Theorem 2.1. (Kuratowski's Theorem) A graph $G$ is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

Proof: Let $G$ be a graph. If $G$ does not contain a subdivision of $K_5$ or $K_{3,3}$, then by Kuratowski's Theorem, $G$ is planar.

Conversely, let $G$ be a planar graph. We need to show that $G$ does not contain a subdivision of $K_5$ or $K_{3,3}$.

Consider a cycle $C$ in $G$. If $C$ contains a subdivision of $K_5$ or $K_{3,3}$, then $G$ contains a subdivision of $K_5$ or $K_{3,3}$.

Therefore, $G$ is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

\[\text{Q.E.D.}\]
\[ Q = D - A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \]

Delete one row and column (last chosen row and col). Then solve for determinants (can replace by col + row). Then expand by row 3.

\[ G = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

(Assume row 3 is the row to expand by.)

\[ G_{e} = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ G - e = H \]

2(b) G
(6) If $G$ is hamiltonian, then certainly $G + uv$ is hamiltonian. (For all we've done is add one edge — no new vertices!) Conversely, say $G + uv$ is hamiltonian, with $\deg u + \deg v > p$. If a ham. cycle in $G + uv$ does not include the edge $uv$, then it is also a ham. cycle in $G$. So say a ham. cycle in $G + uv$ does include edge $uv$. Delete edge $uv$, and we have a ham. $u - v$ path in $G$: $u = u_1, u_2, u_3, \ldots, u_{p-1}, u_p = v$. (These are the $p$ vertices in $G$.)

We must have some $u_i$ with $u \sim u_i$, $v \sim u_{i-1}$, or else, if no removing the $(\deg u)$ vertices in $\{u_2, \ldots, u_p\}$ adjacent to $u$ leaves $(p-1) - (\deg u)$ vertices, and we'd have to have $(p-1) - (\deg u) \geq \deg v$,

ie. $\deg u + \deg v \leq p-1 < p$, a contradiction!

So $\exists$ $u_i$ with $u \sim u_i$ and $v \sim u_{i-1}$:

Now $u, u_2, \ldots, u_i, u_{i+1}, \ldots, u_{p-1}, u_p = v$ is a ham. cycle in $G$. This completes the proof.