MATH4405
PROBLEM SHEET 3

1) If the set $E \subset \mathbb{R}$ has a Lebesgue measure zero, then the set $\{x^2;\, x \in E\}$ has also measure zero.

2) If $E \subset \mathbb{R}$ is a Lebesgue measurable set with finite measure, then there exists a measurable set $A \subset E$ such that $\lambda(A) = \frac{\lambda(E)}{2}$.

3) Let $A \subset \mathbb{R}$. Show that if $f : E \to \mathbb{R}$ is (Lebesgue measurable), then for each $r \in \mathbb{R}$, the set $\{x \in E;\, f(x) = r\}$ is measurable. The converse is false.

Hint. To show that the converse is not valid, consider a nonmeasurable set $A \subset (0,1)$ and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -x & \text{if } x \notin A \\ x & \text{if } x \in A. \end{cases}$$

4) Find the completion of $\mathcal{B}(\mathbb{R})$ under the point mass concentrated at 0.

5) Prove that under Lebesgue measure on $\mathbb{R}^2$

(1) every straight line has a measure zero, and

(2) every circle has a measure zero.