Some Taxicab Geometry Problems (I suggest you have some squared paper handy!)

1. (a) Let \( C = (2, 3) \). Find the set of all points \( P \) such that \( d_T(P, C) = 3 \). (Draw your answer first and then describe the points.)
This would suggest a “circle” of “radius” 3 and centred at \( C \).

(b) Remembering that \( \pi \) is the circumference of a circle divided by its diameter, what is a reasonable numerical value for \( \pi \) in the taxicab geometry? If you’d taken a different “circle” in part (a), would your answer for \( \pi \) in the taxicab geometry change?

2. (a) How could you choose a set of points \( S \) in the taxicab plane so that every point of the plane is within distance 3 of some point of \( S \)?

(b) Let \( S = \{ A \in \mathbb{R}^2 \mid d_T(A, (0, 0)) \leq 12 \} \).
Let \( S' \subseteq S \) be a set of points such that \( A \in S \) implies that there exists \( B \in S' \) with \( d_T(A, B) \leq 4 \). What is the smallest possible number of points in \( S' \)?

3. Let \( A = (-2, -1) \), \( B = (2, 2) \). Sketch the taxicab ellipse \( \{ P \in \mathbb{R}^2 \mid d_T(P, A) + d_T(P, B) = 9 \} \).

Hint: If you have problems with this question or the next, try dividing the \( xy \)-plane up into regions. To choose the regions, let \( P = (x, y) \). Then \( d_T(P, A) = |x + 2| + |y + 1| \), (and work out \( d_T(P, B) \)), and so the requirement that \( d_T(P, A) + d_T(P, B) = 9 \) becomes
\[
|x + 2| + |y + 1| + |x - 2| + |y - 2| = 9.
\]

So to “remove” the absolute value signs, partition the \( xy \)-plane into nine regions, according as
\[
x \leq -2, \quad -2 < x < 2, \quad x \geq 2, \quad \text{or} \quad y < -1, \quad -1 \leq y \leq 2, \quad \text{or} \quad y > 2.
\]
Then for instance when \( x \leq -2 \) and \( -1 \leq y \leq 2 \), (*) simplifies to \( x = -3 \).

4. Let \( A = (-2, -1) \) and \( B = (2, 2) \). Sketch the taxicab hyperbola
\[
\{ P \in \mathbb{R}^2 \mid |d_T(P, a) - d_T(P, B)| = 3 \}.
\]

Some Moulton Plane and Missing Strip Plane Problems

In questions below we take the strip which is missing to be \( \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < 1 \} \).

5. Given the following pairs of points: (i) \((2, 3)\) and \((3, -1)\); (ii) \((0, 3)\) and \((\frac{1}{2}, -2)\); (iii) \((-1, 4)\) and \((2, 7)\).
For both the Moulton Plane and the Missing Strip Plane, if the given pair of points lies in the point set for that geometry, find the line through that pair of points.

6. In the Moulton Plane:

(a) Find where the line \( M_{1,2} \) meets (i) the line \( M_{2,0} \), (ii) the line \( L_{-1,2} \).

(b) What line(s) passing through \((1, 3)\) are parallel to \( M_{1,2} \)?
7. If $S$ denotes the set of points of the missing strip plane, find the lines through the point $(2,0)$ which are parallel in the missing strip plane to:
   (a) the line $L_{-1} \cap S$;       (b) the line $L_{1,0} \cap S$.

8. In the Missing Strip plane $(S, \mathcal{L})$, for the line $\ell = L_{m,b}$, define
   
   $$ g_\ell(x, y) = \begin{cases} 
   f_\ell(x, y) & \text{if } x < 0, \\
   f_\ell(x, y) - \sqrt{1 + m^2} & \text{if } x \geq 1.
   \end{cases} $$

   Verify that $g_\ell : (\ell \cap S) \to \mathbb{R}$ is a bijection.

General Problems

9. Find a ruler $f$ with $f(P) = 0$, $f(Q) >$ for the pair of points:
   (a) $P = (2,3), Q = (2, -5)$, in the Euclidean plane.
   (b) $P = (2,3), Q = (2, -5)$, in the Taxicab plane.
   (c) $P = (2,3), Q = (2,1)$, in the Hyperbolic plane.

10. Find the coordinate of the point $(2,3)$ with respect to:
    (a) the line $x = 2$ in the Euclidean plane;
    (b) the line $y = -4x + 11$ in the Euclidean plane;
    (c) the line $y = -4x + 11$ in the Taxicab plane;
    (d) the line $(x - 1)^2 + y^2 = 10$ in the Hyperbolic plane;
    (e) the line $x = 2$ in the Hyperbolic plane.