

Assignment Number 1

Problem 1 10 marks

Let X and Y be topological spaces, $I = [0, 1]$. Maps $f, g \in C(X, Y)$ are called *homotopic* (write: $f \simeq g$), if there exists a *homotopy* between them, i.e. $F \in C(I \times X, Y)$ with $F(0, \cdot) = f(\cdot)$, $F(1, \cdot) = g(\cdot)$.

a) Show that $\simeq$ defines an equivalence relationship on $C(X, Y)$.

b) Consider $A \subseteq X$. f and g in $C(X, Y)$ with $f|_A = g|_A$ are called *homotopic relative to A* (write: $f \simeq_A g$, or $f \simeq g \text{ rel } A$), if there exists a homotopy F between f and g which further satisfies: $F(\cdot, a) = f(a) (= g(a))$ for all $a \in A$. Show that $\simeq_A$ is also an equivalence relationship on $C(X, Y)$.

c) Now take $X = I$, $A = \partial I$. A *path* in Y is a map in $C(I, Y)$. The composition (concatination) of two paths f and g in Y is defined by

$$(f * g)(t) = \begin{cases} f(2t) & \text{for } 0 \leq t < \frac{1}{2} \\ g(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Obviously, $f * g$ is a path iff: $f(1) = g(0)$. Consider paths f_1, f_2, g_1 and g_2 in Y with $f_1 \simeq_{\partial I} f_2, g_1 \simeq_{\partial I} g_2$. Show that $f_1(1) = g_1(0)$ implies: $f_1 * g_1 \simeq_{\partial I} f_2 * g_2$.

Problem 2 10 marks

Let X be a topological space, and set $I := [0, 1]$. The *inverse* of a path f in X is defined by

$$f^{-1}(t) = f(1 - t) \quad t \in I.$$

The *constant path* in $p \in X$ is defined via $i_p(t) = p$ for all $t \in I$.

a) Does there always hold $f * f^{-1} = f^{-1} * f$?

b) Let f be a path in X with $f(0) = f(1) = p$, (a *loop with basepoint p*). Does there hold $f * f^{-1} = f^{-1} * f = i_p$?

c) The *path class* of a path f , $[f]$, is the equivalence class of f w.r.t. $\simeq_{\partial I}$. In particular, $[f](0), [f](1)$ are well defined.

Fix $p \in X$. Show that the set

$$\{[f] : [f] \text{ is a path class with } [f](0) = [f](1) = p\}$$

is a group with the binary operation $[f][g] := [f * g]$, identity $[i_p]$ and $[f]^{-1} := [f^{-1}]$. The group is called the *fundamental group of x with basepoint p* , and is denoted $\pi_1(X, p)$.

Problem 3 10 marks

a) Let X be a topological space, $x_0 \in X$, and let X_0 be the path-component of X which contains x_0 . Show that the groups $\pi_1(X, x_0)$ and $\pi_1(X_0, x_0)$ are isomorphic. (Hint: The inclusion $j : X_0 \rightarrow X$ induces a map $j_* : \pi_1(X_0, x_0) \rightarrow \pi_1(X, x_0)$ via $[f] \mapsto [jf]$. Show that j_* is an isomorphism.)

b) Let X be a path-connected topological space, with $x_0, x_1 \in X$. Show that the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. (This means that one can speak of $\pi_1(X)$ in this situation.)

Due: Thursday, 17/09/2009, 11 AM