Lecture 3

1 SERIES OF NUMBERS

DEFINITION Let \( \{a_n\} \subset \mathbb{R} \) and define

\[
s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n
\]

for each \( n \in \mathbb{N} \).

The symbol \( \sum_{n=1}^{\infty} a_n \) or \( a_1 + a_2 + \ldots \) is called an infinite series having nth term \( a_n \) and nth partial sum \( s_n \).

DEFINITION A series \( \sum_{n=1}^{\infty} a_n \) is said to converge to \( a \in \mathbb{R} \) if the sequence of partial sums \( s_n = \sum_{k=1}^{n} a_k \) converges to \( a \), and if so we write

\[
a = \lim_{n \to \infty} a_n.
\]

Otherwise we say that \( \sum_{n=1}^{\infty} a_n \) diverges.

**Theorem 1.1** If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \).

**Proof** Let \( \sum_{n=1}^{\infty} a_n = a \). Since \( a_n = s_n - s_{n-1} \), we have \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1} = 0 \).

**Example** Let \( \sum_{n=1}^{\infty} \frac{n+10}{n+1000} \), then \( \lim_{n \to \infty} \frac{n+10}{n+1000} = 1 \), so \( \sum_{n=1}^{\infty} \frac{n+10}{n+1000} \) diverges.

**Cauchy criterion** Let \( \{a_n\} \subset \mathbb{R} \). A series \( \sum_{n=1}^{\infty} a_n \) converges if and only if to every \( \epsilon > 0 \) there corresponds \( n_\epsilon \in \mathbb{N} \) such that

\[
\left| \sum_{n=p+1}^{q} a_n \right| < \epsilon \ \text{whenever} \ q > p \geq n_\epsilon.
\]

**Proof** Apply the Cauchy criterion (condition) to the sequence of partial sums \( \{s_n\} \)

\[
s_q - s_p = a_{p+1} + \ldots + a_q.
\]

\[\square\]
EXAMPLE \[ \sum_{n=1}^{\infty} \frac{1}{n} \] - the harmonic series
Assume that the series converges. Then its partial sums
\[ s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \]
form a Cauchy sequence. Thus there exists \( N \in \mathbb{N} \) such that
\[ n \geq N \text{ implies } |s_n - s_N| < \frac{1}{3}. \]
Therefore
\[ \frac{1}{3} > s_{2N} - s_N = \frac{1}{N+1} + \frac{1}{N+2} + \ldots + \frac{1}{2N} \]
\[ > \frac{1}{2N} + \frac{1}{2N} + \ldots + \frac{1}{2N} = \frac{N}{2N} = \frac{1}{2}, \]
which is impossible.

**Theorem 1.2** If \( \sum_n a_n \) and \( \sum_n b_n \) converge, then
\[ \sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n \]
and
\[ \sum_n ca_n = c \sum_n a_n \text{ for every } c \in \mathbb{R}. \]

**Geometric series.** Let \( a \in \mathbb{R} \) and \( r \in \mathbb{R} \). Then \( \sum_n^\infty ar^n \) converges and its sum is \( \frac{a}{1-r} \) if \( |r| < 1 \). If \( a \neq 0 \) and \( |r| \geq 1 \), then this series diverges.

By the formula for geometric progressions we have
\[ s_n = \sum_{k=0}^{n} ar^k = a \frac{1 - r^{n+1}}{1 - r}. \]
Assuming that \( |r| < 1 \), \( \lim_{n \to \infty} s_n = \frac{a}{1-r} \). If \( a \neq 0 \) and \( |r| \geq 1 \), then \( |ar^n| \geq |a| \neq 0 \) and this shows that the series cannot converge.

**Comparison test** Suppose that \( 0 \leq a_n \leq b_n \) for all \( n \in \mathbb{N} \).

1. If \( \sum_n b_n \) converges, then \( \sum_n a_n \) converges.
2. If \( \sum_n a_n \) diverges, then \( \sum_n b_n \) diverges.

**Proof** (1) Let \( A_n = \sum_{k=1}^{n} a_k \), \( B_n = \sum_{k=1}^{n} b_k \). Both sequences are nondecreasing. By assumption \( B_n \to B \) so it must be bounded. Hence \( \{A_n\} \) in nondecreasing and bounded, so \( \lim_{n \to \infty} A_n \) exists. \( \square \)
EXAMPLES

(1) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges because \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges and \( \frac{1}{n} \leq \frac{1}{\sqrt{n}} \) for every \( n \).

(2) \( \sum_{n=1}^{\infty} \frac{1}{2^n + n}, \frac{1}{2^n} \leq \frac{1}{2^n} \) for every \( n \) and \( \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \), so \( \sum_{n=1}^{\infty} \frac{1}{2^n + n} \) converges.

(3) \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \),

Note \( a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \)

and

\[ s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1} \to 1 \quad \text{as} \quad n \to \infty. \]

Hence \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) converges.

(4) \( \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \) converges because \( \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} \) and \( \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \) converges.

Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}, \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

Using the integral test it will be shown that

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} = \left\{ \begin{array}{ll}
\text{converges if } p > 1, \\
\text{diverges if } p \leq 1.
\end{array} \right. \]

Remark  Comparison test, as stated, does not apply to

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}. \]

2 ABSOLUTE CONVERGENCE

DEFINITION  We say that \( \sum_{n=1}^{\infty} \) converges absolutely, if \( \sum_{n=1}^{\infty} |a_n| \) converges.

Theorem 2.1 If \( \sum_{n=1}^{\infty} |a_n| \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

Proof  Assume \( \sum_{n=1}^{\infty} |a_n| \) converges. Since \( a_n = (a_n + |a_n| - |a_n|) \) we have

\[ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|. \]
We now observe that

\[ 0 \leq a_n + |a_n| \leq 2|a_n| \quad \text{and} \quad \sum_{n=1}^{\infty} 2|a_n| \]

converges, so by the comparison test \( \sum_{n=1}^{\infty} (a_n + |a_n|) \) converges, consequently by (2.1) \( \sum_{n=1}^{\infty} a_n \) converges.

**Remark** The converse is not true.

**Leibnitz’s alternating test** Let \( \{a_n\} \) be a nonincreasing sequence of positive numbers such that \( \lim_{n \to \infty} a_n = 0 \). Then \( \sum_{n=1}^{\infty} (-1)^n a_n \) is convergent.

**EXAMPLE** \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \) converges. If \( a_n = (-1)^n \frac{1}{n} \), then \( |a_n| = \frac{1}{n} \), and \( \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} \). Thus \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \) converges but not absolutely.

**Root test (Cauchy)** Let \( \sum_{n=1}^{\infty} a_n \) and

\[ \rho = \limsup_{n \to \infty} \sqrt[n]{|a_n|} \]

(i) If \( \rho < 1 \), the series converges absolutely,

(ii) If \( \rho > 1 \), the series diverges.

(iii) If \( \rho = 1 \), then the test is inconclusive.

**Proof** Suppose that \( \rho < 1 \). Fix \( \beta \) such that \( \rho < \beta < 1 \). There exists \( n_0 \in \mathbb{N} \) such that \( \sqrt[n]{|a_n|} < \beta \) for every \( n \geq n_0 \). Thus,

\[ \sum_{n=1}^{\infty} |a_n| < \sum_{n=n_0}^{\infty} \beta^n = \frac{\beta^{n_0}}{1 - \beta} < \infty. \]

If \( \rho > 1 \), then \( \sqrt[n]{|a_n|} > 1 \), hence \( |a_n| > 1 \) for infinitely many \( n \), and so it is false that \( a_n \to 0 \). Hence \( \sum_{n=1}^{\infty} a_n \) converges.

**Ratio test (d’Alembert)** Let \( \sum_{n=1}^{\infty} a_n \) be a series with \( a_n \neq 0 \) for every \( n \).

(i) If \( \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \), then the series converges absolutely.

(ii) If \( \liminf_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \), then the series diverges.

(iii) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), then the test is inconclusive.

**Proof** This follows from the following observation: let \( \{b_n\} \) be a sequence of positive numbers. Then

\[ \liminf_{n \to \infty} \frac{b_{n+1}}{b_n} \leq \liminf_{n \to \infty} \sqrt[n]{b_n} \leq \limsup_{n \to \infty} \sqrt[n]{b_n} \leq \limsup_{n \to \infty} \frac{b_{n+1}}{b_n}. \]
We shall prove the first inequality. Let \( a = \liminf_{n \to \infty} \frac{b_{n+1}}{b_n} \). We may assume that \( a > 0 \), since otherwise there is nothing to prove. Let \( 0 < \alpha < a \). There is an integer \( N \) such that \( \frac{b_{n+1}}{b_n} > \alpha \) for every \( n \geq N \). Iterating this inequality we get

\[
\begin{align*}
  b_{N+1} & \geq b_N \alpha \\
  b_{N+2} & \geq b_{N+1} \alpha \geq \alpha^2 b_N \\
  b_{N+3} & \geq b_{N+2} \alpha \geq \alpha^2 b_N \\
  \vdots \\
  b_{N+m} & \geq \alpha^m b_N 
\end{align*}
\]

for every \( m \in \mathbb{N} \). We write the last inequality in the following way: let \( n > N \), then

\[
\frac{b_n}{b_N} > \alpha^{n-N} \quad \text{or} \quad b_n > \alpha^{n-N} b_N,
\]

so

\[
\liminf_{n \to \infty} \sqrt[n]{b_n} \geq \alpha \liminf_{n \to \infty} \alpha^{\frac{n}{n-N}} = \alpha.
\]

But \( \alpha \) was an arbitrary (positive) number \( < a \) and so

\[
\liminf_{n \to \infty} \sqrt[n]{b_n} \geq a = \liminf_{n \to \infty} \frac{b_{n+1}}{b_n}.
\]

\[\square\]

**EXAMPLES**

1. \( \sum_{n=1}^{\infty} \frac{1}{n^{2\pi}}, \quad a_n = \frac{1}{n^{2\pi}}, \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{2(n+1)} \to \frac{1}{2} < 1 \) and the series converges.

2. \( \sum_{n=1}^{\infty} \frac{n}{3^n}, \quad a_n = \frac{n}{3^n} \quad \sqrt[n]{a_n} = \frac{\sqrt[n]{3}}{3} \to \frac{1}{3} \) and the series converges.