NOTE: This sheet (apart from Qus. 11–13) is largely based upon work you should know from MATH2300, which we only cover quickly and briefly for revision in this course. Consequently it is longer than the next four sheets!

(1) Let the vertices of a graph $G$ be labelled with the two-element subsets of the set \{1, 2, 3, 4, 5\}. Join two vertices of $G$ with an edge if the subsets labelling them are disjoint. Show that $G$ is the Petersen graph.

(2) Which of the following degree sequences are graphical? (If a sequence is graphical, draw an example; if a sequence is not graphical, explain carefully why.)

(i) 6, 5, 5, 4, 4, 3, 3.
(ii) 7, 5, 4, 4, 3, 3, 2.
(iii) 7, 5, 4, 4, 3, 3, 2, 2.

(3) Draw an eulerian simple graph with an even number of vertices and an odd number of edges, or prove that this is impossible.

(4) A tree $T$ has $p$ vertices where $p \geq 11$. If $T$ has precisely six vertices of degree 1 and precisely four vertices of degree 3, find the degrees of the other $p - 10$ vertices.

(5) Let $G$ be a simple graph of order $p$. Prove that the following are all equivalent.

(i) $G$ is connected, with no cycles.
(ii) $G$ is connected, with $p - 1$ edges.
(iii) $G$ has $p - 1$ edges, with no cycles.
(iv) for all $u, v \in V(G)$, $G$ has exactly one path from $u$ to $v$.

*Hint: Show (i) $\Rightarrow$ (ii) $\lor$ (iii); (ii) $\Rightarrow$ (iii) $\lor$ (i); (iii) $\Rightarrow$ (i) $\lor$ (ii). Use induction on $p$ for the first of these. Then show (i) $\iff$ (iv) using a contradiction argument.*

You may assume that a tree (which is a connected simple graph with no cycles) of order 2 or more has at least 2 leaves, and that deleting a leaf from a tree of order $p$ yields a tree of order $p - 1$.

(6) Let $T$ be any tree on $k + 1$ vertices. Let $G$ be a simple graph with $\delta(G) \geq k$. Prove that $T$ is a subgraph of $G$.

*Hint: Use induction on $k$.*
(7) A simple graph $G$ is self-complementary if it is isomorphic to its complement $\overline{G}$.

(a) Prove that a self-complementary graph must have order $p \equiv 0$ or $1 \pmod{4}$.
(b) Let $G$ be a self-complementary graph of order $p$ where $p \equiv 1 \pmod{4}$. Prove that $G$ contains at least one vertex of degree $(p - 1)/2$.
(c) Construct self-complementary graphs of order $p$ for all $p \equiv 0$ or $1 \pmod{4}$.

*HINT:* For $p \equiv 0 \pmod{4}$: When $p = 4$, the graph is a path of length 3. Think of this as $K_2$ and $\overline{K_2}$, together with half of the edges between the $K_2$ and the $\overline{K_2}$. Then for $p = 8$, take a $K_4$ on four of the vertices, and leave the other four vertices as $\overline{K_4}$; then take half the edges between the two sets of four vertices. Similarly for any $p \equiv 0 \pmod{4}$.
For $p \equiv 1 \pmod{4}$: add one vertex to your graph constructed in the case $p - 1$, and appropriate edges from this new vertex.

*NOTE:* $p \equiv 1 \pmod{4}$ means $p$ is an integer which leaves remainder 1 when divided by 4, so $p$ is of the form $4s + 1$ for some integer $s$. (Similarly for $p \equiv 0 \pmod{4}$; here $p$ is divisible by 4.)

(8) (a) Prove that every tree with maximum degree $\Delta$ with $\Delta > 1$ has at least $\Delta$ vertices of degree 1 (that is, at least $\Delta$ leaves).
(b) Show that this is best possible by constructing a $p$-vertex tree with maximum degree $\Delta$ and exactly $\Delta$ leaves, for all $p$ and $\Delta$ with $2 \leq \Delta < p$.
(c) In a tree with exactly $\Delta$ leaves where $\Delta \geq 3$, show that there is precisely one vertex of degree $\Delta$.

(9) A $(p,q)$ graph $G$ is called graceful if it is possible to label the vertices of $G$ with distinct elements from the set $\{0, 1, \ldots, q\}$ in such a way that the induced edge labelling, which prescribes the integer $|i - j|$ to the edge joining the vertices labelled $i$ and $j$, assigns the labels $1, 2, \ldots, q$ to the $q$ edges of $G$.
[It has been conjectured that every tree is graceful.]
(a) Prove that every star is graceful. (Recall that a star of order $p$ is a tree of order $p$ with one vertex of degree $p - 1$.)
(b) Prove that every path is graceful.
(c) Show that every tree of order 6 is graceful.

(10) A connected graph has degree sequence
\[
8, 8, 7, 7, 6, 6, 6, 5, 4, 4, 3.
\]
How many edges must be removed from $G$ to produce a spanning tree of $G$? (Explain your answer carefully.)
(11) Recall that one proof of the number of distinct labelled spanning trees in $K_p$, due to Prüfer, sets up a 1–1 correspondence between labelled spanning trees of $K_p$ and sequences of length $p - 2$ formed from $\{1, 2, \ldots, p\}$. FIRST, given the following labelled spanning tree, find the corresponding sequence of length 6.

![Graph](image)

SECONDLY, given the sequence $(2, 3, 3, 4, 5, 2)$, draw the corresponding labelled spanning tree on eight vertices.

(12) Solve the Chinese Postman Problem for the given graph $G$:

![Graph](image)

(13) (a) Construct the de Bruijn digraph $D_{4,2}$ with an alphabet of size 4 and edge sequences of length 2. Use this digraph to find an appropriate de Bruijn sequence.

(b) Solve the rotating drum problem for a drum with 16 sectors.