1. (1 mark) Find the hyperbolic line through the points (0, 2) and (1, 3).

2. (3 marks) Find all lines through (1, 2) and parallel to \(5L\) in the hyperbolic plane \(\mathcal{H}\).

3. (2 marks) If \((S, \mathcal{L}, d)\) is a metric geometry, and \(P \in S\), prove that for all real numbers \(r > 0\), there exists a point in \(S\) which is at distance \(r\) from \(P\). In other words, show that, given a real number \(r > 0\), there is a point \(Q\) with \(d(P, Q) = r\).

For questions 4–6, you will need the following.

Definition: A distance function \(d\) on a set of points \(S\) is said to satisfy the triangle inequality if and only if
\[
d(A, C) \leq d(A, B) + d(B, C), \quad \text{for all } A, B, C \in S.
\]

4. (2 marks) Show that the Euclidean distance function \(d_E\) satisfies the triangle inequality.

Hint: First use the Cauchy-Schwartz inequality to show that \(X, Y \in \mathbb{R}^2\) implies that
\[
||X + Y|| \leq ||X|| + ||Y||.
\]

5. (2 marks) Show that the taxicab distance function \(d_T\) satisfies the triangle inequality.

6. The function \(d_F\) from \(\mathbb{R}^2\) to the reals is defined for all points \(P, Q \in \mathbb{R}^2\) by
\[
d_F(P, Q) = \begin{cases} 
0 & \text{if } P = Q; \\
d_E(P, Q) & \text{if line } L_{PQ} \text{ is not vertical}; \\
3d_E(P, Q) & \text{if line } L_{PQ} \text{ is vertical}.
\end{cases}
\]

(i) (2 marks) Show that \(d_F\) is a distance function on \(\mathbb{R}^2\).

(ii) (2 marks) Show that \((\mathbb{R}^2, \mathcal{L}_E, d_F)\) is a metric geometry.

(iii) (2 marks) Show that \(d_F\) does not satisfy the triangle inequality.

Marks total 16; your mark will be halved to yield a maximum possible mark of 8% towards the final assessment.