1. (1 mark) Rolf Harris has his Jake peg-leg on, so a “set” of socks for him consists of THREE socks the same colour. He has a drawer full of 54 socks, with exactly six socks in each of 9 different colours. How many does a blind friend have to remove from the drawer to be certain that Jake has a monochromatic set of three socks?

2. (2 marks)
   (a) A centipede, Ms Millie, has 110 legs in total, and she wants a set of matching (monochromatic) socks. In her drawer she has 220 socks in red, 120 in blue and 130 in green. How many socks must our blind friend remove to be guaranteed of a matching set for Millie?

   (b) Millie’s friend Freddie also has 110 legs. He has in his drawer 300 red socks, 120 blue socks and 100 green socks. How many must our blind friend remove from Freddie’s drawer to ensure that Freddie has a monochromatic set of socks to wear?

3. (3 marks)
   Let $X$ be a set of five distinct positive integers, none of which exceeds 9.
   (i) Show that the sums of the non-empty subsets of $X$ cannot all be distinct.
   (ii) If $Y$ is a set of four distinct positive integers, none exceeding 8, is the same true about the sums of non-empty subsets of $Y$? (Explain carefully!)

4. (3 marks) Generalise Example (e) in the Notes from $n^2$ to $mn$:
   If $a_1, a_2, \ldots, a_\ell$ is a sequence of more than $mn$ distinct real numbers, show carefully that either there is a decreasing subsequence of more than $m$ terms, or else there is an increasing subsequence of more than $n$ terms.

5. (3 marks)
   (a) Describe the unique triangle-free graph on 9 vertices with $T(9, 3) − 1$ edges.
   (b) We know that a $K_4$-free graph on 7 vertices has no more than $T(7, 4) − 1$ edges. Calculate $T(7, 4) − 1$, and describe the (unique) graph on 7 vertices with this number of edges which contains no $K_4$.
   Then draw any other (non-isomorphic) graph on 7 vertices with this same number of edges, and mark a $K_4$ in your graph.

6. (3 marks) Calculate the Turán number $T(p, 4)$ for each of the cases $p \equiv 0 \pmod{3}$, $p \equiv 1 \pmod{3}$ and $p \equiv 2 \pmod{3}$. (Express your answer as a polynomial in $p$.)
7. (1 mark) Prove that for every positive integer \( n \), \( r(3, n) \leq \frac{n^2 + n}{2} \).

8. (3 marks) Let \( m \geq 2 \) and \( n \geq 2 \) be integers, and let \( p = r(m, n) - 1 \). Suppose that each edge of \( K_p \) is arbitrarily coloured red or blue. Show carefully that:
   (i) \( K_p \) contains a red \( K_{m-1} \) or a blue \( K_n \).
   (ii) \( K_p \) contains a red \( K_m \) or a blue \( K_{n-1} \).

9. (4 marks) Find the generalised Ramsey numbers:
   (i) \( r(K_{1,3}, K_{1,3}) \); (ii) \( r(K_2, K_3, K_3) \); (iii) \( r(K_{1,3}, P_3) \); (iv) \( r(K_{1,3}, K_3) \).
   Explain your answers carefully! (Here \( P_3 \) denotes a path on 3 vertices with 2 edges.)

10. (3 marks) There are eight people in a waiting room. They discover that of these eight, two of them know 2 people in the room, one knows three, two know 6 people, and the last three people know 5 people in the room. Show that there must be three people in the waiting room all of whom know each other.

11. (4 marks) If a graph \( G \) on \( n \) vertices has more than \( \frac{1}{2}n\sqrt{n-1} \) edges, prove that the length of the smallest cycle in \( G \) is at most 4 (i.e. \( G \) has girth at most 4). So show that either \( G \) is not simple (there’s a multiple edge, and so a “cycle” of length 2) or else \( G \) contains a triangle or \( G \) contains a cycle of length 4.
   Hint: Assume not so. Consider the vertices adjacent to some vertex \( x \), say \( y_1, \ldots, y_d \). None can be adjacent (why?). Also show there cannot be a vertex \( z \neq x \) adjacent to more than one of the \( d \) vertices adjacent to \( x \). Then find an inequality relating the degrees of the \( y_i \) and the total number of vertices, and finally let \( x \) vary over all \( n \) vertices.