Lecture 6

1 DIFFERENTIATION OF FUNCTIONS OF ONE VARIABLE

**Definition** Let a function $f$ be defined on some open interval containing $x_0 \in \mathbb{R}$. We say that $f$ is differentiable at $x_0$ if

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. We call $f'(x_0)$ the derivative of $f$ at $x_0$.

Rewriting this condition as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0,$$

we see that the straight line $y = f(x_0) + f'(x_0)(x - x_0)$, called the tangent line to the graph of $f$ at $x_0$, is a good approximation to $f$ near $x_0$, and rewriting it as

$$\lim_{\Delta x \to 0} \left[ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0) \right] = 0,$$

we see that $f'(x_0)$, being the limit of slopes of the secant lines, can be interpreted as the slope of the tangent line to the graph of $f$ at $(x_0, f(x_0))$.

**Proposition 1.1** If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.

**Proof** Indeed,

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left[ \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) + f(x_0) \right] = f'(x_0) \cdot 0 + f(x_0) = f(x_0).$$

\[ \square \]

**Example** Let $f(x) = |x|$. Show that $f$ is continuous but not differentiable at $x_0 = 0$.

The limit of difference quotient

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = 0$$
does not exist, since \( \frac{h^1}{h} \) is 1 if \( h > 0 \) and is –1 if \( h < 0 \). (The function is clearly continuous at \( x_o = 0 \), take \( \delta = \epsilon \) in the definition of the continuity)

**EXAMPLE** Calculate the derivative of \( f(x) = x^3 \) at \( x_o \).

Here

\[
\lim_{h \to 0} \frac{(x_o + h)^3 - x_o^3}{h} = \lim_{h \to 0} \frac{x_o^3 + 3x_o^2h + 3x_o h^2 + h^3 - x_o^3}{h} = \lim_{h \to 0} (3x_o h + 3x_o^2 + h^2) = 3x_o^2.
\]

Thus \( f(x) = x^3 \) is differentiable at each \( x \in \mathbb{R} \).

In practical problems, we don’t calculate derivatives using the definition. Rather we use rules of calculus.

**Some elementary formulas**

**Theorem 1.2** Let \( f, g : I \to \mathbb{R} \) be defined on an open interval \( I \) and differentiable at \( x \in I \). Let \( x_o \in \mathbb{R} \). Then the functions \( \alpha f, f + g, f \cdot g \) and \( \frac{f}{g} \) (provided \( g \neq 0 \)) are differentiable at \( x \). Moreover,

(i) \( (\alpha f)'(x) = \alpha f'(x) \),

(ii) \( (f + g)'(x) = f'(x) + g'(x) \),

(iii) \( (f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x) \),

(iv) \( \left( \frac{f(x)}{g(x)} \right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \).

**Proof** (iv) We write

\[
\frac{1}{h} \left[ \frac{f(x + h)}{g(x + h)} \right] = \frac{f(x)}{g(x)g(x + h)} = \frac{1}{h} \left[ \frac{f(x + h)g(x) - g(x + h)f(x)}{g(x)} \right] = \frac{1}{h} \left[ \frac{f(x + h)g(x) - f(x)g(x) + f(x)g(x) - g(x + h)f(x)}{g(x + h)g(x)} \right]
\]

\[
= \frac{1}{h} \left[ \frac{[f(x + h) - f(x)]g(x) - [g(x + h) - g(x)]f(x)}{g(x + h)g(x)} \right].
\]

The result follows by letting \( h \to 0 \). \( \Box \)

**Theorem 1.3** (Chain rule) Let \( f : I \to J \) and \( g : J \to \mathbb{R} \), where \( I \) and \( J \) are open intervals. Suppose that \( f \) is differentiable at \( c \in I \) and that \( g \) is differentiable at \( f(c) \). Then the composite function \( g \circ f : I \to \mathbb{R} \) defined by \( g \circ f(x) = g(f(x)) \) is differentiable at \( c \) and

\[
(g \circ f)'(c) = g'(f(c))f'(c).
\]
Proof Write $d = f(c)$. Define functions $\alpha : I \to \mathbb{R}$ and $\beta : J \to \mathbb{R}$ by

$$
\alpha(x) = \begin{cases} 
    \frac{f(x) - f(c)}{x - c} - f'(c), & x \neq c, \\
    0, & x = c
\end{cases}
$$

and

$$
\beta(x) = \begin{cases} 
    \frac{g(y) - g(d)}{y - d} - g'(d), & y \neq d, \\
    0, & y = d.
\end{cases}
$$

By the definition of the derivative $\lim_{x \to c} \alpha(x) = 0$ and $\lim_{y \to d} \beta(y) = 0$. Making substitution $y = f(x)$ we have

$$
g \circ f(x) = g \circ f(c) = g(y) - g(d) = (y - d)(g'(d) + \beta(y))
$$

$$
= (f(x) - f(c))(g'(f(c)) + \beta(f(x)))
$$

$$
= (x - c)(f'(c) + \alpha(x))(g'(f(c)) + \beta(f(x))).
$$

Hence

$$
\frac{(g \circ f(x) - g \circ f(c)}{x - c} = (f'(c) + \alpha(x))(g'(f(c)) + \beta(f(x)))
$$

and letting $x \to c$ we get $(g \circ f)'(c) = f'(c)g'(f(c))$. \qed

2 LOCAL EXTREMA

DEFINITION Let $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}$. We say that $f$ has a local maximum (local minimum) at $c$ if there exists a neighbourhood $U$ of $c$ such that $f(x) < f(c)$ ($f(x) > f(c)$) for every $x \in U$. If $f$ has either a local maximum or a local minimum at $c$, we say that $f$ has a local extremum at $c$.

The next theorem gives a necessary, but not sufficient, condition that a local extremum exists at a given point.

Theorem 2.1 Let $a < c < b$ and $f : [a, b] \to \mathbb{R}$ be given. If $f$ has a local extremum at $c$ and $f'(c)$ exists, then $f'(c) = 0$.

Proof We suppose that $f$ has a local maximum at $c$ (otherwise consider $-f$). There exists a $\delta > 0$ such that $a \leq c - \delta < c + \delta \leq b$ and $f(x) \leq f(c)$ whenever $|x - c| < \delta$. Thus, the difference quotient

$$
q(x) = \frac{f(x) - f(c)}{x - c}
$$

satisfies

$$
q(x) \geq 0 \text{ if } c - \delta < x < c,
$$

$$
q(x) \leq 0 \text{ if } c < x < c + \delta.
$$

Since $q(x)$ has the limit $f'(c)$ as $x \to c$, it follows $f'(c) = 0$. \qed
Remark 2.2 (a) It is important that $c$ is not an endpoint of $[a, b]$. For instance the function $f(x) = \sqrt{x}$ on $[0, 1]$ has a local minimum at 0 and a local maximum at 1, $f'(0) = \infty$ and $f'(1) = \frac{1}{2}$.
(b) The function $f(x) = x^3$ on $(-1, 1)$ satisfies $f'(0) = 0$ but does not have a local extremum at 0.
(c) Theorem assures that if we are seeking all local extrema of a differentiable function on an open interval, then we need only consider, as candidates, those $c$ for which $f'(c) = 0$.
(d) If $f(x) = |x|$ for $x \in \mathbb{R}$, then $f$ has a local minimum at $c = 0$, but $f'(0)$ does not exist.

3 MEAN VALUE THEOREMS

Theorem 3.1 (Rolle’s theorem) Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on $(a, b)$ and $f(a) = f(b)$. Then there exists a number $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Proof If $f(x) = f(a)$ for every $x \in [a, b]$, then any $\xi \in (a, b)$ will suffice. Thus, we suppose that $f(x_0) > f(a)$ for some $x_0 \in (a, b)$ (if $f(x) \leq f(a)$ for every $x \in [a, b]$, consider $-f$ instead of $f$). There exists $\xi \in (a, b)$ such that $f(\xi) = \max_{x \in [a, b]} f(x)$ and consequently $f'(\xi) = 0$. \[\square\]

Theorem 3.2 (Lagrange) If $[a, b] \rightarrow \mathbb{R}$ is continuous and differentiable on $(a, b)$, then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof Let

$$\phi(x) = f(x) - f(a) - (x - a)\frac{f(b) - f(a)}{b - a}$$

and apply Rolle’s theorem. \[\square\]

Corollary 3.3 If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f'(x) = 0$ for every $x \in (a, b)$, then $f$ is constant.

Proof Applying mean value theorem to $f$ on $[z, x]$ gives a point $c$ such that $f(x) - f(a) = f'(c)(x - a) = 0$, so that $f(x) = f(a)$ for every $x \in [a, b]$, and therefore $f$ is constant. \[\square\]

EXAMPLE Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable and $f'(x) \leq M$. Prove that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in (a, b)$.

By the mean value theorem

$$f(x) - f(y) = f'(c)(x - y)$$
for some $c \in (x, y)$. Taking the absolute values gives the result.

To see how powerful the mean value theorem is, consider a function $f : [a, b] \to \mathbb{R}$ that is differentiable on $(a, b)$ and continuous on $[a, b]$ and suppose that $f'(x) \geq 0$ for every $x \in (a, b)$. Then for $x_1, x_2 \in (a, b)$, with $x_1 < x_2$, we can use the mean value theorem on $f$ restricted to $[x_1, x_2]$. We find that there is some $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. But $f'(c) \geq 0$ by our assumption. Thus $f(x_2) \geq f(x_1)$. We have shown that if $x_1, x_2 \in (a, b)$ and $x_1 < x_2$, then $f(x_1) \leq f(x_2)$. Thus $f$ is increasing on $[a, b]$. This proves (i) of the next theorem.

**Theorem 3.4** Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

(i) If $f'(x) \geq 0$ for every $x \in (a, b)$, then $f$ is increasing on $[a, b]$.

(ii) If $f'(x) \leq 0$ for every $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

(iii) If $f'(x) > 0$ for every $x \in (a, b)$, then $f$ is strictly increasing on $[a, b]$.

(iv) If $f'(x) < 0$ for every $x \in (a, b)$, then $f$ is strictly decreasing on $[a, b]$.

(v) If $f'(x) = 0$ for every $x \in (a, b)$, then $f$ is constant on $[a, b]$.

**Theorem 3.5** Suppose that $f$ is continuous on $[a, b]$ and is twice differentiable on $(a, b)$, and that $x_o \in (a, b)$.

(i) If $f'(x_o) = 0$ and $f''(x_o) > 0$, then $x_o$ is a strict local minimum of $f$.

(ii) If $f'(x_o) = 0$ and $f''(x_o) < 0$, then $x_o$ is a strict local maximum of $f$.

**Proof** By the definition of the second order derivative we have

$$f''(x_o) = \lim_{x \to x_o} \frac{f'(x) - f'(x_o)}{x - x_o} = \lim_{x \to x_o} \frac{f'(x)}{x - x_o}.$$  

Let $\epsilon > 0$ be such that $f''(x_o) - \epsilon > 0$. Then there is a $\delta > 0$ such that

$$f'(x) > f''(x_o) - \epsilon > 0$$

for $x_o - \delta < x < x_o$ and $x_o < x < x_o + \delta$. Thus, $f'(x) < 0$ for $x_o - \delta < x < x_o$ and $f'(x) > 0$ for $x_o < x < x_o + \delta$. This means that $f(x) > f(x_o)$ if $x_o - \delta < x < x_o$ and $f(x) < f(x_o)$ if $x_o < x < x_o + \delta$. \qed