MATH4405
PROBLEM SHEET 4

1) Show that there is a Lebesgue measurable subset of $\mathbb{R}^2$ whose projection on $\mathbb{R}$ under the map $(x, y) \to x$ is not Lebesgue measurable.

2) Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere on $\mathbb{R}$, then its derivative f' is Borel measurable.

3) Let $(X, \mathcal{A})$ be measurable space and let $A \in \mathcal{A}$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone and $g : A \to \mathbb{R}$ is measurable, then $f \circ g$ is measurable. (Here $(f \circ g)(x) = f(g(x))$ for $x \in A$).

4) Let $\{x_n\}$ be a sequence of real numbers, and define μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by $\mu = \sum_m \delta_{x_m}$. Show that functions f, $g : \mathbb{R} \to \mathbb{R}$ agree μ-almost everywhere if and only if $f(x_m) = g(x_m)$ holds for each m.

5) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, \infty]$ be $\mathcal{A}$-measurable.

(a) Show that if each value of f is a nonnegative integer or $+\infty$, then $\int f \, d\mu = \sum_{n=1}^{\infty} \mu(\{x;\ f(x) \geqq n\})$.

(b) Now suppose that the values of f are arbitrary elements of $[0, \infty]$ and that μ is finite. Show that f is integrable if and only if the series $\sum_{n=1}^{\infty} \mu(\{x \in X;\ f(x) \geqq n\})$ has a finite sum.