MATH4405
PROBLEM SHEET 2

1) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. If C is the set of all points at which f is continuous, then C is a G_δ set.

2) Let $(X, \mathcal{A})$ be a measure space. Show that if $\{\mu_n\}$ is an increasing sequence of measures on $(X, \mathcal{A})$ (here "increasing" means that $\mu_n(A) \leq \mu_{n+1}(A)$ holds for each A and each n), then the formula $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ defines a measure on $(X, \mathcal{A})$.

3) Let $\{x_n\}$ be a sequence of real numbers, and define a measure on $(X, \mathcal{B}(\mathbb{R}))$ by $\mu = \sum_n \delta_{x_n}$. Show that μ assigns finite values to the subintervals of $\mathbb{R}$ if and only if $\lim_{n\to\infty} |x_n| = \infty$.

4) Let μ be a measure on $(X, \mathcal{A})$, and let $\{A_n\}$ be sequence of $\mathcal{A}$-measurable sets such that $\sum_n \mu(A_n) < \infty$. Show that the set of points that belong to A_k for infinitely many values of k has measure 0 under μ.

5) Let $(X, \mathcal{A}, \mu)$ be a measure space, and define $\mu^* : \mathcal{A} \to [0, \infty]$ by

$$\mu^*(A) = \sup\{\mu(B);\ B \subset A,\ B \in \mathcal{A}, \mu(B) < \infty\}.$$

(1) Show that μ^* is a measure on $(X, \mathcal{A})$.

(2) Show that if μ is σ-finite, then $\mu^* = \mu$.

(3) Find μ^* if X is non-empty and μ is the measure defined by

$$\mu(A) = \begin{cases} +\infty & \text{if } A \in \mathcal{A} \text{ and } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$