Matrix Product States

Ian McCulloch

University of Queensland
Centre for Engineered Quantum Systems

28 August 2017

Hilbert space

(Hilbert) space is big. Really big. You just won't believe how vastly, hugely, mind-bogglingly big it is.
Douglas Adams

System of N particles

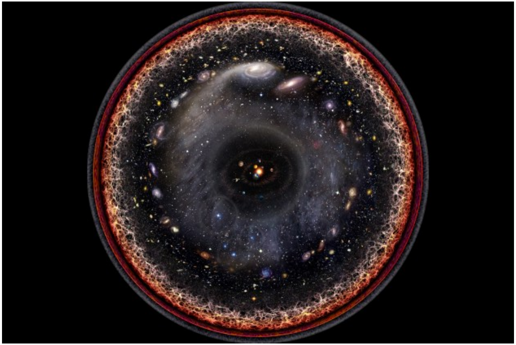
Ordinary material, $N \sim$ Avogadro number, $\sim O(10^{23})$

Number of basis states in the Hilbert space $\sim O\left(10^{10^{23}}\right)$

Number of basis states in the Hilbert space $\sim O\left(10^{10^{23}}\right)$

Compare to...

Number of atoms in the observable universe $\sim O(10^{80})$



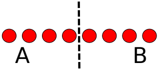
Entanglement

- A generic quantum state has a d^N dimensional Hilbert space

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_N} \psi_{j_1,j_2,\ldots,j_N} |j_1\rangle|j_2\rangle|j_3\rangle\ldots|j_N\rangle \quad , \quad j_n = 1\ldots d$$

- Partition the state into two pieces (Schmidt decomposition)

$$|\psi\rangle = \sum_{i,j} C_{ij}|i\rangle_A|j\rangle_B = \sum_\alpha \lambda_\alpha|\alpha\rangle_A|\alpha\rangle_B$$



- Entanglment entropy is a measure for the amount of entanglement
  $S = -\sum_\alpha \lambda_\alpha^2 \ln \lambda_\alpha^2$
  Equivalent to $S = -\mathrm{Tr}\rho_A \ln \rho_A$ with $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|$

Entanglement

- Product state

$$|\psi\rangle = \frac{1}{2}\left(|\uparrow\rangle_A + |\downarrow\rangle_A\right)\left(|\uparrow\rangle_B + |\downarrow\rangle_B\right)$$

$$S = 0$$

One non-zero Schmidt value
Not entangled.

- Entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

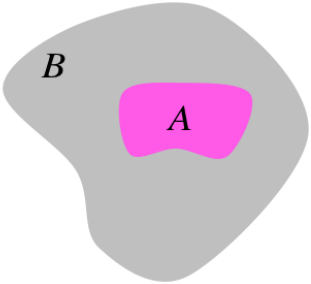
$$S = \ln 2$$

Two non-zero Schmidt values

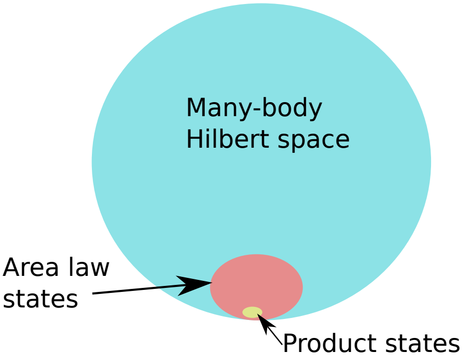
Entanglement

Area law

The entropy of a region of a ground state of a local Hamiltonian is proportional to the area of the boundary (up to logarithmic corrections)



$S \sim \partial A$



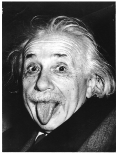
Compression of quantum states

- Wavefunction

$$|\psi\rangle = \sum_{i,j} C_{ij} |i\rangle_A |j\rangle_B = \sum_{\alpha} \lambda_\alpha |\alpha\rangle_A |\alpha\rangle_B$$

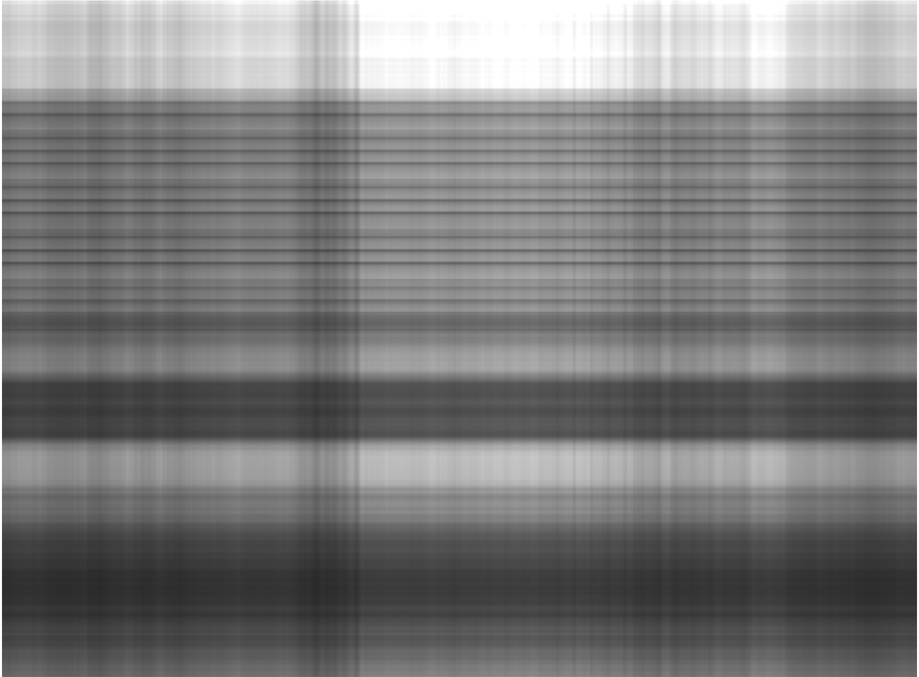
- Matrix C_{ij} as an image

$$\begin{pmatrix} 0.78 & \cdots & 0.22 \\ \vdots & \ddots & \vdots \\ 0.91 & \cdots & 0.66 \end{pmatrix} =$$

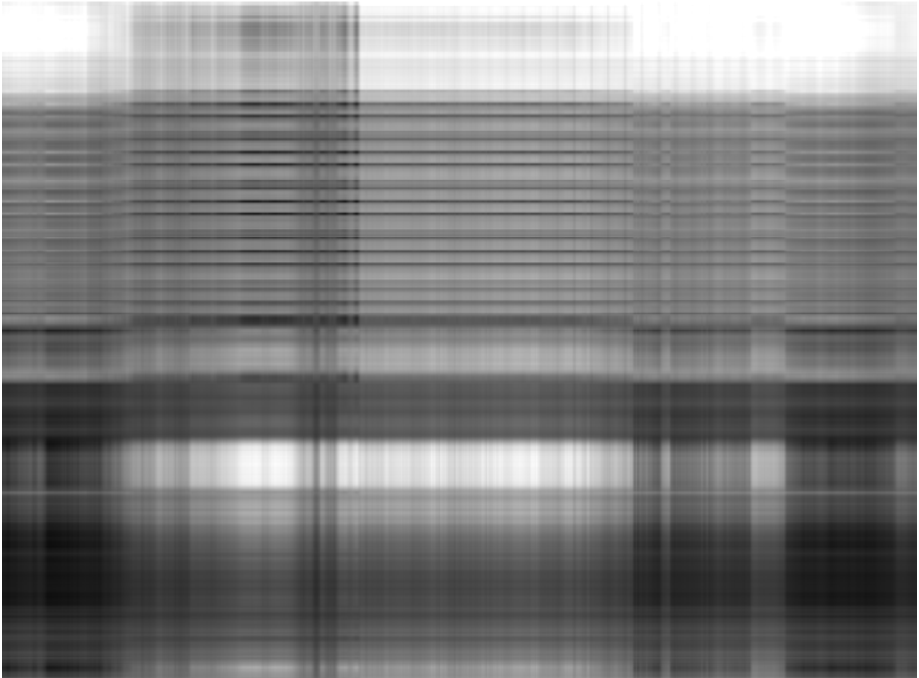


- Reconstruct the image (matrix) from a small number of Schmidt vectors (Singular Value Decomposition):

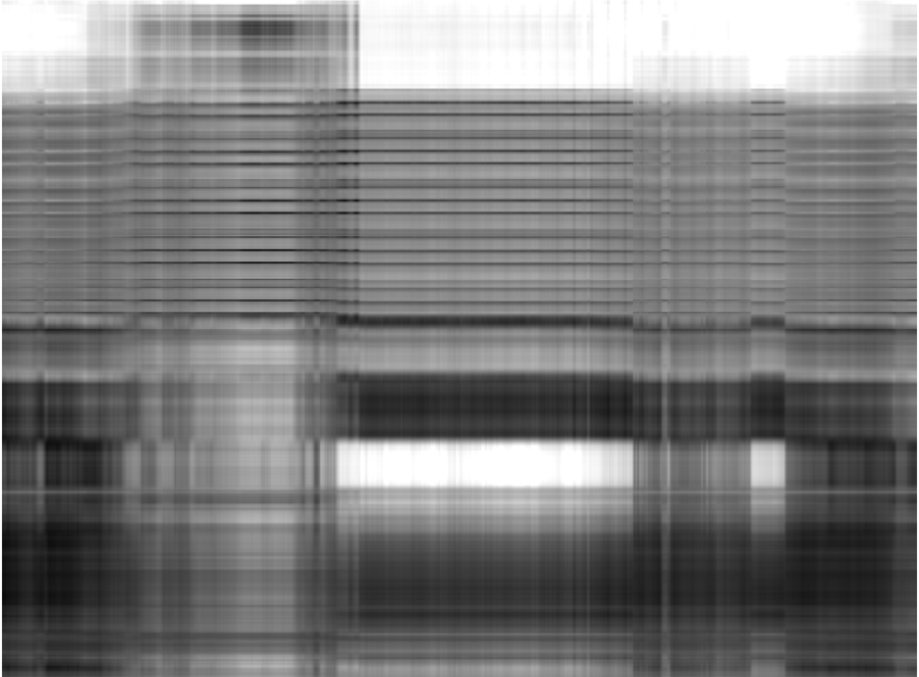
$D = 1$



$D = 2$



$D = 3$



$D = 4$

$D = 5$

$D = 6$

$D = 7$

$D = 8$

$D = 9$

$D = 10$

$D = 11$

$D = 12$

$D = 13$

$D = 14$

$D = 15$

$D = 16$

$D = 17$

$D = 18$

$D = 19$

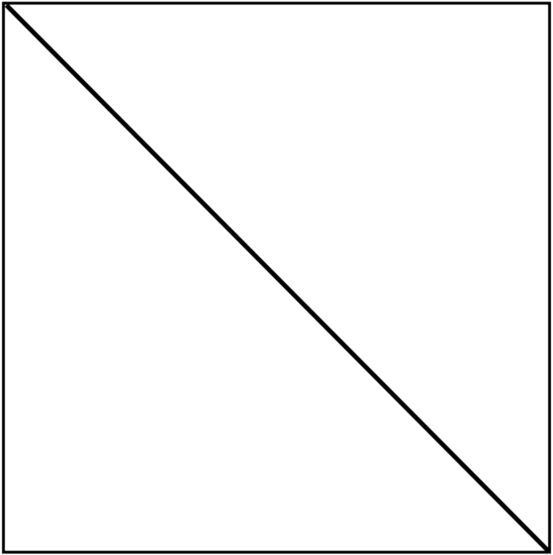
$D = 20$

$D = 50$

$D = 100$

$D = 600$ maximum

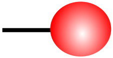
Worst case scenario: no compression (maximally entangled)



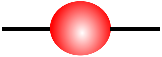
Tensor networks

Graphical representation for a state vector

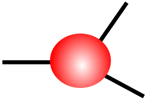
Vector $\vec{v}$



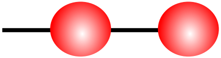
Matrix A_{ij}



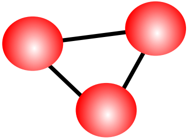
Tensor T_{ijk}



Matrix-vector multiplication

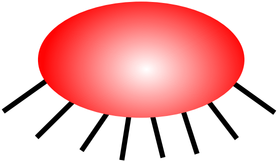


$\mathrm{Tr}(ABC)$ (Scalar)

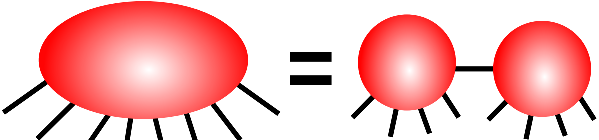


Tensor networks

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_N} \psi_{j_1,j_2,\ldots,j_N} |j_1\rangle|j_2\rangle|j_3\rangle\ldots|j_N\rangle$$



Decompose into smaller tensors via Schmidt decomposition



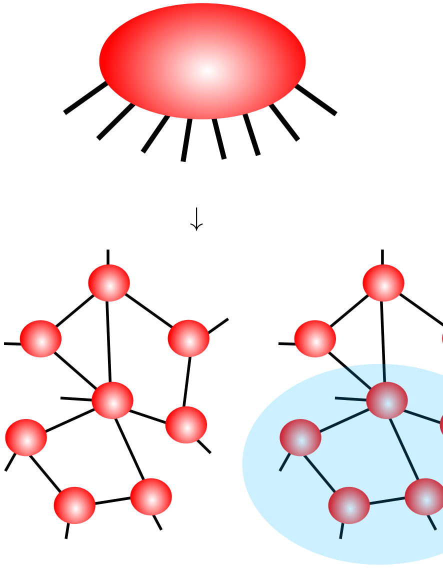
Tensor networks

Continue this recursively ...

Choose a decomposition that matches the geometric distribution of entanglement

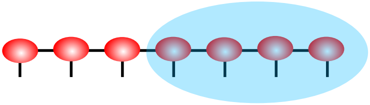
Satisfies area law: maximum entropy of a partitionis proportional to the number of cut bonds

Example: 3 bonds cut

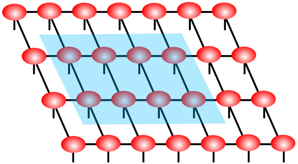


Entanglement and tensor networks

Entanglement scaling is encoded in the shape of a tensor network



Partition cuts chain at single point
$\rightarrow S(L) \sim$ const



2D - Tensor Product State or
PEPS

Constructing MPS: Method 1: quantize a classical state

Start from a *classical* (product) state

$$|\psi\rangle = |s^1\rangle\, |s^2\rangle\, |s^3\rangle\, |s^4\rangle\, \cdots$$

Each $|s^i\rangle$ is a classical vector, with real (or c-number) coefficients in some basis

$$|s^i\rangle = a_i^x|x\rangle + a_i^y|y\rangle + a_i^z|z\rangle$$

Turn our (commuting) numeric coefficients into a matrix

$$|s^i\rangle_{jk} = A_{jk}^x|x\rangle + A_{jk}^y|y\rangle + A_{jk}^z|z\rangle$$

We can recover an amplitude at the end by taking the trace, or arranging that the boundary matrices are $1 \times D$ and $D \times 1$.

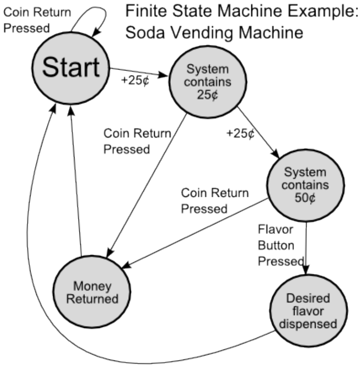
$$|\psi\rangle = \mathrm{Tr} \sum_{s_i} A^{s_1} A^{s_2} A^{s_3} A^{s_4} \cdots |s^1\rangle\, |s^2\rangle\, |s^3\rangle\, |s^4\rangle \cdots$$

Method 2: quantum finite-state machines

What is a Matrix Product State?

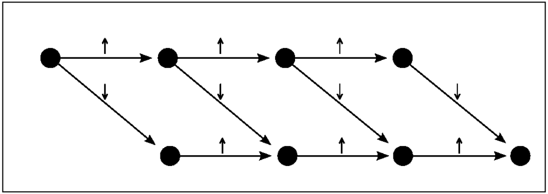
- Another way to visualizing them (from Greg Crosswhite)

A *finite-state machine* is a model of a system that can transition between a finite number of states.



A classical finite-state machine is always in one discrete state.

In a *quantum* finite-state machine, we choose every possible transition with some probability amplitude



(from Crosswhite and Bacon, Phys. Rev. A 78, 012356 (2008))

$$|\psi\rangle = \begin{cases} |\uparrow\rangle \\ |\downarrow\rangle \end{cases}$$

$$|\psi\rangle = \begin{cases} |\uparrow\uparrow\rangle \\ |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle \end{cases}$$

$$|\psi\rangle = \begin{cases} |\uparrow\uparrow\uparrow\rangle \\ |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle \end{cases}$$

$$|\psi\rangle = |\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle$$

Matrix Product States

This quantum finite-state machine has a transition matrix associated with it

- W-state

$$|\psi\rangle = \frac{1}{\sqrt{N}}(|\downarrow\uparrow\uparrow\uparrow\ldots\rangle + |\uparrow\downarrow\uparrow\uparrow\ldots\rangle + |\uparrow\uparrow\downarrow\uparrow\ldots\rangle + \ldots)$$

$$A = \begin{pmatrix} |\uparrow\rangle & 0 \\ |\downarrow\rangle & |\uparrow\rangle \end{pmatrix}$$

Practically all prototype wavefunctions studied in quantum information have a low-dimensional MPS representation

- GHZ state – long-range entangled, $S = \ln 2$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle)$$

$$A = \begin{pmatrix} |\uparrow\rangle & 0 \\ 0 & |\downarrow\rangle \end{pmatrix}$$

- AKLT state

$$A = \begin{pmatrix} \sqrt{1/3}|0\rangle & -\sqrt{2/3}|+\rangle \\ \sqrt{2/3}|-\rangle & -\sqrt{1/3}|0\rangle \end{pmatrix}$$

Spin 1 Chains

The AKLT Model: A prototypical Resonating Valence Bond groundstate

- $H = \sum_{<ij>} \left[\vec{S}_i \cdot \vec{S}_j + \beta (\vec{S}_i \cdot \vec{S}_j)^2 \right]$
- $\beta = 0$: usual Heisenberg spin chain
  - Haldane: unlike half-integer spin chains, integer spin chains have a gap
  - string order parameter: $S_0^z \ \exp[i\pi \sum_{m=1}^{n-1} S_m^z] \ S_n^z \rightarrow$ constant
  - free Z_2 parameter at the boundary: effective spin-$1/2$ edge states
- $\beta = 1/3$: exactly solvable groundstate

Matrix product realization:

- $A = \begin{pmatrix} \sqrt{1/3} & |0\rangle & -\sqrt{2/3} & |+\rangle \\ \sqrt{2/3} & |-\rangle & -\sqrt{1/3} & |0\rangle \end{pmatrix}$