(1) **Taxicab plane**: given \( d_T(A,C) = d_T(A,B) + d_T(B,C) \)

Not necessarily true that \( A-B-C \) because this requires \( A,B,C \) to be collinear. So an example suffices. Let \( A=(0,0), \ B=(1,0), \ C=(1,2) \).

Then \( d_T(A,C) = |x_A-x_C|+|y_A-y_C| = |0-1|+|0-2| = 3 \),
and \( d_T(A,B) = 1 \), \( d_T(B,C) = 2 \), so \( d_T(A,C) = d_T(A,B) + d_T(B,C) \).

But \( A,B,C \) are not collinear.

(2) \( d_s((x_1,y_1),(x_2,y_2)) = \max \{ |x_1-x_2|, |y_1-y_2| \} \).

Circle, radius 1 and centre \((0,0)\) is:
\[ \{(x,y) \mid d_s((x,y),(0,0)) = 1\} = \{(x,y) \mid \max \{ |x|, |y| \} = 1\} \]

Sketch:

(3) \( M \) is midpt of \( \overline{AB} \) iff \( AM=MB \) (here \( AM \) means \( d(A,M) \)).

(a) **Claim**: \( A-M-B \).

Since \( A,M,B \) are collinear, one of the following holds:
\[ A-M-B \quad \text{or} \quad A-B-M \quad \text{or} \quad M-A-B \]

Now \( A-B-M \) means \( AM=AB+BM \), but \( AM=MB \) so \( AB=0 \) or \( A=B \), not so.

And \( M-A-B \) means \( MA+AB=MB \), but \( MB=MA \), so \( AB=0 \), or \( A=B \), not so.

Hence \( A-M-B \), as required.

(b) **Claim**: \( \overline{AB} \) has a midpt \( M \), and \( M \) is unique. (Let \( d=\overline{AB} \).)

Let \( f : \ell \to \mathbb{R} \) be rule for \( \ell \) with \( f(A)=0, f(B)=b>0 \).

Define \( M \in \overline{AB} \) by \( M = f^{-1}(\frac{b}{2}) \). Since \( f \) is onto \( \mathbb{R} \), \( M \) exists, and since \( f \) is 1-1, \( M \) is uniquely determined.

And \( d(A,M)=|f(A)-f(M)| = |0-\frac{b}{2}| = \frac{b}{2} \), while \( d(M,B)=|f(M)-f(B)| = \frac{b}{2} - 0 = \frac{b}{2} \).

Hence \( AM=MB \), and \( M \) is the unique midpt. of \( \overline{AB} \).
Ass't 2, Sols (cont'd)

(3)(c) \( A = (0,9) \) and \( B = (0,1) \). Want midpoint of \( AB \) in
   
   (i) \( \mathbb{E} \), Euclidean plane; (ii) \( \mathbb{H} \), Hyperbolic plane.

   (i) Noting \( A, B \) are on \( L_0 \), and \( 9-1 = 4 \),
   \( M = (0,5) \) because
   \( d_E((0,9),(0,5)) = |9-5| = 4 \) and \( d_E((0,5),(0,1)) = |5-1| = 4 \).

   (ii) Line is \( \ell_0 \) in \( \mathbb{H} \). Let \( M = (0,m) \).
   Then \( f(M) = \ln(m) \), and we want
   \[ \ln m - \ln 1 = \ln 9 - \ln m \]
   \[ \ln m = \ln \frac{9}{m} \]
   \[ m^2 = 9, \; m = 3. \]
   (Check: \( \ln \frac{9}{3} = \ln 3 \)).

   So \( (0,3) = M \) is the midpoint in \( \mathbb{H} \).

(4) In \( \mathbb{H} \), \( A = (-1,3) \), \( B = (0,2\sqrt{2}) \), \( C = (1,\sqrt{5}) \).

   Is \( A-B-C \) true?

   Must check whether \( A, B, C \) are collinear AND whether
   \( d_H(A,B) + d_H(B,C) = d_H(A,C) \).

   Say line is \( cL_3 \). Then \( (x-c)^2 + y^2 = r^2 \).
   
   \[ A \text{ on line } \Rightarrow \; (-1-c)^2 + 9 = r^2 \] \( \Rightarrow \; 1 + 2c + 9 = 8 \), \( 1+2c+9=8 \);
   \[ B \text{ on line } \Rightarrow \; c^2 + 8 = r^2 \] \( \Rightarrow \; c = -1, \; r = 3 \).

   So \( AB \) is \( -1L_3 \), or \( (x+1)^2 + y^2 = 9 \):

   Check pt \( C \) : is this on \( \overleftrightarrow{AB} \)?
   \( (1+1)^2 + 5 = 4 + 5 = 9 \), so yes, \( C \in -1L_3 = AB \).

   So \( A, B, C \) are indeed collinear points.

   Now \( d_H(A,B) = \ln \left| \frac{c - x_A + r}{\frac{y_A}{c - x_A + r}} \right| = \ln \left| \frac{\frac{-1+3}{3}}{\frac{1+3}{2\sqrt{2}}} \right| = \ln \sqrt{2} = \frac{1}{2} \ln 2 \),

   and \( d_H(B,C) = \ln \left| \frac{-1+3}{\frac{2\sqrt{2}}{-1+3}} \right| = \ln \sqrt{5} = \frac{1}{2} \ln 5 \),

   while \( d_H(A,C) = \ln \frac{1}{\sqrt{5}} = \ln \sqrt{5} = \frac{1}{2} \ln 5 \),

   and \( \frac{1}{2} \ln 2 + \frac{1}{2} \ln \frac{3}{2} = \frac{1}{2} \ln 2 + \frac{1}{2} \ln 5 - \frac{1}{2} \ln 2 = \frac{1}{2} \ln 5 = d(A,C) \).
(5) $S = \{ (x,y) | 0 \leq x < 1 \}$, Missing Strip Plane

Want lines through $(-1,1)$, parallel in Missing Strip Plane, to (i) $L_2 \cap S$, (ii) $L_{-1,2} \cap S$.

(i) Only $L_{-1,0} \cap S$

is parallel to $L_2 \cap S$.

(ii) Now parallel to $L_{-1,2} \cap S$:

One line is of course $L_{-1,0} \cap S$.

But there are many more parallel lines, going through the "gap"! We find these now:

$L_m, b \text{ in } y = mx + b \text{ going through } (-1,1) \text{ means } 1 = -m + b$.

or $m = b - 1$.

So $L_{b-1, b}$. Now $b$, the intercept on the $y$-axis, can go from 2 down to but not including 1,

$1 < b \leq 2$. (See figure!)

So $L_{b-1, b} \cap S$ passes through $(-1,1)$ and doesn't meet $L_{-1,2} \cap S$ provided $1 < b \leq 2$.

(And $L_{-1,0} \cap S$ is also parallel to $L_{-1,2} \cap S$ and is through $(-1,1)$. )
Claim: In $\mathbb{H}$, \( \{(x, y) \in \mathbb{H} \mid x^2 + (y-5)^2 = 9\} \) is the circle with centre $(0, 4)$ and radius $3 \sqrt{2}$. 

Proof: Let $A = \{(x, y) \in \mathbb{H} \mid x^2 + (y-5)^2 = 9\}$, and let $B = \{(x, y) \in \mathbb{H} \mid d_H((x, y), (0, 4)) = 3 \sqrt{2}\}$. We'll show $A \subseteq B$ and then $B \subseteq A$, for equality.

First, $A \subseteq B$:
Let $(a, b) \in A$. If $a = 0$, then $b - 5 = \pm 3$ so $b = 3$ or $2$.
And $d_H((0, 2), (0, 4)) = d_H((0, 3), (0, 4)) = 3 \sqrt{2}$.
Now say $(a, b) \in A$ and $a \neq 0$. Must find $c$ and $r$ so that $(a, b) \in cL + r$ and $(0, 4) \in cL + r$, in order to find $d_H((a, b), (0, 4))$.

\[(x-c)^2 + y^2 = r^2 \Rightarrow (a-c)^2 + b^2 = r^2 = c^2 + 16 \Rightarrow a^2 - 2ac + b^2 = 16 \Rightarrow c = \frac{a^2 + b^2 - 16}{2a}, \quad r = \sqrt{c^2 + 16}.\]

Now $(a, b) \in A$ means $a^2 + (b-5)^2 = 9$, so $a^2 + b^2 - 16 = 2ac$ and $a^2 + b^2 - 16 = 9 - 25 + 10b - 16 = 10b - 32$.
So $c = \frac{10b - 32}{2a} = \frac{5b - 16}{a}$.

And $r = \sqrt{c^2 + 16} = \sqrt{\left(\frac{5b - 16}{a}\right)^2 + 16a^2} = \sqrt{\frac{25b^2 + 16(a^2 - 10b + 16)}{a^2}} = \sqrt{\frac{9b^2 + 16(a^2 - 10b + 9)}{a^2}} = \frac{3b}{|a|}$ (always $b > 0$).

Now $d_H((a, b), (0, 4)) = \frac{\left|a + \frac{4}{b} - c + \frac{4}{b} - r\right|}{|b|}$ and

\[
\frac{a - c + r}{b} = \frac{a - \frac{5b - 16}{a} + \frac{3b}{|a|}}{b} \quad \text{so} \quad \frac{a - c + r}{b} = \begin{cases} 
\frac{4}{b(-5b + 16 \pm 3b)} & \text{if } a > 0 \\
\frac{4}{b(16 - 8b)} & \text{if } a < 0
\end{cases}.
\]
\[
\begin{cases}
\frac{t}{2} = 2 & \text{if } a > 0 \\
\frac{t}{2} = \frac{1}{a} & \text{if } a < 0.
\end{cases}
\]

So \(d_{\mathbb{H}}((a,b),(0,1)) = |\ln 2| = |\ln \frac{1}{a}|\), so \((a,b) \in B\) so \(A \subseteq B\), since \((a,b)\) was any pt in \(A\).

To show \(B \subseteq A\):

Let \((a,b) \in B\), and let \(l\) be the hyperbolic line through \((a,b)\) and \((0,1)\). Then \(l\) is either part of a vertical Euclidean line or a Euclidean circle. Now from knowledge of Euclidean geometry, \(l \cap A\) is exactly two points. Likewise, \(l \cap B\) has two points.

Since \(A \subseteq B\), \(l \cap A \subseteq l \cap B\). And since \(l \cap A, l \cap B\) are each just two points, \(l \cap A = l \cap B\) and \((a,b) \in l \cap B = l \cap A\), so \((a,b) \in l \cap A\) so \((a,b) \in A\).

Hence \(B \subseteq A\).

Thus we have the required equality.