Lecture 7

1 INTEGRALS OF FUNCTIONS OF ONE VARIABLE

**Definition** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a bounded function. We partition \([a, b]\), which means we choose an integer \( n \) and points \( x_0, x_1, \ldots, x_{n-1}, x_n = b \) in such a way that \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \). Denote such a partition by \( P \), that is, let \( P = \{x_0, x_1, \ldots, x_n\} \). Then form two sums

\[
U(f, P) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i), \quad \text{where } M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)
\]

and

\[
L(f, P) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i), \quad \text{where } m_i = \inf_{x \in [x_i, x_{i+1}]} f(x),
\]

called the upper and lower sums, respectively (upper Riemann sum and lower Riemann sum).

Since \( f \) is bounded, say \(-M \leq f(x) \leq M\) for every \( x \in [a, b] \), we see that

\[
-(b - a)M \leq L(f, P) \leq U(f, P) \leq (b - a)M
\]

for every partition \( P \) of \([a, b]\).

It seems reasonable to expect that as the size of the intervals in \( P \) gets smaller, \( U(f, P) \) decreases while \( L(f, P) \) increases.

**Definition** If \( P \) and \( P' \) are partitions of \([a, b]\) with \( P \subset P' \), then \( P' \) is called a refinement of \( P \).

**Lemma 1.1** If \( P' \) is a refinement of \( P \), then \( L(f, P) \leq L(f, P') \) and \( U(f, P) \geq U(f, P') \).

According to the inequality (1.1) Riemann sums are bounded, therefore we can introduce the following notation

\[
\int_a^b f(x) \, dx = \inf\{U(f, P); P \text{ is any partition of } [a, b]\},
\]
the upper Riemann integral, and
\[\int_a^b f(x) \, dx = \sup\{L(f, P); P \text{ is any partition of } [a, b]\},\]
the lower Riemann integral.

**Lemma 1.2** Let \( P_1 \) and \( P_2 \) be any partitions of \([a, b]\). Then \( L(f, P_1) \leq U(f, P_2) \).

**Proof** Indeed, let \( P = P_1 \cup P_2 \), so \( P \) is refinement of both. Hence
\[L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).\]

\[\square\]

**Corollary 1.3**
\[\int_a^b f(x) \, dx \leq \int_a^b f(x) \, dx.\]

**Definition** We say that \( f : [a, b] \to \mathbb{R} \) is Riemann integrable (or just integrable or that the integral exists) if
\[\int_a^b f(x) \, dx = \int_a^b f(x) \, dx.\]
The common value is denoted by \( \int_a^b f(x) \, dx \).

**Theorem 1.4** A function \( f : [a, b] \to \mathbb{R} \) is integrable on \([a, b]\) if given any \( \epsilon > 0 \) there is a partition \( P \) such that
\[U(f, P) - L(f, P) < \epsilon.\]

**Proof** Given \( \epsilon > 0 \), let \( P \) be a partition of \([a, b]\) such that
\[U(f, P) - L(f, P) < \epsilon.\]
Then
\[0 \leq \int_a^b f(x) \, dx - \int_a^b f(x) \, dx \leq U(f, P) - L(f, P) < \epsilon.\]
Since \( \epsilon > 0 \) is arbitrary the result follows. \(\square\)
EXAMPLES

(1) Compute $\int_0^1 x \, dx$.

Let $f(x) = x$ for $x \in [a, b]$. Define a partition $P_n$ by $x_i = \frac{i}{n}$, $i = 0, \ldots, n$. Then

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = \frac{i - 1}{n}, \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = \frac{i}{n},$$

and

$$L(x, P_n) = \sum_{i=1}^{n} \left( \frac{i - 1}{n} \right) \left( \frac{i}{n} - \frac{i - 1}{n} \right) = \sum_{i=1}^{n} \frac{i - 1}{n} \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} (i - 1)$$

$$= \frac{1}{n^2} \frac{n(n - 1)}{2} = \frac{n - 1}{2n} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty,$$

$$U(x, P_n) = \sum_{i=1}^{n} \frac{i}{n} \left( \frac{i}{n} - \frac{i - 1}{n} \right) = \sum_{i=1}^{n} \frac{i}{n} \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \frac{n(n + 1)}{2}$$

$$= \frac{n + 1}{2n} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty.$$

Hence $\int_0^1 x \, dx \geq \frac{1}{2}$, since $\int_0^1 x \, dx$ is an upper bound for the lower sums and $\int_0^1 x \, dx \leq \frac{1}{2}$ since $\int_0^1 x \, dx$ is a lower bound for upper sums. Therefore

$$\frac{1}{2} \leq \int_0^1 x \, dx \leq \int_0^1 x \, dx \leq \frac{1}{2}$$

and hence

$$\int_0^1 x \, dx = \int_0^1 x \, dx = \frac{1}{2}.$$

(2) Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational}, \\ 0 & \text{if } x \text{ is rational}. \end{cases}$$

We observe that on any interval, $f$ is always 1 at some points and 0 at others, so that $U(f, P) = 1$ and $L(f, P) = 0$ for every partition $P$ of the interval $[0, 1]$. This means that

$$\int_0^1 f(x) \, dx = 0 \quad \text{and} \quad \int_0^1 f(x) \, dx = 1$$

and $f$ is not integrable.
Theorem 1.5  
(i) If $f : [a, b] \to \mathbb{R}$ is bounded and continuous at all but finitely many points of $[a, b]$, then $f$ is integrable on $[a, b]$.

(ii) Any increasing (decreasing) function on $[a, b]$ is integrable on $[a, b]$.

Proof  (i) (Only for continuous functions) In this case $f$ is uniformly continuous on $[a, b]$. Given $\epsilon > 0$ we can find $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b - a} \quad \text{whenever} \quad |x - y| < \delta.$$ 

Let $P$ be a partition consisting of points $x_0 = a < x_1 < \ldots < x_m = b$ such that $x_i - x_{i-1} < \delta$. Then

$$U(f, P) - L(f, P) = \sum_{i=1}^{m} (M_i - m_i)(x_i - x_{i-1}) \leq \frac{\epsilon}{b - a} \sum_{i=1}^{m} (x_i - x_{i-1})$$

$$= \frac{\epsilon (b - a)}{\epsilon (b - a)} = \epsilon.$$ 

(ii) If $f$ is increasing on $[a, b]$, then $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Thus

$$L(f, P) = \sum_{i=1}^{m} f(x_{i-1})(x_i - x_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{m} f(x_i)(x_i - x_{i-1}).$$

Pick a partition $P$ such that all subintervals have a length $\frac{b-a}{n}$. Then

$$U(f, P) - L(f, P) = \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$

$$= \frac{b-a}{n} (f(x_n) - f(x_0)) = \frac{b-a}{n} (f(b) - f(a)).$$

Choose $n > \frac{(b-a)(f(b)-f(a))}{\epsilon}$. Then

$$U(f, P) - L(f, P) < \epsilon$$

so $f$ is integrable on $[a, b]$.  

\[\square\]
2 PROPERTIES OF INTEGRALS

Theorem 2.1  (i) If is integrable on $[a, b]$ and $k \in \mathbb{R}$, then $k \cdot f$ is integrable on $[a, b]$ and
\[ \int_a^b k \cdot f(x) \, dx = k \int_a^b f(x) \, dx. \]
(ii) If $f$ and $g$ are integrable on $[a, b]$, then $f + g$ is integrable on $[a, b]$ and
\[ \int_a^b (f + g) \, dx = \int_a^b f \, dx + \int_a^b g \, dx. \]
(iii) If $f$ and $g$ are integrable on $[a, b]$ and $f(x) \leq g(x)$ for every $x \in [a, b]$, then
\[ \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx. \]
(iv) If is integrable on $[a, b]$ and $[b, c]$, then $f$ is integrable on $[a, c]$ and
\[ \int_a^c f \, dx = \int_a^b f \, dx + \int_b^c f \, dx. \]

Theorem 2.2 (Mean value theorem) If $f$ is continuous on $[a, b]$, then
\[ \int_a^b f(x) \, dx = f(c)(b - a) \]
for some $c \in [a, b]$.

Proof  Let
\[ m = \min\{f(x); a \leq x \leq b\} = f(x_m) \text{ for some } x_m \in [a, b], \]
\[ M = \max\{f(x); a \leq x \leq b\} = f(x_M) \text{ for some } x_M \in [a, b]. \]
Since $m \leq f(x) \leq M$ for every $x \in [a, b]$, we have
\[ m \leq \frac{\int_a^b f(x) \, dx}{b - a} \leq M. \]
Then by the Intermediate Value Theorem applied to $f$ on $[a, b]$ with $\frac{1}{b-a} \int_a^b f(x) \, dx$, there exists $c \in [a, b]$ such that
\[ f(c) = \frac{1}{b - a} \int_a^b f(x) \, dx. \]
\[ \square \]