 2 + 2 + r + s = 16, \]
That is, \( r + s = 16 - 9 = 7 \). We’re also given that \( 3 \leq r \leq s \), and of course \( r \) and \( s \) are
integers; therefore we must have \( r = 3 \) and \( s = 4 \).
You’re not asked for a picture, but here is such a tree, with its degrees labelled:

4. (6 marks) Which of the following graphs contains an Euler circuit, which contains an Euler
path, and which contains neither? (Explain your answers briefly, and give any conditions on
\( n, q \) and \( r \).)
\[ K_4, K_5, K_n, K_{2,4}, K_{2,q}, K_{3,r}. \]
(i) \( K_4 \) has 4 vertices of degree 3, all odd, so there’s no Euler circuit and no Euler path either.
(ii) \( K_5 \) has all vertices of degree 4 (even), and so this graph has an Euler circuit.
(iii) \( K_n \) has all vertices of degree \( n - 1 \). So for all odd positive integers \( n \) there is an Euler
circuit. (Of course when \( n = 1 \) there are no edges!) Also when \( n = 2 \) there are just two vertices
of odd degree, and we have an Euler path. (The graph \( K_2 \) is trivially a single edge!)
(iv) \( K_{2,4} \) has all vertices of degrees 2 or 4, even, and so there is an Euler circuit in this graph.
(v) \( K_{2,q} \) has vertices of degrees 2 and \( q \). There is an Euler circuit if and only if \( q \) is even, \( q \geq 2 \),
for then all vertices have even degree.
Also if \( q \) is odd, there are only two vertices of odd degree \( q \), and the other \( q \) vertices have even
degree 2. So for odd \( q \) there is an Euler path.
(vi) \( K_{3,r} \) has degrees 3 and \( r \).
If \( r \) is even, and if \( r = 2 \), there are just 2 vertices of degree 3 and 3 of degree 2, so there is an
Euler path.
If \( r \) is odd, we have at least 4 vertices of odd degree, so there is no Euler path nor an Euler
circuit.)