ORTHOGONAL BISECTIONAL CURVATURE AND THE GENERALISED FRANKEL CONJECTURE

HUY THE NGUYEN

ABSTRACT. In the paper [SY80] it was shown that a Kähler manifold with strictly positive bisectional curvature was biholomorphic to \( \mathbb{CP}^m \). In this paper, we use the techniques developed by [SY80], to prove that a compact Kähler manifold with positive orthogonal bisectional curvature is biholomorphic to \( \mathbb{CP}^m \), a condition strictly weaker than positive bisectional curvature. This gives a direct elliptic proof of this theorem, which was proved by [Che07] by applying the Kähler Ricci flow and the Siu-Yau theorem.

1. Introduction

In the paper, Siu-Yau [SY80], following theorem was proved, which was known as the Frankel conjecture.

**Theorem 1.1** ([SY80]). Let \((M^m, g_{\alpha \beta})\) be a compact \(m\)-dimensional Kähler manifold with positive bisectional curvature. Then \((M, g_{\alpha \beta})\) is biholomorphic to \(\mathbb{CP}^m\).

This was also proved by S. Mori [Mor79] using techniques from algebraic geometry. In fact, Mori proved the Hartshorne conjecture, that is

**Theorem 1.2** ([Mor79]). Let \(M^m\) be an irreducible non-singular projective variety with ample tangent bundle defined over an algebraic closed field \(K\) of characteristic \(\chi(K) \geq 0\), then \(M^m\) is isomorphic to projective space \(\mathbb{PK}^m\).

In Mori’s proof, rational curves were constructed using the deformation theory of curves together with the algebraic geometry of positive characteristic.

Here we will use the techniques of [SY80] to provide a proof of the following theorem.

**Theorem 1.3.** Let \((M^n, g_{\alpha \beta})\) be a compact \(m\)-dimensional Kähler manifold with positive orthogonal bisectional curvature. Then \((M, g_{\alpha \beta})\) is biholomorphic to \(\mathbb{CP}^m\).

Note that while positive bisectional curvature implies that the tangent bundle is positive in the sense of Griffiths, positive orthogonal bisectional curvature does not. Hence, we can not recover this result by directly appealing to Mori’s proof of the Hartshorne conjecture. We also note that by showing that positive orthogonal bisectional curvature is preserved by the Kähler Ricci flow, then this theorem was proved by [Che07], see also [GZ10] and [Wil10]. However, this proof shows that along the Kähler-Ricci flow, the bisectional curvature then becomes positive and then appeals to the Theorem of [SY80]. The proof of this theorem in this paper shows directly that the Kähler manifold is biholomorphic to \(\mathbb{CP}^m\).
By a result of Kobayashi-Ochiai, the complex projective space is characterised as a Kähler manifold $M$ such that its first Chern class satisfies $c_1(M) = \lambda c_1(F)$ where $\lambda \geq m + 1$ and $F$ is a positive holomorphic line bundle over $M$. Unlike the case of positive bisectional curvature, positive orthogonal bisectional curvature does not imply that the curvature has positive Ricci curvature. Hence, it is not immediate that the first Chern class is positive or that the Ricci curvature is strictly bounded below by a positive constant. However, by applying the Bochner formula as in Bishop-Kobayashi [GK67], we will show that the second better number $b^{1,1} = 1$. This also show sthat the first Chern class is positive, which together with Yau’s theorem, shows that the surface admits a positive Ricci curvature metric. In particular, we may replace $M$ with its universal cover and assume that $M$ is simply connected. Hence to prove the Frankel conjecture, it will suffice to show that $c_1(M)$ is $\lambda$ times the generator of $H_2(M, \mathbb{Z})$ for some $\lambda \geq 1 + \dim M$. By a result of Grothendieck, the tangent bundle of $M$ over a rational curve splits into a direct sum of holomorphic line bundles over the rational curve. Hence if we can show that the free part of $H_2(M, \mathbb{Z})$ can be represented by a rational curve, then this will prove the theorem.

To construct the desired rational curve, we use a theorem of [SY80] which uses results of [SU81] and [MY80] to prove that the infimum of the energy of maps from $S^2$ to $M$ representing the generator of $\pi_2(M)$ can be represented a sum of stable harmonic maps $f_i : S^2 \to M$, $1 \leq i \leq n$. The key step is to show that these maps are holomorphic or anti-holomorphic. Here we use the second variation formula of the energy. The 2-parameter deformation of the second variation formula is used to imitate the holomorphic deformation of a curve. Here we carefully analyse the curvature term in the deformation, and the together with the Riemann-Roch theorem, we find that it suffices to have positive orthogonal bisectional curvature. Then after this key step, we show that there is only one stable harmonic map. In order to do this, we show that if $n > 1$ then applying a holomorphic deformation of some holomorphic $f_i$ and some anti-holomorphic $f_j$ we have that they are tangential at a point. By removing a small disc centred at the contact point, and joining the boundary of the discs by a suitable surface, we obtain a map from $S^2$ to $M$ with strictly smaller energy, which is a contradiction. Hence we see that $n = 1$ and the image of $f_1$ is a rational curve representing the generator of the free part of $H_2(M, \mathbb{Z}) = \pi_2(M)$ as we assumed that $M$ is simply connected.

We now give an outline of the paper. On section 2, we gather the necessary definitions and previous results that we will require in the following parts of the paper. In section 3, we show that the hypotheses show that the first Chern class is positive. This is not immediate, as unlike the case of positive bisectional curvature, positive orthogonal bisectional curvature does not imply positive Ricci curvature. Then in section 4, we show that energy minimising maps from the Riemann sphere are either holomorphic or anti-holomorphic. Here show that we can replace the hypothesis of positive bisectional curvature with positive orthogonal bisectional curvature. In section 5 we show that in Proposition 5.1 (cf Proposition 3 [SY80]) we show that we can replace the condition that the tangent bundle is positive and hence ample (which is implied by positive bisectional curvature) with the condition orthogonal bisectional curvature is positive, which shows that the normal bundle to a rational curve is ample. Finally in section 6, we give a proof of the main theorem.
2. Preliminaries

We now review some basic facts of Kähler Geometry


Definition 2.1 (Kähler Manifold). A Kähler manifold is an almost complex manifold, $(M,J)$ with a $J$-invariant metric (or Hermitian)

$$g(JX,JY) = g(X,Y)$$

and $J$ is parallel,

$$\nabla J = 0, \quad \nabla X(JY) = J(\nabla X Y), \quad g_{\alpha \overline{\beta}} = g_{\beta \alpha}, \quad g_{\alpha \beta} = g_{\overline{\beta} \overline{\alpha}} = 0.$$

for all $X,Y \in \mathfrak{X}(M)$, where $\nabla$ is the Riemannian covariant derivative.

Proposition 2.2 (Kähler Identities).

$$\frac{\partial}{\partial z^\gamma} g_{\alpha \overline{\beta}} = \frac{\partial}{\partial z^\overline{\beta}} g_{\alpha \gamma}, \quad \Gamma^\gamma_{\alpha \beta} = \Gamma^\gamma_{\beta \alpha}, \quad \Gamma^\gamma_{\overline{\alpha} \overline{\beta}} = \overline{\Gamma^\gamma_{\alpha \beta}},$$

The only nonvanishing components of the curvature tensor are

$$R_{\alpha \overline{\beta} \gamma \delta}, R_{\alpha \gamma \overline{\beta} \delta}, R_{\overline{\gamma} \beta \overline{\alpha} \delta}, R_{\overline{\gamma} \overline{\alpha} \beta \delta}.$$

In particular, the Bianchi identity gives us the following identity,

$$R_{\alpha \overline{\beta} \gamma \delta} = R_{\alpha \gamma \overline{\beta} \delta}.$$

The curvature tensor is $J$-invariant that is

$$\text{Rm}(X,Y)JZ = J(\text{Rm}(X,Y)Z).$$

Definition 2.3 (Bisectional Curvature and Orthogonal Bisectional Curvature). We define the bisectional curvature as follows,

$$K_{\alpha \overline{\beta}} = R_{\alpha \gamma \overline{\beta} \overline{\gamma}}.$$

where $e_\alpha, e_\beta \in SM \otimes \mathbb{C}$. That is they are unit vectors but are not necessarily orthogonal. We define the orthogonal bisectional curvature as follows, let $e_\alpha, e_\beta \in SM \otimes \mathbb{C}$ such that $\langle e_\alpha, e_\beta \rangle = 0$ then

$$F_{\alpha \overline{\beta}} = R_{\alpha \gamma \overline{\beta} \overline{\gamma}}.$$

A simple consequence of non-negative bisectional curvature is non-negative holomorphic sectional curvature and analogously, non-negative orthogonal bisectional curvature implies two non-negative holomorphic sectional curvature, that is for any unitary two-frame $e_1, e_2$, we have

$$R_{1111} + R_{2222} \geq 0.$$
2.2. Energy, $\overline{\partial}$-Energy and $\partial$-Energy. Suppose that $M, N$ are compact Kähler manifolds whose Kähler metrics are respectively,

$$ds^2_M = 2\Re \rho_{\alpha \beta}dz^\alpha d\bar{z}^\beta, \quad ds^2_N = 2\Re \sigma_{ij}dw^i d\bar{w}^j.$$

**Definition 2.4 ($\overline{\partial}$-Energy, $\partial$-Energy).** We define the $\overline{\partial}$ energy density as

$$|\overline{\partial}f|^2 = \sigma_{ij} f^\alpha_i f^\beta_j h_{\alpha \beta}$$

and the $\partial$-energy density as

$$|\partial f|^2 = \sigma_{ij} f^\alpha_i f^\beta_j h_{\alpha \beta}$$

where $f^\alpha_i = \frac{\partial f^\alpha}{\partial w^i}$ and $f^\beta_j = \frac{\partial f^\beta}{\partial w^j}$. Therefore the energy density $e(f)$ of $f$ which is defined as the trace of $f^*ds^2_M$ with respect to $ds^2_N$ is equal to $|\overline{\partial}f|^2 + |\partial f|^2$. Now we assume that $N$ is a complex curve, that is $\dim_{\mathbb{C}} N = 1$. Then the pull back of the Kähler form under $f$ is

$$\sqrt{-1} h_{\alpha \beta} df^\alpha \wedge df^\beta = \sqrt{-1} h_{\alpha \beta} \left( \frac{\partial f^\alpha}{\partial w} \frac{\partial f^\beta}{\partial \bar{w}} - \frac{\partial f^\alpha}{\partial \bar{w}} \frac{\partial f^\beta}{\partial w} \right) dw \wedge d\bar{w}.$$

Hence we have that

$$\int_N |\partial f|^2 - \int_N |\overline{\partial}f|^2 = \int_N \sqrt{-1} h_{\alpha \beta} df^\alpha \wedge df^\beta$$

which is equal to the Kähler class $\omega(M)$ of $M$ evaluated at the homology class $[f(N)]$ defined by $f : N \to M$. Hence it then follows that

$$\int_N |\overline{\partial}f|^2 = \frac{1}{2} \int_N e(f) - \frac{1}{2} \omega(M)[f(N)],$$

$$\int_N |\partial f|^2 = \frac{1}{2} \int_N e(f) + \frac{1}{2} \omega(M)[f(N)].$$

As a consequence, the energy minimizing maps from $N$ to $M$ are precisely the same as the $\overline{\partial}$ energy minimizing maps. This was first observed by Lichnerowicz.

2.3. The Second Variation Formula. Suppose that $\dim_{\mathbb{C}} N \geq 1$. Let $f(t) : N \to M, t \in \mathbb{C}, |t| < \epsilon$ be a family of smooth maps parameterised by an open disc in $\mathbb{C}$. We choose local holomorphic co-ordinates at $P$ and $Q = f(P)$ such that

$$dg_{ij} = 0 \quad \text{at} \quad P$$

and

$$dh_{\alpha \beta} = 0 \quad \partial_\alpha \partial_\beta h_{\alpha \beta} \quad \text{at} \quad Q.$$

Direct computation gives us

$$\frac{\partial^2}{\partial t \partial \bar{t}} |\overline{\partial}f|^2 = 2\Re g^{\alpha \beta} \left( \frac{\partial^2}{\partial t \partial \bar{t}} f^\alpha_i \right) f^\beta_j h_{\alpha \beta} + g^{\alpha \beta} \frac{\partial}{\partial t} f^\alpha_i h_{\alpha \beta}$$

$$+ g^{\beta \gamma} \frac{\partial}{\partial \bar{t}} f^\beta_j h_{\alpha \beta} + g^{\alpha \beta} f^\gamma_i f^\delta_j \partial_\gamma \partial_\delta f^\alpha_i f^\beta_j$$

$$+ g^{\beta \gamma} f^\gamma_i f^\delta_j \partial_\gamma \partial_\delta f^\beta_j.$$
Let us consider the vector field $\xi$ defined on $N$ by

$$\xi^i = g^{ij} \left( \frac{\partial}{\partial t} \frac{\partial}{\partial t} f^\alpha \right) T_j^\alpha h_{\alpha \beta} = g^{ij} \left( \frac{\partial^2}{\partial t \partial t} f^\alpha + \Gamma_{\beta \gamma}^{\alpha, M} \frac{\partial f^\beta}{\partial t} \frac{\partial f^\gamma}{\partial t} T_j^\alpha h_{\alpha \beta} \right)$$

where $\frac{\partial}{\partial t}$ is the covariant derivative with respect to the canonical connection of the tangent bundle $TM$ and $\Gamma_{\beta \gamma}^{\alpha, M}$ is the associated Christoffel symbols. We take the divergence of this vector field at $P$ to get

$$\nabla_{\xi^i} = g^{ij} \left( \frac{\partial^2}{\partial t \partial t} f^\alpha \right) T_j^\alpha h_{\alpha \beta} + g^{ij} \left( \frac{\partial^2}{\partial t \partial t} \right) \frac{\partial f^\alpha}{\partial t} h_{\alpha \beta}$$

$$+ g^{ij} (\partial_\mu \partial_\nu h_{\alpha \beta}) T_j^\alpha \frac{\partial f^\mu}{\partial t} \frac{\partial f^\alpha}{\partial t}.$$

Now let us assume that $f$ is harmonic at $t = 0$. Then at $P$ and at $t = 0$ we have the following

$$\frac{\partial^2}{\partial t \partial t} \int_N |\nabla f|^2 = \int_N g^{ij} \left( \frac{\partial D f^\alpha}{\partial t} \right) \left( \frac{\partial D f^\alpha}{\partial t} \right) h_{\alpha \beta} + \int_N g^{ij} \left( \frac{\partial D f^\alpha}{\partial t} \right) \left( \frac{\partial D f^\alpha}{\partial t} \right) h_{\alpha \beta}$$

$$+ \int_N g^{ij} f^\alpha f^\alpha R_{\mu \nu \alpha \beta} \frac{\partial f^\mu}{\partial t} \frac{\partial f^\nu}{\partial t} + \int_N g^{ij} f^\alpha f^\alpha R_{\mu \nu \alpha \beta} \frac{\partial f^\mu}{\partial t} \frac{\partial f^\nu}{\partial t}$$

$$= 2 R \int_N g^{ij} R_{\mu \nu \alpha \beta} f^\alpha \frac{\partial f^\mu}{\partial t} \frac{\partial f^\nu}{\partial t},$$

where $R_{\mu \nu \alpha \beta}$ is the curvature tensor on $M$.

2.4. Existence of Energy Minimising Maps. The following is an existence theorem for energy-minimising maps from $S^2$ to a compact Riemannian manifold. In particular, the manifold need not be Kähler or have positive orthogonal bisectional curvature. The following theorems are proved in [SY80].

**Theorem 2.5.** There exists $\epsilon > 0$ and $\alpha_0 > 1$ such that if $E(s) < \epsilon$, $1 \leq \alpha < \alpha_0$ and $s$ a critical map of $E_\alpha$, then $z \in N_0$ and $E(s) = 0$.

The following theorem shows that for a $C^1$ mapping from the Riemann sphere into a compact manifold there exists energy minimising maps whose sum are homotopic to $f$.

**Proposition 2.6 ([SY80, Proposition 2]).** For every $C^1$ map $f : S^2 \to M$, there exists energy minimizing maps $f_i : S^2 \to M, 1 \leq i \leq n$ such that the sum of $f_i$ is homotopic to $f$ and $E([f]) = \sum_{i=1}^n E(f_i)$.

2.5. Algebraic Geometry. In the following we will require the Riemann-Roch theorem, which allows us to compute the dimension of the space of holomorphic sections.

**Theorem 2.7** (Riemann-Roch Theorem, [Gun67]). If $M$ is a compact Riemann surface of genus $g$ and $\xi \in H^1(M, \mathcal{O}^*)$ is a complex line bundle on $M$ then

$$\dim H^0(M, \mathcal{O}(\xi)) - \dim H^1(M, \mathcal{O}(\xi)) - c(\xi) = 1 - g;$$

or equivalently

$$\dim \Gamma(M, \mathcal{O}(\xi)) - \dim \Gamma(M, \mathcal{O}(\kappa \xi^{-1})) = c(\xi) + 1 - g$$

where $\kappa$ is the canonical bundle.
In particular, we have that if $g = 0$ and $c(\xi) \geq 1$, then $\gamma(\xi) \geq 2$ and if $g = 0$ and $c(\xi) = 0$, then $\gamma(\xi) \geq 1$. We will also require a theorem due to Grothendieck.

**Theorem 2.8** (Theorem 2., [Gro57]). Let $X$ be the Riemann sphere. Then every holomorphic vector bundle $E$ on $X$ is the direct sum of complex line bundles $L_i$, that is

$$E = L_1 \oplus \cdots \oplus L_i.$$  

The line bundles $L_i$ are determined up to a permutation of the indices.

### 2.6. Kähler Ricci Flow.

In this section we gather the necessary facts on Kähler-Ricci flow. The Kähler-Ricci flow is the following quasi-linear parabolic system,

$$\partial_t g_{\alpha\beta} = -R_{\alpha\beta}, \quad g_{\alpha\beta} |_{t=0} = g^0_{\alpha\beta}.$$  

If we choose the initial Kähler metric to have Chern class $c_1(X)$, then the flow preserves this class. Kähler-Ricci flow (and Kähler geometry) has the special property that the quasi-linear parabolic system above, may be reduced to a single fully nonlinear parabolic PDE for the Kähler potential.

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega^\varphi}{\omega^n} + \varphi - h_\omega,$$

where $h_\omega$ is defined by

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega, \quad \text{and} \quad \int_X (e^{h_\omega} - 1) \omega^n = 0.$$  

The evolution of the bisectional curvature is given by the following simple formula,

$$\partial_t K_{\alpha\beta\gamma\delta} = \sum_{\nu,\mu} |R_{\alpha\nu\beta\gamma}|^2 - |R_{\alpha\nu\beta\gamma}|^2 + R_{\alpha\nu\beta\gamma} R_{\mu\nu\beta\gamma}$$

$$- \sum_{\mu=1}^n \Re\{R_{\alpha\nu\gamma\delta} R_{\mu\nu\gamma\delta} + R_{\gamma\delta} R_{\alpha\nu\gamma\delta}\}.$$  

This a follows from a simple manipulation of the evolution equation for the Ricci flow. For a derivation, see [CCG+07]. Note that for the evolution of the orthogonal bisectional curvature, the last term can be made to vanish by Uhlenbeck’s trick.

### 3. Positivity of the First Chern Class

In this section, we prove that if $M$ is a compact Kähler manifold with positive orthogonal bisectional curvature, then we can show that the first Chern class is in fact positive. This will then be used to show that the Kähler manifold then admits a metric with positive Ricci curvature. first we state a lemma that we will require.

**Lemma 3.1** ([GK67, Lemma 1]). Let $\xi$ be a real form bi-degree $(1,1)$ on a Kähler manifold. There there exists a local field of orthonormal co-ordinates $X_1, \ldots, X_m$, $JX_1, \ldots, JX_m$ such that

$$\xi(X_i, JX_j) = 0, \quad \text{for} \ i \neq j.$$  

Applying this lemma, together with a Bochner type argument, we can then show that the first Chern class is positive.
Theorem 3.2 (cf [GK67]). Let \((M^n, g_{\alpha\beta})\) be a compact Kähler manifold with positive orthogonal bisectional curvature. Then all real harmonic \((1,1)\) forms are parallel. This implies that
\[ b_{1,1}(M) = \dim H^{1,1}(M) = 1 \]
which shows us that \(c_1(M) > 0\).

Proof. Now for any two form \(\xi\) on a compact Riemannian manifold, we can define the Bochner curvature as
\[ F(\xi) = 2R_{ij} \xi^i \xi^j - R_{ijkl} \xi^i \xi^j \xi^k \xi^l. \]
by the Bochner formula, we know that if \(\xi\) is harmonic and if \(F(\xi) \geq 0\) then \(F(\xi) = 0\) and \(\xi\) is parallel. By Lemma 3.1, set \(\xi_{\alpha\beta} = \xi(X_\alpha, JX_\beta)\). By a simple calculation, we obtain that
\[ F(\xi) = 2 \sum_{i,j} R_{i\alpha j\beta}(\xi_{\alpha\beta} - \xi_{\beta\alpha})^2, \]
where
\[ R_{i\alpha j\beta} = R(X_i, JX_\alpha, X_j, JX_\beta). \]
Hence if the orthogonal bisectional curvature is positive, we have that \(F(\xi) \geq 0\). Assume that \(\xi\) is harmonic, then \(F(\xi) = 0\) implies that \(\xi_{\alpha\beta} = \xi_{\beta\alpha}\) at each point \(i, j = 1, \ldots, n\). Hence we have that \(\xi = f\omega\) where \(\omega\) is the Kähler form. Since \(\xi\) is parallel, \(f\) is constant hence \(\dim C H^{1,1} = 1\). If \(b^{1,1}(M) = \dim H^{1,1}(M)\), let \(\rho\) and \(\omega\) denote the Ricci form and the Kähler form, then by Hodge theory, we have that
\[ \rho = \lambda \omega + \eta, \]
where \(\lambda\) is a real number and \(\int_M \langle \omega, \eta \rangle = 0\). This gives us that
\[ \int_M \langle \rho, \omega \rangle = \frac{1}{4} \int_M R = \lambda \|\omega\|^2 > 0. \]
as the scalar curvature is a sum of orthogonal bisectional curvatures, \(R_{\alpha\beta\gamma\delta} > 0 \implies R > 0\). This shows that \(c_1 > 0\).

By Yau’s theorem, we can then show that there exists a Kähler metric on \(M\) that represents the positive Ricci curvature metric.

Theorem 3.3 (Yau’s Theorem, [Yau78]). Let \(M\) be a compact Kähler manifold with Kähler metric,
\[ g_{\alpha\beta} dz^\alpha \otimes d\bar{z}^\beta. \]
Let \(\tilde{R}dz^\alpha \otimes d\bar{z}^\beta\) be a tensor whose associated \((1,1)\) form,
\[ \frac{\sqrt{-1}}{2\pi} \tilde{R}_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta \]
represents the first Chern class of \(M\). Then we can find a Kähler metric \(\tilde{g}_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta\) whose Ricci tensor is given by \(\tilde{R}_{\alpha\beta}\). Furthermore we can require that this Kähler metric has the same Kähler class as the original one. In this case the required Kähler metric is unique.
Combining this theorem with the positivity of the first Chern class gives the following.

**Corollary 3.4.** Let \((M^m, g_{ij})\) be a compact Kähler manifold with positive orthogonal bisectional curvature. Then \(M^m\) admits a Riemannian metric \(\tilde{g}_{ij}\) with strictly positive Ricci curvature, that is there exists a \(C > 0\) such that \(\tilde{R}_{ij} \geq Cg_{ij}\).

By Yau’s theorem, the metric above can be chosen to be Kähler. However, this fact will not be used in the following.

### 4. Complex-Analyticity of Energy-Minimising Maps

Let \(M\) be a compact Kähler manifold with positive orthogonal holomorphic bisectional curvature. We will now show that if \(f_0 : \mathbb{P}_1 \to M\) is an energy minimizing map then \(f_0\) is either holomorphic or anti-holomorphic. This generalises a proposition of Siu-Yau. In the proof, we use the second variation formula for energy minimizing maps, where a 2 parameter family of variations is used as a substitute for holomorphic deformation. This proof follows Siu-Yau, [SY80, Proposition 1]. The key difference here is that we must be careful in the construction the non-trivial holomorphic section of \(L_i\), to use the orthogonal bisectional curvature we must have that \(\frac{\partial f}{\partial w}\) is orthogonal to \(\frac{\partial f}{\partial z}\).

**Proposition 4.1.** Let \(f_0 : \mathbb{P}_1 \to M\) be an energy minimizing map. If \(f^*c_1(M)\) evaluated at \(\mathbb{P}_1\) is non-negative (respectively non-positive) then \(f_0\) is holomorphic (respectively conjugate holomorphic).

**Proof.** Let \(T_M\) be the holomorphic tangent bundle of \(M\) and let \(w\) be a local co-ordinate of \(\mathbb{P}_1\) and let \(\frac{\partial}{\partial w}\) be covariant differentiation in the antiholomorphic direction, of local cross sections of \(f_0^*T_M\) with respect to the connection of \(T_M\). Let \(\mathcal{F}\) be the sheaf of germs of local cross sections of \(f_0^*T_M\) with \(\frac{\partial}{\partial w}s = 0\). Clearly \(\mathcal{F}\) is an analytic sheaf over \(\mathbb{P}_1\). Using the Cauchy kernel and the standard classical iteration process of Korn-Lichtenstein, we can show that for an arbitrary point \(P\) of \(\mathbb{P}_1\) there exist local cross sections \(s_1, \ldots, s_m\) at \(P\) such that \(\frac{\partial}{\partial w}s_i \equiv 0, 1 \leq i \leq m\), and \(s_1(P), \ldots, s_m(P)\) form a basis for the fibre \(f_0^*T_M\) at \(P\) where \(m = \dim_{\mathbb{C}} M\). Therefore \(\mathcal{F}\) is a locally free and the holomorphic vector bundle associated to \(\mathcal{F}\) is topologically isomorphic to \(f_0^*T_M\).

The pullback bundle \(f_0^*T_M\) is a holomorphic vector bundle, hence by Theorem 2.8, we can write it as a sum of line bundles,

\[
  f_0^*T_M = L_1 \oplus \cdots \oplus L_m, \quad m = \dim_{\mathbb{C}} M.
\]

Note that, as we assumed that \(f^*c_1(T_M)\) then as the first Chern class is additive,

\[
  f^*(c_1(T_M)) = c_1(L_1) + \cdots + c_1(L_m)
\]

we have either that there exists \(c_1(L_i) \geq 1\) or \(c_1(L_i) = 0\) for each \(i = 1, \ldots m\). This implies by the theorem of Riemann-Roch, Theorem 2.7 that there exists at least two nontrivial global holomorphic sections

\[
  s = \sum \frac{\partial}{\partial z^\alpha}s^\alpha, \quad \text{of } L_i \text{ and hence of } f_0^*T_M \text{ over } \mathbb{P}_1.
\]

Now as \(f_0 : \mathbb{P}_1 \to M\) is energy minimizing, \(\frac{\partial f}{\partial w}\) is holomorphic with respect to \(\frac{\partial}{\partial w}\). In particular, as there are at least two holomorphic sections, we can choose \(s\) to be orthogonal to \(\frac{\partial f}{\partial w}\) except on a finite set where either \(s\) or \(\frac{\partial f}{\partial w}\) is zero.
Let \( f(t) : \mathbb{P}_1 \to M, t \in \mathbb{C}, |t| < \epsilon \) be a variation of \( f \) so that \( f(0) = f_0 \) and so that \( t = 0 \) we have that
\[
\frac{\partial}{\partial t} f^\alpha(t) \bigg|_{t=0} = 0, \quad \text{and} \quad \frac{\partial}{\partial t} f^\alpha(t) = s^\alpha.
\]

As \( s \) is a holomorphic section, we get that
\[
\frac{D}{\partial t} \frac{\partial}{\partial \bar{w}} f^\alpha = \frac{D}{\partial \bar{w}} \frac{\partial}{\partial t} f^\alpha = \frac{D}{\partial \bar{w}} s^\alpha = 0.
\]
And at \( t = 0 \) we have directly from the definition,
\[
\frac{D}{\partial t} \frac{\partial}{\partial \bar{w}} f^\alpha = \frac{D}{\partial \bar{w}} \frac{\partial}{\partial t} s^\alpha = 0.
\]
The second variation formula then gives us that
\[
\frac{\partial^2}{\partial t \partial \bar{t}} \int_{\mathbb{P}_1} |\partial f|^2 = \int_{\mathbb{P}_1} \frac{\partial f^\alpha}{\partial \bar{w}} \frac{\partial f^{\alpha'}}{\partial \bar{w}} R_{\nu \mu \rho \sigma} \frac{\partial f^\mu}{\partial t} \frac{\partial f^\nu}{\partial t} \sqrt{-1} dw \wedge d\bar{w}.
\]
But recall that \( f_0 \) is energy minimising and hence by Lichnerowicz is also \( \partial \)-energy minimizing, hence we have that
\[
\frac{\partial^2}{\partial t \partial \bar{t}} \int_{\mathbb{P}_1} |\partial f|^2 \geq 0.
\]
Now as \( \frac{\partial}{\partial t} f^\alpha(t) = s^\alpha \) is a holomorphic cross section that is orthogonal to \( \frac{\partial f}{\partial z} \), then as the orthogonal bisectional curvature is positive, that is
\[
R_{\mu \nu \alpha \beta} \xi^\nu \eta^\alpha \eta^\beta < 0.
\]
for \( \xi^\nu, \eta^\alpha \in \mathbb{C}^m \setminus \{0\} \). It follows that we have that
\[
\frac{\partial f}{\partial \bar{w}} = 0
\]
on \( \mathbb{P}_1 \setminus Z \) where \( Z = \{ z \mid s(z) = 0 \} \). As \( Z \) is a finite set this shows that \( \frac{\partial f}{\partial \bar{w}} = 0 \) on \( \mathbb{P}_1 \) and hence is holomorphic. \( \square \)

5. Holomorphic Deformation of Rational Curves

The following proposition, in its is true for any complex manifold. However, in our case we will require that the manifold is Kähler. Note that since \( c_1(M) > 0 \), there exists a Kähler metric with positive Ricci curvature on the compact manifold \( M \). By compactness, there exists \( c > 0 \) such that this metric has \( R_{\alpha \beta} \geq cg_{\alpha \beta} \). In particular, by the theorem of Bonnet-Myers this shows that the universal cover of \( M \) is compact. Furthermore,

**Proposition 5.1.** Let \( M \) be a compact Kähler manifold with positive orthogonal bisectional curvature. Let \( C_0 \) be a rational curve in \( M \) and \( F : \mathbb{P}_1 \to C_0 \) be its normalization. Then there exists a proper subvariety \( Z \) of \( \mathbb{P}(T_M) \) with the following property. If \( y \in M \) and \( \xi \in TM_y \setminus \{0\} \) define an element of \( \mathbb{P}(T_M) \setminus Z \), then there exists a holomorphic map \( f' : \mathbb{P}_1 \to M \) homotopic to \( f \) (when \( f \) is regarded as a map from \( \mathbb{P}_1 \) to \( M \)) such that \( y \) is a regular point of \( f'(\mathbb{P}_1) \) and the tangent vector of \( f'(\mathbb{P}_1) \) at \( y \) is a non zero multiple of \( \xi \).
Proof. Let $C_0$ be a rational curve in $M$ possibly with singularities that is $C_0$ is the image of a holomorphic map $f_0 : \mathbb{P}_1 \to M$ which is the normalisation of $C_0$. Denote by $V$ the graph of the $f_0$ in $\mathbb{P}_1 \times M$. Then consider the projections

$$
\pi : \mathbb{P}_1 \times M \to \mathbb{P}_1,
\sigma : \mathbb{P}_1 \times M \to M,
$$

be the natural projections. Since $T_{\mathbb{P}_1} \times M$ is isomorphic to $\pi^* T_{\mathbb{P}_1} \oplus \sigma^* T_M$ and $T_V$ is isomorphic to the normal bundle $N_V$ of $V$ in $\mathbb{P}_1 \times M$, that is $f_0^* T_M$ is isomorphic to $N_V$.

Let $\tilde{D}$ be the moduli space when $V$ is deformed as a subspace of $\mathbb{P}_1 \times M$. Let $D$ be the irreducible component of $\tilde{D}$ which contains the point $x_0$ of $\tilde{D}$ that contains the point $x_0$ of $\tilde{D}$ corresponding to $V$. The infinitesimal deformation of $V$ is given by $\Gamma(V, N_V)$ We claim that $H^1(V, N_V)$ vanishes so that all infinitesimal deformations are realised as actual deformations and $x_0$ is a regular point of $\tilde{D}$ of dimension equal to the dimension of $\Gamma(N, N_V)$.

The idea is to use that $M$ has positive orthogonal bisectional curvature. Consider the curvature of the normal bundle and the Ricci formula,

$$
\langle \overline{R}(X, Y) \eta, \zeta \rangle = (R^i(X, Y) \eta, \zeta) + \langle [S_\eta, S_\zeta]X, Y \rangle,
$$

where $[S_\eta, S_\zeta]$, denotes the operator $S_\eta \circ S_\zeta - S_\zeta \circ S_\eta$. In general the last quantity is not zero. However, since $f_0$ is a holomorphic curve, $J$ respects the tangent and normal bundle. Furthermore as $M$ is a Kähler manifold, $\nabla J = 0$. This implies the the second fundamental form is $J$ invariant in the sense that $S_J \eta = JS_\eta$. In particular, this implies that for vectors of the form $X, JX, \eta, J\eta$, we have that

$$
\langle \overline{R}(X, JX) \eta, J\eta \rangle = \langle R^i(X, JX) \eta, J\eta \rangle.
$$

Hence this implies that the curvature of the normal bundle is positive hence the normal bundle is positive in the sense of Griffiths. This implies that there exists some $\mu$ such that $T^\mu_M$ of $T_M$ is ample. Take $y = \pi(v)$ as the map $f_0$ is an immersion. Then for $\mu \geq \mu_0$ we have an exact sequence

$$
0 \to F \to \Gamma(V, N^\mu_V) \to (N^\mu_V)_v \to 0.
$$

so that the natural map $F \to (N^\mu_V)_v \otimes (T^*_V)_v$ is surjective. By Grothendieck, Theorem 2.8, the normal bundle $N_v$ splits into a sum of line bundles,

$$
N_V = L_1 \oplus \ldots \oplus L_m,
$$

over $V$. Then for $\mu \geq \mu_0$ and $1 \leq i \leq m$, we have an exact sequence

$$
0 \to F_i \to \Gamma(V, L^\mu_i) \to (L^\mu_i)_v \to 0
$$

so that the natural map $f_i \to (L^\mu_i)_v \otimes (T^*_V)_v$ is surjective. Hence each $L_i$ is a positive holomorphic line bundle over $V$. By the Riemann-Roch, Theorem 2.7, $H^1(V, L_i) = 0$ for each $i$. It follows that $H^1(V, N_V) = 0$, hence every infinitesimal deformation is realised as an actual deformation.

We have a complex subspace $\mathcal{C}$ of $D \times \mathbb{P}_1 \times M$ with the following property. Let

$$
\alpha : \mathcal{C} \to D
\beta : \mathcal{C} \to \mathbb{P}_1 \times M
$$

be the natural projections. Since $T_{\mathbb{P}_1} \times M$ is isomorphic to $\pi^* T_{\mathbb{P}_1} \oplus \sigma^* T_M$ and $T_V$ is isomorphic to the normal bundle $N_V$ of $V$ in $\mathbb{P}_1 \times M$, that is $f_0^* T_M$ is isomorphic to $N_V$.

Let $\tilde{D}$ be the moduli space when $V$ is deformed as a subspace of $\mathbb{P}_1 \times M$. Let $D$ be the irreducible component of $\tilde{D}$ which contains the point $x_0$ of $\tilde{D}$ that contains the point $x_0$ of $\tilde{D}$ corresponding to $V$. The infinitesimal deformation of $V$ is given by $\Gamma(V, N_V)$ We claim that $H^1(V, N_V)$ vanishes so that all infinitesimal deformations are realised as actual deformations and $x_0$ is a regular point of $\tilde{D}$ of dimension equal to the dimension of $\Gamma(N, N_V)$.

The idea is to use that $M$ has positive orthogonal bisectional curvature. Consider the curvature of the normal bundle and the Ricci formula,

$$
\langle \overline{R}(X, Y) \eta, \zeta \rangle = (R^i(X, Y) \eta, \zeta) + \langle [S_\eta, S_\zeta]X, Y \rangle,
$$

where $[S_\eta, S_\zeta]$, denotes the operator $S_\eta \circ S_\zeta - S_\zeta \circ S_\eta$. In general the last quantity is not zero. However, since $f_0$ is a holomorphic curve, $J$ respects the tangent and normal bundle. Furthermore as $M$ is a Kähler manifold, $\nabla J = 0$. This implies the the second fundamental form is $J$ invariant in the sense that $S_J \eta = JS_\eta$. In particular, this implies that for vectors of the form $X, JX, \eta, J\eta$, we have that

$$
\langle \overline{R}(X, JX) \eta, J\eta \rangle = \langle R^i(X, JX) \eta, J\eta \rangle.
$$

Hence this implies that the curvature of the normal bundle is positive hence the normal bundle is positive in the sense of Griffiths. This implies that there exists some $\mu$ such that $T^\mu_M$ of $T_M$ is ample. Take $y = \pi(v)$ as the map $f_0$ is an immersion. Then for $\mu \geq \mu_0$ we have an exact sequence

$$
0 \to F \to \Gamma(V, N^\mu_V) \to (N^\mu_V)_v \to 0.
$$

so that the natural map $F \to (N^\mu_V)_v \otimes (T^*_V)_v$ is surjective. By Grothendieck, Theorem 2.8, the normal bundle $N_v$ splits into a sum of line bundles,

$$
N_V = L_1 \oplus \ldots \oplus L_m,
$$

over $V$. Then for $\mu \geq \mu_0$ and $1 \leq i \leq m$, we have an exact sequence

$$
0 \to F_i \to \Gamma(V, L^\mu_i) \to (L^\mu_i)_v \to 0
$$

so that the natural map $f_i \to (L^\mu_i)_v \otimes (T^*_V)_v$ is surjective. Hence each $L_i$ is a positive holomorphic line bundle over $V$. By the Riemann-Roch, Theorem 2.7, $H^1(V, L_i) = 0$ for each $i$. It follows that $H^1(V, N_V) = 0$, hence every infinitesimal deformation is realised as an actual deformation.

We have a complex subspace $\mathcal{C}$ of $D \times \mathbb{P}_1 \times M$ with the following property. Let

$$
\alpha : \mathcal{C} \to D
\beta : \mathcal{C} \to \mathbb{P}_1 \times M
$$

be the natural projections. Since $T_{\mathbb{P}_1} \times M$ is isomorphic to $\pi^* T_{\mathbb{P}_1} \oplus \sigma^* T_M$ and $T_V$ is isomorphic to the normal bundle $N_V$ of $V$ in $\mathbb{P}_1 \times M$, that is $f_0^* T_M$ is isomorphic to $N_V$. Let
be the natural projections. Then \( \mathcal{C} \) is \( \alpha \)-flat, and for every \( x \in D, \beta \) maps \( \alpha^{-1}(x) \) biholomorphically onto the complex subspace of \( \mathbb{P}_1 \times M \). In particular, \( \beta \) maps \( \alpha^{-1}(x_0) \) biholomorphically onto \( V \).

Let \( \mathcal{C}' \) be the subset of \( \mathcal{C} \) consisting of all \( w \in \mathcal{C} \) such that

1. the structure sheaf of \( \mathcal{C} \) is reduced at \( w \),
2. \( D \) is regular at \( \alpha(w) \) and,
3. the map \( \alpha \) is a submersion at \( w \).

Clearly \( \mathcal{C} - \mathcal{C}' \) is a subvariety of \( \mathcal{C} \). The fibre \( \alpha^{-1}(x_0) \) is contained in \( \mathcal{C}' \). Then \( D_1 \) is a proper subvariety of \( D \).

Let \( \Omega^1_{D} \), respectively \( \Omega^1_{C} \), be the sheaf of germs of holomorphic 1-forms on \( D \) (respectively \( \mathcal{C} \)) in the sense that it is locally the sheaf of germs of holomorphic 1-forms on an ambient manifold modulo the defining functions and their differentials. Let \( \mathcal{L} \) be isomorphic to \( \Omega^1_{D} - \mathcal{C} \) on \( \mathcal{C} \), i.e., the sheaf of germs of holomorphic functions on \( \mathcal{L} \) which are linear forms along the fibres of \( \mathcal{L} \) is isomorphic to \( \Omega^1_{D} - \mathcal{C} \) on \( \mathcal{C} \). Let \( \mathbb{P}(\mathcal{L}) \) be obtained by replacing each fibre of \( \mathcal{L} \) by the projective space of all complex lines i.e. \( \mathbb{P}(\mathcal{L}) \) is the orbit space of \( \mathcal{L} \) under the \( \mathbb{C}^* \) action on the fibres. The projection \( \sigma \beta : \mathcal{C} \rightarrow M \) induces a holomorphic map

\[
(\sigma \beta)_* : \mathbb{P}(\mathcal{L}) \rightarrow \mathbb{P}(T_M),
\]

where \( \mathbb{P}(T_M) \) is the projectivization of the tangent bundle \( T_M \). We claim that \( (\sigma \beta)_* \) is surjective.

Since \( D \) is compact, it follows that \( \mathcal{C} \) is compact and \( \mathbb{P}(\mathcal{L}) \) is compact. Hence the image of \((\sigma \beta)_* \) is open at some point. Take a regular point \( y_0 \) of \( C_0 \). Then it corresponds to a point \( w_0 \) in \( \mathcal{C} \) i.e. \( \alpha(w_0) = x_0 \) and \( (\sigma \beta)(w_0) = y_0 \). Let \( \tilde{y}_0 \in \mathbb{P}_1 \) be the point such that \( f_0(\tilde{y}_0) = y_0 \). Let \( \zeta \) be a local co-ordinate of \( \mathbb{P}_1 \) at \( \tilde{y}_0 \) vanishing at \( y_0 \). Then \( w_0 \) and \( \left( \frac{\zeta}{\tilde{y}_0} \right) \) determine a point \( \theta_0 \in \mathbb{P}_1(\mathcal{L}) \).

**Claim 5.2.** \((\sigma \beta)_* \) is open at \( \theta_0 \).

**Proof of Claim.** Let \( E \) be the divisor \( df_0 \) of \( f_0 \), and let \([E] \) be the line bundle over \( \mathbb{P}_1 \) associated to the divisor \( E \). Then \( T_{\mathbb{P}_1} \otimes [E] \) is a line sub bundle of \( f_0^*TM \). Let \( Q \) be the quotient bundle \( f_0^*TM - \mathbb{P}_1 \otimes [E] \) is a sub bundle of the normal bundle of \( f_0^*TM \), which we have previously shown is positive, hence by Grothendieck, Theorem 2.8 \( Q \) splits as a sum of positive line bundles \( Q_\nu, \nu = 1, \ldots, m \). Let \( \tilde{y}_0 \) be the line bundle over \( \mathbb{P}_1 \) associate to the divisor \( \tilde{y}_0 \). For \( 2 \leq \nu \leq m \), there exists a holomorphic section \( s_\nu \) of \( Q_\nu \otimes [\tilde{y}_0]^{-1} \) over \( \mathbb{P}_1 \) which does not vanish at \( \tilde{y}_0 \). We can regard \( s_\nu \) as a holomorphic section of \( Q \otimes [\tilde{y}_0]^{-1} \). Consider the following exact sequence

\[
\Gamma(\mathbb{P}_1, f_0^*TM \otimes [\tilde{y}_0]^{-1}) \rightarrow \Gamma(\mathbb{P}_1, Q \otimes [\tilde{y}_0]^{-1}) \rightarrow H^1(\mathbb{P}_1, T_{\mathbb{P}_1} \otimes [E] \otimes [\tilde{y}_0]^{-1})
\]

coming from the exact sequence

\[
0 \rightarrow T_{\mathbb{P}_1} \otimes [E] \otimes [\tilde{y}_0]^{-1} \rightarrow (f_0^*TM) \otimes [\tilde{y}_0]^{-1} \rightarrow Q \otimes [\tilde{y}_0]^{-1} \rightarrow 0.
\]

Since \( H^1(\mathbb{P}_1, T_{\mathbb{P}_1} \otimes [E] \otimes [\tilde{y}_0]^{-1}) = 0 \)

by the vanishing theorem of Kodaira, the sections \( s_\nu, 2 \leq \nu \leq m \) can be lifted to holomorphic sections \( \tilde{s}_\nu \) of \( f_0^*TM \otimes [\tilde{y}_0]^{-1} \) over \( \mathbb{P}_1 \). Let \( \nu \) be the holomorphic section of \([\tilde{y}_0] \) over \( \mathbb{P}_1 \) whose divisor is \( \tilde{y}_0 \). Let \( u_\nu = \tilde{s}_\nu t \). Then \( u_\nu \) is a holomorphic section of \( f^*TM \) over \( \mathbb{P}_1, 2 \leq \nu \leq m \), and these sections have the property that
\( \frac{1}{\zeta} u_\nu, 2 \leq \nu \leq m \) form a basis of \((f^* T_M \setminus T_M)_{\tilde{y}_0}\). For \(1 \leq \nu \leq m - 1\), there exists a section \(s_{m-\nu}\) of \(Q_{\nu+1}\) over \(\mathbb{P}_1\) which does not vanish at \(\tilde{y}_0\). We can regard \(s_{m+\nu}, 1 \leq \nu \leq m - 1\), as a holomorphic cross section of \(Q\). Consider the following exact sequence,

\[
\Gamma(\mathbb{P}_1, f^* T_M) \to \Gamma(\mathbb{P}_1, Q) \to H^1(\mathbb{P}_1, T_{\mathbb{P}_1} \otimes [E])
\]

coming from the exact sequence

\[
0 \to T_{\mathbb{P}_1} \otimes [E] \to f_0^* T_M \to Q \to 0.
\]

Since

\[
H^1(\mathbb{P}_1, T_{\mathbb{P}_1} \otimes [E]) = 0
\]

by the vanishing theorem of Kodaira, the sections \(s_{m+\nu}, 1 \leq \nu \leq m - 1\), can be lifted up to a holomorphic section \(u_{m+\nu}, 1 \leq \nu \leq m - 1\) of \(f_0^* T_M\) over \(\mathbb{P}_1\).

Each \(u_\nu, 1 \leq \nu \leq 2m - 1\) defines a tangent vector \(v_\nu\) of \(D\) at \(x_0\) and defines \(\xi_\nu \in \Gamma(\alpha^{-1}(x_0), T_C)\) such that

1. \((da)(\xi_\nu) = v_\nu\) are every point of \(\alpha^{-1}(x_0)\) and
2. \((d(\sigma \beta))(\xi_\nu)\) at \(w \in \alpha^{-1}(x_0)\) equals \(u_\nu\) at \(\pi \beta(w)\).

Choose a local submanifold \(R\) of \(D\) at \(x_0\) such that the tangent space of \(R\) at \(x_0\) is spanned by \(v_2, \ldots, v_{2m-1}\). At \(w_0\), we choose a local coordinate system \(t_1, \ldots, t_{2n-1}\) of \(\alpha^{-1}(R)\) such that

1. \(t_\nu\) is of the form \(t' \circ \alpha\) for \(2 \leq \nu \leq 2m - 1\),
2. \(t_1(w_0) = \cdots = t_{2m-1}(w_0) = 0\),
3. \(\partial_{t_\nu} = \xi_\nu\) for \(2 \leq \nu \leq 2m - 1\) at \(\alpha^{-1}(x_0)\), and
4. \(t_1 = \zeta \circ f_0^{-1} \circ (\sigma \beta)\) on \(\alpha^{-1}(x_0)\).

Choose a local co-ordinate system \(z_1, \ldots, z_m\) of \(M\) at \(y_0\) such that

1. \(C_0\) is defined by \(z_2 = \cdots = z_m = 0\), and
2. \(z_1 = \zeta \circ f_0^{-1}\) on \(C_0\).

Let

\[
u_\nu = \sum_{\mu=1}^{m} u_{\nu \mu} \frac{\partial}{\partial z_\mu}
\]

where \(u_{\nu \mu}\) is a holomorphic function on \(\mathbb{P}_1\) near \(\tilde{y}_0\). Since \(\frac{1}{\zeta} u_\nu, 1 \leq \nu \leq m\) form a basis of \((f_0^* T_M \setminus T_M)_{\tilde{y}_0}\), it follows that the matrix

\[
\left(\frac{1}{\zeta} u_{\nu \mu}\right)_{2 \leq \nu, \mu \leq m}
\]

is non singular at \(\tilde{y}_0\). Let

\[
a_{\nu \mu} = \left(\frac{1}{\zeta} u_{\nu \mu}\right) \circ f_0^{-1}
\]

on \(C_0\) near \(y_0\). Then on \(C_0\) near \(y_0\) we have that

\[
(d(\sigma \beta))\frac{\partial}{\partial t_1} = \frac{\partial}{\partial z_1},
\]

\[
(d(\sigma \beta))\frac{\partial}{\partial t_\nu} = z_1 \sum_{\mu=1}^{m} a_{\nu \mu} \frac{\partial}{\partial z_\mu}, \quad (2 \leq \nu \leq m),
\]
Let
\[(d(\sigma\beta))\frac{\partial}{\partial t^\nu} = \sum_{\mu=1}^{m} b_{\nu\mu} \frac{\partial}{\partial z^{\mu}}, \quad (m < \nu < 2m).\]

Since \(u_\nu, m < \nu < 2m\) form a basis of \((f^*_{\theta_0}T_M \mid T_M)_{\bar{y}_0}\) it follows that the matrix
\[
(b_{\nu\mu})_{m<\nu<2m}
\]
is non singular at \(y_0\). Now we calculate that the Jacobian matrix of the map \((\alpha\beta)_*\) restricted to \(\mathbb{P}(\mathcal{L}) \mid R\) with respect to local coordinate systems we are going to describe and verify that the Jacobian matrix has rank \(2m - 1\) over \(\mathbb{C}\) at \(\theta_0\). Since \(\alpha^{-1}(x)\) is of complex dimension 1 for every \(x \in D\) of \(w \in \mathcal{C}'\) the fibre of \(\mathbb{P}(\mathcal{L})\) at \(w\) consists only of a single point. Hence we can identify \(\mathbb{P}(\mathcal{L}) \mid \mathcal{C}'\) with \(\mathcal{C}'\) and use \(t_1, \ldots, t_{2m-1}\) as local co-ordinates at \(w_0\) for \(\mathbb{P}(\mathcal{L})\) after identification of \(\mathbb{P}(\mathcal{L}) \mid \mathcal{C}'\) with \(\mathcal{C}'\). Every element \(\eta\) of \(T_M\) may be written as \(\eta_\nu \frac{\partial}{\partial z^{\nu}}\) and we use local co-ordinates of \(\mathbb{P}(T_M)\) near \((y_0, \frac{\partial}{\partial z^1})\) the functions
\[
z_1, \ldots, z_m, \frac{\eta_2}{\eta_1}, \ldots, \frac{\eta_m}{\eta_1}.
\]
The map \(\sigma\beta\) makes \(z_1, \ldots, z_m\) functions of \(t_1, \ldots, t_{2m-1}\). The Jacobian matrix of \((\sigma\beta)_*\) restricted to \(\mathbb{P}(\mathcal{L}) \mid R\) with respect to the co ordinates systems
\[
t_1, \ldots, t_{2m-1}
\]
and
\[
z_1, \ldots, z_m, \frac{\eta_2}{\eta_1}, \ldots, \frac{\eta_m}{\eta_1}.
\]
equals to the \((2m - 1) \times (2m - 1)\) matrix \((A, B)\) where
\[
A = \begin{pmatrix} \frac{\partial z^\mu}{\partial t^\nu} \end{pmatrix}_{1 \leq \nu \leq 2m-1, 1 \leq \mu \leq m}, \quad B = \begin{pmatrix} \frac{\partial z^\mu}{\partial t^\nu} \frac{\partial z^1}{\partial z^1} \end{pmatrix}_{1 \leq \nu \leq 2m-1, 2 \leq \mu \leq m}
\]
WE have
\[
\frac{\partial z^\mu}{\partial t^1} = \delta^\mu_1, \quad \text{for } 1 \leq \mu \leq m,
\]
\[
\frac{\partial z^\mu}{\partial t^\nu} = t_1 a_{\nu\mu}, \quad \text{for } 2 \leq \nu \leq m, 1 \leq \mu \leq m,
\]
\[
\frac{\partial z^\mu}{\partial t^\nu} = b_{\nu\mu}, \quad \text{for } m < \nu < 2m, 1 \leq \mu \leq m
\]
on \(\alpha^{-1}(x_0)\) near \(w_0\). Now
\[
\frac{\partial}{\partial t^\nu} \left( \frac{\partial z^\mu}{\partial t^1} \right) = a_{\nu\mu}
\]
at \(w_0\) for \(2 \leq \nu \leq m, 2 \leq \mu \leq m\). Since the matrices
\[
(a_{\nu\mu})_{2 \leq \nu, \mu \leq m}
\]
are nonsingular at \( y_0 \) it follows that \((A, B)\) is nonsingular at \( w_0 \). This shows that \((\sigma \beta)_*\) is open at \( \theta_0 \). □

Let \( G \) be the subset of \( \mathbb{P}(T_M) \) consisting of all points \( \hat{y} \) of \( \mathbb{P}(T_M) \) such that \((\sigma \beta)_* (\hat{y})\) is entirely contained in the restriction of \( \mathbb{P}(\mathcal{L}) \) to \( C' - \alpha^{-1}(D_1) \). Let \( Z = \mathbb{P}(T_M) - G \). Then \( Z \) is a proper subvariety of \( \mathbb{P}(T_M) \). This proves the theorem. □

6. Proof of Generalised Frankel Conjecture

The proof of the main theorem now follows from arguments similar to [SY80].

\( \text{Proof} \). Let \( M \) be an \( n \)-dimensional compact Kähler manifold with positive orthogonal bisectional curvature. By Corollary 3.4, \( M \) admits a metric with positive Ricci curvature strictly bounded away from zero. Hence by the Bonnet-Myers theorem, the universal cover of \( M \) is also compact. As \( \mathbb{CP}^m \) has no fixed point free automorphism, we see that by replacing \( M \) with its universal cover, we can assume that \( M \) is simply connected. Hence we have that \( \pi_2(M) \) is isomorphic to \( H_2(M, \mathbb{Z}) \). There exists a positive line bundle \( F \) over \( M \) whose first Chern class \( c_1(F) \) is a generator of \( H^2(M, \mathbb{Z}) \). Let \( g \) be a generator of the free part of \( H_2(M, \mathbb{Z}) \) such that the value of \( c_1(F) \) at \( g \) is 1. Let \( f : \mathbb{P}_1 \to M \) be a smooth map so that the element in \( \pi_2(M) \) defined by \( f \) corresponds \( g \) in the isomorphism between \( \pi_2(M) \) and \( H_2(M, \mathbb{Z}) \).

By Theorem 2.5, there exists energy-minimizing maps \( f_i : \mathbb{P}_1 \to M, 0 \leq i \leq k \) such that the sum of \( f_i, 0 \leq i \leq k \) is a homotopic to \( f \) and \( E([f]) = \sum_{i=1}^{k} E(f_i) \).

By Proposition 4.1 each such map is either holomorphic or conjugate holomorphic. If \( k > 0 \), then at least on \( f_j \) is conjugate holomorphic.

\textbf{Case} \( k = 0 \):

Suppose now that \( E \) is the divisor of the differential of \( f_0 \) and let \( E \) be the line bundle over the complex curve \( \mathbb{P}_1 \) associated to \( E \). The bundle \( T_M \otimes [E] \) is a sub bundle of \( f_0^* T_M \) and the quotient bundle \( f^* T_M / (T_M \otimes [E]) \) splits into a direct sum of line bundles \( Q_2, \ldots, Q_m \). Each \( Q_i \) is a positive bundle hence we have that

\[ c_1(f_0^* T_M) = c_1(T_M) = c_1([E]) + \sum_{i=2}^{m} c_1(Q_i). \]

As \( c_1(T_M) \) is evaluated at \( g \) is greater than \( \geq n + 1 \). That is \( c_1(T_M) = \lambda c_1(F) \) for some integer \( \lambda = m + 1 \). By the theorem of Koybashi-Ochiai [KO70], \( M \) is biholomorphic to \( \mathbb{CP}^m \).

\textbf{Case} \( k > 0 \):

This case follows exactly that of [SY80], which shows that the case \( k > 0 \) cannot happen. Let us assume that \( f_0 \) is holomorphic and that \( f_1 \) is conjugate holomorphic. Then by Proposition 5.1 there exists a proper subvariety \( Z \) of \( \mathbb{P}(T_M) \) for the rational curve \( f_1(\mathbb{P}_1) \) of \( M \). Let \( y \in M \) and let \( \xi \) be a tangent vector of \( M \) at \( y \) so that the corresponding point in the bundle \( \mathbb{P}(T_M) \) satisfies \( (y, \xi) \notin Z \cup Z' \). Then the map \( f_0 \) is homotopic to a holomorphic map \( f_0' \) \( : \mathbb{P}_1 \to M \) such that \( y \) is a regular point of \( f_0'(\mathbb{P}_1) \) where \( y \) is a regular point of \( f_0'(\mathbb{P}_1) \) with tangent vector which is a multiple of \( \xi \). Similarly for the map \( f_1 \) there exists a homotopic map \( f_1' \) \( : \mathbb{P}_1 \to M \) that is anti-holomorphic with the same properties. Since for \( \nu = 0, 1 \) we have that \( E(f_\nu) \) and \( E(f_\nu') \) are equal to the absolute value of the Kähler class of \( M \) at
[\nu(P_1)]}, by replacing \( f_\nu \) with \( f'_\nu \) we can assume that \( f_\nu = f'_\nu \). We then choose a local coordinate system \( z_1, \ldots, z_n \) at \( y \) so that \( y \) is the origin and that \( \xi = \frac{\partial}{\partial z_1} \).

Furthermore we choose a local co-ordinate system \( \zeta \) of \( P_1 \) so that both \( f^{-1}_0(y) \) and \( f^{-1}_1(y) \) correspond to \( \xi = 0 \) and that \( f_\nu(n_\nu = 0, 1) \) is of the form

\[
\begin{align*}
z_1 &= \zeta, \\
z_\mu &= f'_\mu(\zeta) \quad (2 \leq \mu \leq m)
\end{align*}
\]

near \( \zeta = 0 \). Choosing \( \delta > 0 \), we let \( D_\delta \) be a closed disk in \( \mathbb{C} \) or radius \( \delta \) centred at \( 0 \). Then for sufficiently small \( \delta \), we can deform the image of the disk \( f_\nu(\Delta_\delta) \) from \( f_\nu(P_1), (\nu = 0, 1) \) and join the images with the following surface \( S_\delta \) defined by

\[
\begin{align*}
z_1 &= \delta e^{i\theta}, \\
z_\mu &= tf_0(\delta e^{i\theta}) + (1-t)f_1(\delta e^{i\theta}) \quad \text{for } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq t \leq 1.
\end{align*}
\]

The surface, \( S_\delta \cap \bigcup_{\nu=1,2} f_\nu(P_1 \setminus D_\delta) \) is then the image of a map \( \tilde{f} : \mathbb{S}^2 \to M \) which is homotopic to the sum of \( f_0 \) and \( f_1 \). Furthermore, if we choose \( \delta \) sufficiently small, the area of this surface is strictly less than the combined area of \( f_0(P_1) \) and \( f_1(P_1) \). Hence after smoothing out \( \tilde{f} \) and composing \( \tilde{f} \) with an appropriate diffeomorphism of \( \mathbb{S}^2 \), we get a conformal map \( \tilde{f} \) with

\[
E(\tilde{f}) < E(f_0) + E(f_1).
\]

The sum of \( \tilde{f} \) and \( f_2, \ldots, f_k \) is then homotopic to \( f \) but we have that

\[
E(\tilde{f} + \sum_{i=1}^k E(f_i) < \sum_{i=0}^k E(f_i) = E([f]),
\]

which is a contradiction. Hence \( k = 0 \) and this shows that \( M \) is biholomorphic to \( \mathbb{P}_m \). \( \square \)

7. Kähler Ricci Flow

We now apply a theorem of Tian-Zhu [TZ07]to show that for any compact Kähler manifold with positive orthogonal bisectional curvature, the Kähler Ricci flow converges to \( \mathbb{C} \mathbb{P}^m \).

Theorem 7.1 ([TZ07]). Let \( M \) be a compact Kähler Ricci soliton which admits a Kähler-Ricci soliton \( (g_{KS}, X) \). Then any solution of the Kähler-Ricci flow will converge to the \( g_{KS} \) in the sense of Cheeger-Gromov if the initial Kähler metric \( g_0 \) is \( K_X \)-invariant.

Corollary 7.2 ([TZ07]). If \( M \) admits a Kähler-Einstein metric, then any solution of the Kähler Ricci flow will converge to a Kähler-Einstein metric in the sense of Cheeger-Gromov for any initial Kähler metric with Kähler class \( c_1(M) \).

By the main theorem, and applying the corollary, we see that the Kähler-Ricci flow converges in the sense of Cheeger-Gromov to

Theorem 7.3. If \( M \) be a compact Kähler manifold with positive orthogonal bisectional curvature. Then for any initial metric, the Kähler Ricci flow converges to \( \mathbb{C} \mathbb{P}^m \).
In general, we note that Cheeger-Gromov convergence may not preserve the Kähler structure in the limit. However, as $\mathbb{CP}^m$ has a unique Kähler structure then this structure must be preserved into the limit.

References


Mathematics Institute Zeeman Building University of Warwick Coventry CV4 7AL UK
E-mail address: H.T.Nguyen@warwick.ac.uk