

Benchmarks of Generalized Hydrodynamics for 1D Bose Gases

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Generalized hydrodynamics (GHD) is a recent theoretical approach that is becoming a go-to tool for characterizing out-of-equilibrium phenomena in integrable and near-integrable quantum many-body systems. Here, we benchmark its performance against an array of alternative theoretical methods, for an interacting one-dimensional Bose gas described by the Lieb-Liniger model. In particular, we study the evolution of both a localized density bump and dip, along with a quantum Newton’s cradle setup, for various interaction strengths and initial equilibrium temperatures. We find that GHD generally performs very well at sufficiently high temperatures or strong interactions. For weak interactions and low temperatures, we highlight situations where GHD, while not capturing interference phenomena on short lengthscales, can describe a coarse-grained behaviour based on convolution averaging that mimics finite imaging resolution in ultracold atom experiments. In the quantum Newton’s cradle setup, we find that GHD performs well at early to intermediate times, but predicts a much slower thermalization rate compared to the predictions of c -field approaches.

Introduction.—The study of dynamics of integrable and near-integrable quantum many-body systems has been a thriving area of research for more than a decade since the landmark experiments on relaxation in the quantum Newton’s cradle setup [1] and in coherently split one-dimensional (1D) Bose gases [2]. During this time, an in-depth understanding of the mechanisms of thermalization and emergent out-of-equilibrium phenomena within these systems has been developed [3–8]. A recent breakthrough in this area has been the discovery of the theory of generalized hydrodynamics (GHD) [9, 10] (for recent reviews, see [11, 12]). This new theory is capable of simulating large-scale dynamics of integrable and near-integrable systems across a significantly broader range of particle numbers and interaction strengths than those accessible using previous approaches [13, 14]. Because of its broad applicability, GHD is currently regarded as well on its way to becoming “a standard tool in the description of strongly interacting 1D quantum dynamics close to integrable points” [15].

In the years since its discovery, GHD has been rapidly developed to include diffusive terms [16–21], particle loss [22], calculations of quantum and Euler-scale correlations [23–28], as well as the incorporation of numerous beyond-Euler scale effects [29–32] (for other recent developments, see Refs. [33–36] in a special issue). Recently, GHD applied to a 1D Bose gas has been experimentally verified in a variant of the quantum Newton’s cradle setup in the weakly interacting regime [14], and in a harmonic trap quench in the strongly interacting regime [15]. In both cases, GHD provided an accurate coarse-grained model of the dynamics, exceeding conventional (classical) hydrodynamics. In addition to comparisons with experiments, GHD was benchmarked against alternative established theoretical approaches—most prominently for the 1D Bose gas and XXZ spin chain [9, 10, 13, 14, 23, 26, 32, 37–41]. As the purpose of these initial benchmarks was to validate GHD, the typical dynamical scenarios considered were in regimes where GHD was expected to be a valid theory. In all such cases GHD has indeed been validated by showing very good agreement with the alternative approaches. On the other hand, in scenarios involving, for example, short wavelength

density oscillations due to interference phenomena (which are not captured by GHD), it was conjectured that GHD would nevertheless adequately describe spatial coarse-grained averages of the more accurate theories [13, 14, 31]. More generally, it is of significant interest to scrutinise the performance of GHD by extending its benchmarks to a more challenging set of dynamical scenarios. This is important for understanding how exactly GHD breaks down when it is pushed towards and beyond the limits of its applicability.

In this Letter, we systematically benchmark the performance of GHD for the 1D Bose gas in several paradigmatic out-of-equilibrium scenarios. In particular, we focus on the regime of dispersive quantum shock waves emanating from a localized density bump of the type explored recently in Ref. [42]. We use an array of theoretical approaches, including finite temperature c -field methods, the truncated Wigner approximation, and the exact infinite matrix product state (iMPS) method, spanning the entire range of interaction strengths, from the nearly ideal Bose gas to the strongly interacting Tonks-Girardeau regime. We also analyse the dynamics of a localized density dip which sheds grey solitons, hence benchmarking GHD in scenarios not previously considered. In doing so we address the question of how well GHD predictions agree with coarse-grained averaging of the results of the more accurate theoretical approaches. Additionally, we explore the dynamics of a thermal quasicondensate in a quantum Newton’s cradle setup [14, 43, 44] using Navier-Stokes type diffusive GHD [16, 19, 45], and address the question of characteristic thermalization rates [18].

Expansion from a localized density bump.—We begin our analysis by considering dispersive quantum shock waves of the type studied recently in Ref. [42]. More specifically, we first focus on the weakly interacting regime of the 1D Bose gas of N particles, and consider the dynamics of the oscillatory shock wave train generated through a trap quench from an initially localized perturbation on top of a flat background to free propagation in a uniform box of length L with periodic boundary conditions [46, 47]. The weakly interacting regime is characterized by the Lieb-Liniger [48, 49] dimensionless

interaction parameter $\gamma_{\text{bg}} = mg/\hbar^2 \rho_{\text{bg}} \ll 1$, defined with respect to the background particle number density, ρ_{bg} , where $g > 0$ is the strength of delta-function contact repulsive interactions and m is the mass of the particles. The initial density profile [see Fig. 1 (a)], in dimensionless units, is set to [50]

$$\bar{\rho}(\xi, \tau=0) = \bar{\rho}_{\text{bg}} (1 + \beta e^{-\xi^2/2\bar{\sigma}^2})^2, \quad (1)$$

where the dimensionless coordinate, time, and density are introduced, respectively, according to $\xi \equiv x/L$, $\tau \equiv \hbar t/mL^2$, and $\bar{\rho}(\xi, \tau) \equiv \rho(x, t)L$, with $\bar{\rho}_{\text{bg}} = \rho_{\text{bg}}L = N_{\text{bg}}$ being the dimensionless background density equivalent to the total number of particles in the background, $N_{\text{bg}} = N/(1 + \frac{\sqrt{\pi}\beta\sigma}{L}[\beta \operatorname{erf}(\frac{L}{2\sigma}) + 2\sqrt{2} \operatorname{erf}(\frac{L}{2\sqrt{2}\sigma})])$ from the normalization. In addition, the width and amplitude of the bump above the background are characterized by the dimensionless parameters $\bar{\sigma} \equiv \sigma/L$ and $\beta > 0$, respectively.

In our first example, we consider the case of a large total number of particles, $N = 2000$, and $\gamma_{\text{bg}} = 0.01$, so that the gas is in the Thomas-Fermi regime where the interaction energy per particle dominates the kinetic energy per particle. We assume that the gas is initialized in the zero-temperature ($T = 0$) ground state of a dimple trap that results in the density profile of Eq. (1). At time $\tau = 0$, the dimple trap is suddenly switched off, and we follow the evolution of the density profile in a uniform trap; in Fig. 1 (a) and (b) we plot snapshots of the evolving density profiles at different times. Here and hereafter the GHD results [51] are represented using black dashed lines, whereas the results of alternative approaches are as per the figure legends. In Figs. 1 (a) and (b), the alternative approaches are the mean field Gross-Pitaevskii equation (GPE) and the truncated Wigner approximation (TWA) which incorporates the effect of quantum fluctuations ignored in the GPE (see [42] for details). The snapshot at $\tau = 0.00014$ corresponds to the onset of a shock front due to the large density gradient, and the agreement between the different approaches is generally good across the entire profile due to the equivalence of their non-dispersive (pre-shock) dynamics at this stage [13, 52–54]. Past this time, the GPE and TWA show the formation of an oscillatory shock wave train, which has been identified in Ref. [42] as a result of self-interference of the expanding density bump with its own background. The interference contrast in this regime is generally large, even though the quantum fluctuations present in the TWA approach cause a visible reduction in contrast compared with the mean-field GPE result. The GHD prediction, on the other hand, completely fails to capture the oscillations, as these occur on a microscopic lengthscale. The characteristic period of oscillations here (which we note are chirped towards the shorter wavelengths) is given approximately by the healing length $l_h = \hbar/\sqrt{mg\rho_{\text{bg}}}$ ($l_h/L = 0.0057$) which is smaller than the width σ ($\sigma/L = 0.02$) of the initial bump and hence represents the shortest lengthscale of the problem in the bulk of the shock wave train. Thus, even though the local density approximation (required for GHD to be applicable to an inhomogeneous system in the first place) is valid for the initial Thomas-

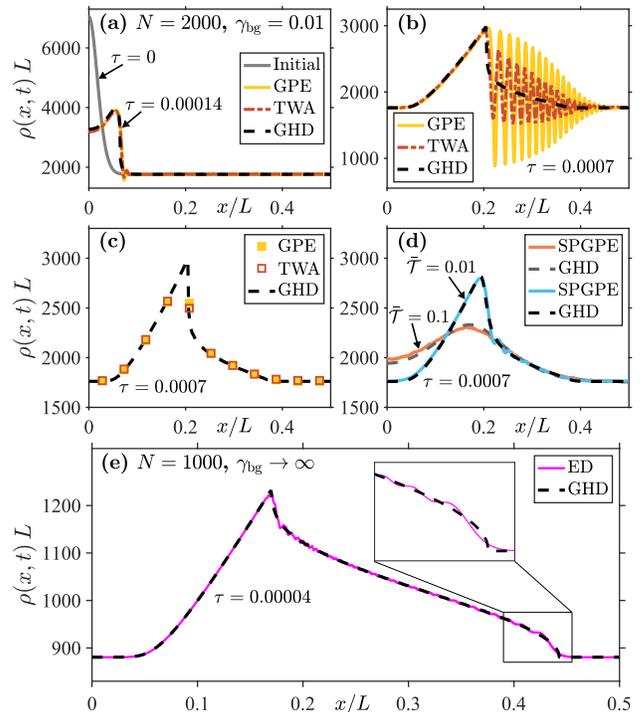


FIG. 1. Density profiles of quantum shock waves in the 1D Bose gas. In (a), we show the initial ($\tau = 0$) and time-evolved ($\tau = 0.00014$) density profiles of a weakly interacting gas in the Thomas-Fermi regime at zero temperature, for $\gamma_{\text{bg}} = 0.01$ and $N = 2000$ ($N_{\text{bg}} \simeq 1761$). Due to the symmetry about the origin, we only show the densities for $x > 0$. In (b), the time-evolved profile is shown at $\tau = 0.0007$. Panel (c) demonstrates the results of finite resolution averaging [56] of both GPE and TWA data from (b) and compares them with the same GHD result. Panel (d) shows the same system as in (b), but at finite temperatures, simulated using the stochastic projected GPE (SPGPE) [42]. The dimensionless temperature parameter \bar{T} is defined according to $\bar{T} = T/T_d$, where $T_d = \hbar^2 \rho_{\text{bg}}^2 / 2mk_B$ [49]. Panel (e) compares GHD predictions with exact diagonalization (ED) results in the Tonks-Girardeau regime ($\gamma_{\text{bg}} \rightarrow \infty$) for $N = 1000$ ($N_{\text{bg}} \simeq 884$), at $\tau = 0.00004$. In all examples, the initial profiles are characterized by $\beta = 1$ and $\bar{\sigma} = 0.02$ in Eq. (1).

Fermi density profile, the failure of GHD at later times is expected since it is not supposed to capture phenomena on microscopic lengthscales, which emerge here dynamically [55].

Despite this failure, an intriguing feature in the GHD result for the fully formed shock wave train is that it seems to adequately capture the average density along the oscillations. This is consistent with the expectation that GHD in an interfering region would correspond to a coarse-grained average density [13, 14, 31]. To quantitatively assess this expectation, we perform a type of convolution averaging that mimics the finite resolution of *in-situ* imaging systems used in quantum gas experiments [56]. As the imaging resolution is usually unable to resolve wavelengths on the order of the healing length, one expects that such averaging will smear out the interference fringes seen in the GPE and TWA data—just as GHD implicitly does. In Fig. 1(c) we show the re-

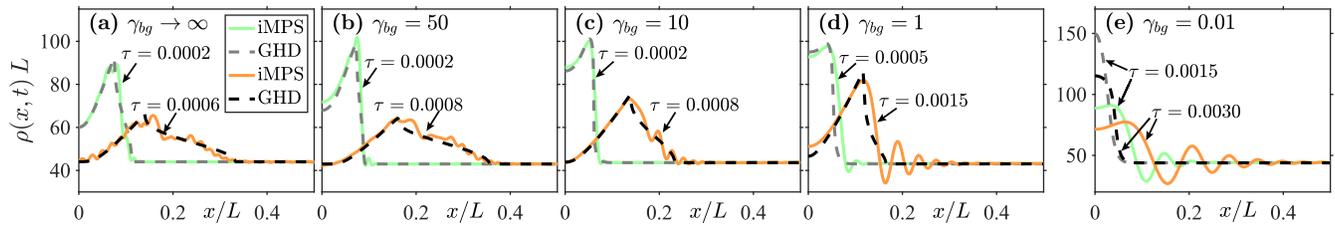


FIG. 2. Quantum shock waves at zero temperature for $N = 50$ particles ($N_{\text{bg}} \simeq 44.03$), over the entire range of interaction strengths. In all examples, the initial density profiles (not shown) closely match Eq. (1) [42], with $\beta = 1$ and $\bar{\sigma} = 0.02$. In all panels, we show the GHD (dashed lines) and iMPS (full lines) results for the evolved density profiles at two time instances. In (e), the shortest length scale in the problem is the width of the initial Gaussian bump σ (with $\bar{\sigma} \equiv \sigma/L = 0.02$), which is much smaller than the healing length l_h ($l_h/L = 0.227$).

sults of convolution-averaged density profiles performed on the GPE and TWA data of Fig. 1(b) and compare them with the same GHD curve. As we see, the agreement between the three curves is now remarkable, thus highlighting the success of GHD in describing coarse-grained averages when interference or short-wavelength phenomena are present.

In our second set of examples, we consider the same shock wave scenario, except this time for a phase fluctuating quasicondensate at finite temperatures. We expect GHD to perform better here as the effect of thermal fluctuations leads to a smearing of the interference contrast due to a reduced thermal phase coherence length in the system, $l_T = \hbar^2 \rho_{\text{bg}} / m k_B T$ [57–59]. We indeed observe such smearing in Fig. 1(d), using finite-temperature c -field SPGPE simulations, and thus very good agreement of GHD with these c -field results.

Our third example is shown in Fig. 1(e) and lies in the Tonks-Girardeau regime of infinitely strong interactions ($\gamma_{\text{bg}} \rightarrow \infty$). This example further illustrates the same observation—that the performance of GHD improves with the loss of phase coherence in the system, wherein interference phenomena are suppressed. Here, we compare the predictions of GHD for the shock wave scenario at $T = 0$ with the results of exact diagonalization. In the Tonks-Girardeau regime, the system does not possess phase coherence beyond the mean interparticle separation $1/\rho_{\text{bg}}$, hence the absence of interference fringes in the evolution of a density bump whose initial width is larger than $1/\rho_{\text{bg}}$ [42]. Accordingly, we see very good agreement of GHD with exact diagonalization, ignoring the small-amplitude density ripples that can be seen in the exact result. Such density ripples (which we note have different origin to Friedel oscillations) have been predicted to occur in the ideal Fermi gas by Bettelheim and Glazman [60] (see also [61]). By the Fermi-Bose mapping [62, 63], these same ripples should emerge in the Tonks-Girardeau gas, which we confirm here through exact diagonalization. However, their description lies beyond the scope of GHD which is a large-scale theory, hence their absence in the GHD result [64].

The final set of examples for the evolution of a density bump is shown in Fig. 2. Here, we consider a range of interaction strengths, starting from very strong and going back [from (a) to (e)] to the weakly interacting regime, all at zero temperature. We compare the GHD results with those of numerically

exact iMPS simulations [42], for $N = 50$. At this relatively low particle number, the strongly interacting regime displays Friedel oscillations which appear in the iMPS result and are, as expected, absent from the prediction of GHD. However, there is generally good agreement between GHD and iMPS at large scale, in the coarse-grained averaging sense. As the interaction strength is reduced, and hence the phase coherence of the gas increases, the Friedel oscillations gradually disappear and interference fringes return, which now have period $\sim \sigma$ (with $\sigma < l_h$) since the gas is no longer in the Thomas-Fermi regime. The worst performance of GHD is observed for $\gamma_{\text{bg}} = 0.01$, which lies in the nearly ideal (noninteracting) Bose gas regime for $N = 50$ particles. In this regime, the local density approximation [49] intrinsic to GHD [13–15] is no longer valid even for the initial density profile, which explains the failure of GHD to agree with iMPS results even in the coarse-grained averaging sense.

Dynamics of a localized density dip.—In this section, we consider the evolution of a localized density depression, after quenching (at time $t = 0$) a suitably tailored trap potential (with a localized barrier) to uniform. We assume that the initial density profile is given by Eq. (1), with β satisfying $-1 < \beta < 0$. In Figs. 3(a) and (b), we show the results of the GPE, TWA, and GHD simulations at different time instances, for a weakly interacting gas with $N = 1688$ and the same $N_{\text{bg}} \simeq 1761$ as in Fig. 1(a). In this scenario, the steep gradient of the shock front forms as the background fluid flows inward and tries to fill the density depression. As a result, one first observes the emergence of large-amplitude structures forming multiple density troughs, which then evolve into a train of grey solitons propagating away from the origin [46, 47, 54, 65–67]. The differences between the TWA and pure mean-field GPE results are consistent with previous observations [68–70] that quantum fluctuations lower the mean soliton speed and fill in the soliton core. The GHD result, on the other hand, fails to capture the solitonic structures (which lie beyond its intended range of applicability, as the characteristic solitonic length-scale is given by the microscopic healing length), however, it still manages to provide an approximate coarse-grained description of the average density across the soliton train (see [71] for details).

The next two examples, shown in Figs. 3(c) and (d),

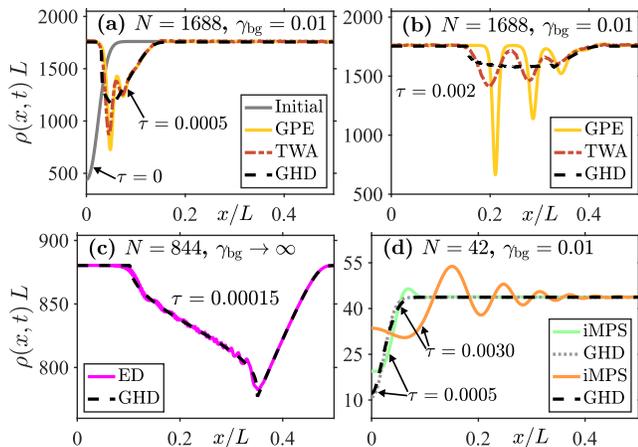


FIG. 3. Evolution of a density dip in a 1D Bose gas. Panels (a) and (b) show the initial and time-evolved density profiles for $\gamma_{\text{bg}} = 0.01$ and $N = 1688$ ($N_{\text{bg}} \simeq 1761$). In panel (c), we show a time-evolved snapshot of the density profile in the Tonks-Girardeau regime, $\gamma_{\text{bg}} \rightarrow \infty$, $N = 844$ ($N_{\text{bg}} \simeq 880.5$), and compare the GHD result with that of exact diagonalization (ED). Panel (d) is in the nearly ideal Bose gas regime, with $\gamma_{\text{bg}} = 0.01$, $N = 42$ ($N_{\text{bg}} \simeq 44$). In all examples, the initial density profile is given by Eq. (1) with $\beta = -0.5$ and $\bar{\sigma} = 0.02$.

correspond, respectively, to the strongly interacting Tonks-Girardeau and the nearly ideal Bose gas regimes. The overall behaviour and conclusions about the performance of GHD in these examples are the same as in the equivalent scenario of the density bump illustrated in Figs. 1 (e) and 2 (a).

Quantum Newton’s cradle in a thermal quasicondensate.— Our final scenario for benchmarking GHD is in a variant of the quantum Newton’s cradle setup for a weakly interacting 1D Bose gas in the quasicondensate regime. We focus our analysis on the release from a symmetric double-well trap to harmonic trap, similar to the type utilized for benchmarking GHD with experiment in Ref. [14]. Here, we use the SPGPE to simulate collisional dynamics and eventual thermalization, as in Ref. [43], and for the sake of one-to-one comparison, we also simulate the same system using the Navier-Stokes type of diffusive GHD [16, 17, 45]. Comparison of the results using the two methods are shown in Fig. 4. While the broad qualitative features of the GHD dynamics are in excellent agreement with those obtained using the SPGPE at short and intermediate times, the GHD dynamics takes a significantly longer time to relax to the final thermal state compared to the SPGPE simulation. The final relaxed state in GHD is assessed from analysis of the total thermodynamic entropy per particle [18, 71], which plateaus upon reaching a thermalized state, whereas in the SPGPE dynamical relaxation is assessed by monitoring the evolution of the peak density averaged over the small region $x/l_{\text{ho}} \in [-2, 2]$ [71].

We have also simulated the original quantum Newton’s cradle scenario of Ref. [1], implemented by splitting the initial quasicondensate into two counter-propagating halves using a Bragg pulse, and observed a similar difference in thermalization times [71]. Understanding the reason for these discrepan-

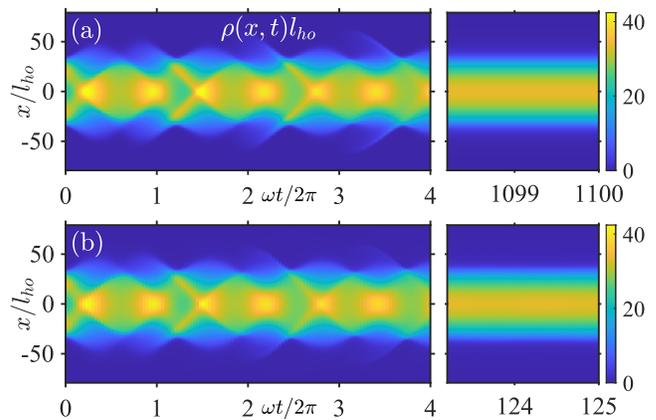


FIG. 4. Evolution of the density distribution $\rho(x, t)$ upon thermalization in a quantum Newton’s cradle setup initialized from a double-well to single-well trap quench, simulated using (a) diffusive GHD and (b) SPGPE. The initial cloud of $N = 1824$ atoms is prepared in a thermal equilibrium state of a symmetric double-well trap potential $\tilde{V}(\tilde{x}) = 6.3 \times 10^{-4} \tilde{x}^4 - 2.7 \times 10^{-1} \tilde{x}^2$, and is characterized by a dimensionless interaction strength at maximum density $\gamma_{\text{max}} = 0.04$ and dimensionless temperature $\tilde{T} = T/(\hbar\omega/k_B) = 23.3$, where we use harmonic oscillator units, with $\tilde{x} = x/l_{\text{ho}}$, $\tilde{V} = V/\hbar\omega$, and $l_{\text{ho}} = \sqrt{\hbar/m\omega}$, where ω is the post-quench harmonic trap frequency.

cies warrants a further study.

Summary.— We have benchmarked GHD in a variety of out-of-equilibrium scenarios in a 1D Bose gas against alternative theoretical approaches which are not limited to long-wavelength excitations. In particular, we have focused on systems supporting dispersive quantum shock waves and soliton trains, demonstrating that GHD generally agrees with the predictions of these approaches at sufficiently high temperatures and strong interactions. Here, the good agreement stems from a reduced phase coherence length of the gas, which in turn leads to a suppression of interference phenomena and therefore an absence of high-contrast short-wavelength interference fringes in the density. At low temperatures and weak interactions, where interference phenomena are more pronounced, the predictions of GHD only agree with a coarse-grained convolution averaging approximation. The effect of such averaging is similar to having finite imaging resolution in quantum gas experiments, and explains why GHD may perform well when compared to experiments, whilst departing from the predictions of theoretical approaches that are valid at short wavelengths. We have also benchmarked diffusive GHD in two quantum Newton’s cradle setups for a finite-temperature quasicondensate. While the transient dynamics compare well with c -field simulations, the long-time dynamics of GHD leads to slower rates of thermalization. The reason for the slower thermalization in diffusive GHD is currently not understood.

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Navier-Stokes scales, along with the finite-resolution averaging method and its application to the density dip. Additionally, further analysis of the quantum Newton's cradle is presented, including the case of a Bragg pulse protocol.

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Supplemental Material for: Benchmarks of Generalized Hydrodynamics for 1D Bose Gases

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I. GENERALIZED HYDRODYNAMICS OF THE ONE-DIMENSIONAL BOSE GAS

We briefly review the various formulations of generalized hydrodynamics (GHD) used in the main text and its relation to the Lieb-Liniger model of the 1D Bose gas. In particular we cover Euler-scale GHD, its zero-entropy subspace method, and its relation to the Wigner function formulation of the free Fermi gas. We also give a brief introduction to Navier-Stokes GHD and numerical tools used to simulate the dynamics of thermalization in Newton's cradle setups.

The one-dimensional Bose gas.—The Lieb-Liniger model of a 1D Bose gas with repulsive contact interaction is a paradigmatic integrable quantum model, described by the following second-quantized Hamiltonian [S1, S2]

$$\hat{H} = -\frac{\hbar^2}{2m} \int dx \hat{\Psi}^\dagger \frac{\partial^2 \hat{\Psi}}{\partial x^2} + \frac{g}{2} \int dx \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi}. \quad (\text{S1})$$

Here, $\hat{\Psi}^\dagger(x)$ and $\hat{\Psi}(x)$ are the boson creation and annihilation field operators, obeying the canonical bosonic commutation relations $[\hat{\Psi}(x), \hat{\Psi}^\dagger(x')] = \delta(x - x')$, m is the bosonic mass, g is the one-dimensional interaction strength which is taken to be positive for repulsive interactions. For a uniform system of 1D density ρ , the exact eigenstates of the Lieb-Liniger model and its ground state properties can be found through the Bethe ansatz [S1], whereas the finite temperature equilibrium properties can be treated using Yang-Yang thermodynamic solutions [S3]. The dimensionless interaction strength that characterises the system is introduced via $\gamma = mg/\hbar^2\rho$.

In the presence of an external trapping potential, $V(x)$, the Hamiltonian acquires an additional term, $\int dx V(x) \hat{\Psi}^\dagger \hat{\Psi}$, and the model becomes generically non-integrable outside of the ideal (noninteracting) Bose gas limit or the Tonks-Girardeau limit of infinitely strong interactions. However, the exact solutions for a uniform gas can be still utilized for finding thermodynamic properties of inhomogeneous gases in the local density approximation (LDA) [S4], wherein the local dimensionless interaction strength $\gamma(x) = mg/\hbar^2\rho(x)$ acquires position-dependence through the density profile $\rho(x)$.

Euler-scale generalized hydrodynamics.—GHD for the 1D Bose gas is expressed using the language of Yang and Yang's thermodynamic Bethe ansatz [S3, S5–S7]. Here we present a brief formulation of first order (or Euler-scale) GHD in terms of the density of quasiparticles, $f_p(\lambda; x, t)$ [S3]. Through $f_p(\lambda; x, t)$ we express the core ‘Bethe-Boltzmann’ equation of Euler-scale GHD in an inhomogeneous external potential [S8, S9],

$$\begin{aligned} \partial_t f_p(\lambda; x, t) + \partial_x (v^{\text{eff}}[f_p(\lambda; x, t)] f_p(\lambda; x, t)) \\ = \frac{\partial_x V(x)}{m} \partial_\lambda f_p(\lambda; x, t), \end{aligned} \quad (\text{S2})$$

where λ is the rapidity or quasi-velocity variable of the quasi-particle excitations. Through the occupation number function, $n(\lambda; x, t)$, related to the density of quasiparticle and quasi-hole excitations, $n = f_p/(f_p + f_h)$ [S3], one may express this hydrodynamic equation of motion in the form [S8, S9]

$$\begin{aligned} \partial_t n(\lambda; x, t) + v^{\text{eff}}[n(\lambda; x, t)] \partial_x n(\lambda; x, t) \\ = \frac{\partial_x V(x)}{m} \partial_\lambda n(\lambda; x, t). \end{aligned} \quad (\text{S3})$$

Though they are equivalent, Eq. (S3) is more convenient for simulating Euler-scale GHD, as it can be solved through the method of characteristics [S10, S11]. The effective velocity $v^{\text{eff}}[n]$ is written in terms of the quasiparticle energy E and momentum p [S6, S7],

$$v^{\text{eff}}[n(\lambda; x, t)] = \frac{(\partial_\lambda E)^{\text{dr}}[n(\lambda; x, t)]}{(\partial_\lambda p)^{\text{dr}}[n(\lambda; x, t)]}, \quad (\text{S4})$$

and is the group velocity, $v^{\text{gr}} = \partial_\lambda E/\partial_\lambda p$, of the excitations renormalized by interactions through the dressing operation [S7, S11],

$$g^{\text{dr}}(\lambda) = g(\lambda) - \int \frac{d\lambda'}{2\pi} \theta(\lambda - \lambda') n(\lambda') g^{\text{dr}}(\lambda'), \quad (\text{S5})$$

where we suppress dependence on space and time variables for clarity. The scattering kernel, $\theta(\lambda)$, is the first of only two model-dependent quantities utilized in GHD, given in the Lieb-Liniger model by $\theta(\lambda) = 2\hbar g/(g^2 + (\hbar\lambda)^2)$ [S1, S3, S12]. The dressing equation may be seen as a generalization of the type of equation present throughout the thermodynamic Bethe ansatz; in particular, one may express the total state density, $f = f_p + f_h$, through the relation $f(\lambda) = \frac{1}{2\pi m} (\partial_\lambda p)^{\text{dr}}(\lambda) = \frac{1}{2\pi} (1)^{\text{dr}}(\lambda)$ [S3, S7].

The second model-dependent quantity are the single-particle eigenvalue functions, $h_i(\lambda)$, $i = 0, 1, 2, \dots$, given in the Lieb-Liniger model by polynomials, $h_i(\lambda) \propto \lambda^i$, with $h_0 = 1$, $h_1(\lambda) = p(\lambda) = m\lambda$ being the quasiparticle momentum, and $h_2(\lambda) = E(\lambda) = m\lambda^2/2$ – the quasiparticle energy, as examples [S1, S3]. Correspondingly, the effective velocity simplifies to $v^{\text{eff}}(\lambda) = \text{id}^{\text{dr}}(\lambda)/1^{\text{dr}}(\lambda)$, where $\text{id}^{\text{dr}}(\lambda)$ is the identity function $\text{id}(\lambda) = \lambda$, dressed according to Eq. (S5), likewise $1^{\text{dr}}(\lambda)$ is the dressed unit function $1(\lambda) = 1$ [S13]. The evaluation of average charge densities takes a simple integral form [S3, S6, S7],

$$\langle q_i \rangle = \frac{m}{\hbar} \int d\lambda f_p(\lambda) h_i(\lambda) = \frac{m}{\hbar} \int \frac{d\lambda}{2\pi} n(\lambda) h_i^{\text{dr}}(\lambda), \quad (\text{S6})$$

where, for example, $\langle q_0 \rangle(x, t) = \rho(x, t)$ is the particle number density, $\langle q_1 \rangle(x, t)$ is the momentum or mass-current density (where, in conventional hydrodynamics,

$\langle q_1 \rangle(x, t)/m\rho(x, t) = v(x, t)$ would correspond to the velocity field), and $\langle q_2 \rangle(x, t) = e(x, t)$ is the energy density.

Evaluation of an initial equilibrium state requires knowledge of the local effective chemical potential, $\mu(x) = \mu_0 - V(x)$, where μ_0 is the global equilibrium chemical potential, simplifying GHD to a local density approximation [S4, S13]. In our GHD simulations of systems evolving from finite-temperature thermal equilibrium states, we used the *iFluid* software package [S14], which is an efficient, easily expandable open-source numerical framework based in Matlab. Simulations of systems evolving from zero-temperature ground states, on the other hand, were carried out using zero-entropy subspace methods found in Ref. [S13] and detailed below.

Zero-entropy subspace methods and the hard-core limit.—At zero temperature the occupation number simplifies to an indicator function, $n_{T=0}(\lambda) = \chi(\lambda \in [-\lambda_F, \lambda_F])$, which is 1 when λ is in the interval $[-\lambda_F, \lambda_F]$, and 0 outside this region, where λ_F is the Fermi rapidity and is generally dependent on the local effective chemical potential [S13]. Thus, dynamics of the occupation number is restricted to its edges, or its ‘Fermi contour’ in systems of finite size [S15, S16]. Numerically solving the GHD equation of motion of these systems may be accomplished through the zero-entropy algorithm presented in the supplementary material of Ref. [S13]. This algorithm works with the finite set of curves defined by the edges of the zero-temperature occupation number, with each point on these curves given by a position and rapidity coordinate (x_i, λ_i) . For a single time-step, δt , the position x_i is shifted by an amount $v_{[\lambda]}^{\text{eff}}(\lambda_i)\delta t$, where calculating the effective velocity given in Eq. (S4) through the dressing operation in Eq. (S5) is simplified due to the form of the zero-temperature occupation number described above.

In the Tonks-Girardeau limit of hard-core bosons, $\gamma \rightarrow \infty$, at zero temperature there is an equivalence between the occupation number and the Wigner function in the semiclassical formulation for the free Fermi gas [S17]. Here, the Fermi rapidity is equivalent to the Fermi pseudovelocity $\lambda_F(x) = \sqrt{2\mu(x)/m}$ [S13]. In this limit, the GHD equation of motion is equivalent to the evolution of the Wigner function, and is given in the form of a Boltzmann equation [S13, S17],

$$\partial_t n(\lambda) + \lambda \partial_x n(\lambda) = 0. \quad (\text{S7})$$

Quantum corrections to the semiclassical Wigner function description were analysed by Bettelheim *et al.*, demonstrating the presence of long-lived ‘quantum ripples’ in the dynamics [S17–S19]. Analysis presented in the supplementary material of [S13] demonstrated that these corrections to the local density approximation are negligible at large scales in the regime where

$$\rho(x)^{-1} \ll \left(\frac{m\lambda_F''(x)}{8\hbar} \right)^{-1/3}. \quad (\text{S8})$$

In this limit, the Wigner function oscillates rapidly as a function of λ such that, when calculating observables using Eq. (S6), the integration over rapidity averages over these fast oscillations, and eliminates the oscillatory terms [S13].

Navier-Stokes GHD.—Higher order corrections to the Euler-scale hydrodynamics presented above have recently

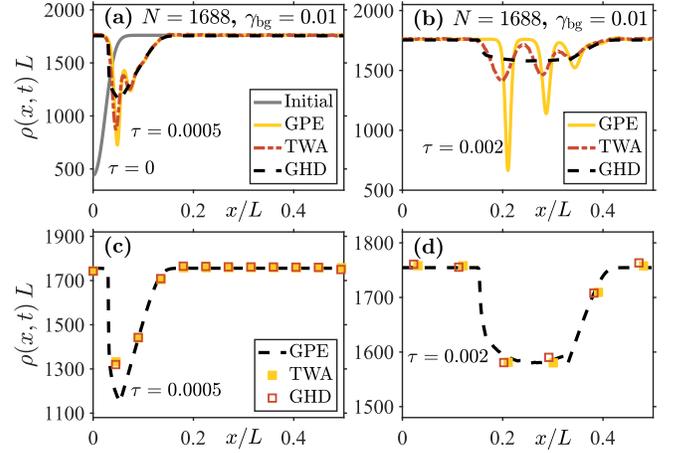


FIG. S1. Finite resolution averaging of the density dip; panels (a) and (b) repeat the results of Fig. 3 in the main text. Panels (c) and (d) show the outcomes of such averaging of both GPE and TWA results, with the same normalized Gaussian resolution function width of $1\mu\text{m}$, used in Fig. 1(c) of the main text. Similarly to Figs. 1 and 3 of the main text, we have ^{87}Rb atoms in mind as an example species for the relevant parameter values (see footnote [41] of the main text.) For panel (c), we use the same pixel size, $\Delta = 4.5\mu\text{m}$, as before, whereas for panel (d) we use $\Delta = 9.0\mu\text{m}$ due to the presence of grey solitons (see text). Here and in Fig. 1(c) of the main text the data is presented given at the centre of the pixels.

been formulated, extending GHD to the Navier-Stokes scale and incorporating diffusive effects [S11, S20–S24]. At this level, the hydrodynamic equation of motion is modified through the incorporation of a diffusion operator, \mathcal{D} , arising through two-body scattering processes among quasiparticles [S25]. Subsequently, it was shown in Ref. [S22], and later fully justified in Ref. [S23], that the diffusive hydrodynamic equation for the quasiparticle density, f_p , in the presence of an external trapping potential is given by

$$\partial_t f_p + \partial_x (v^{\text{eff}} f_p) = \frac{\partial_x V(x)}{m} \partial_\lambda f_p + \partial_x (\mathcal{D} \partial_x f_p), \quad (\text{S9})$$

where $(\mathcal{D} \partial_x f_p)(\lambda; x, t) = \int d\lambda' \mathcal{D}(\lambda, \lambda'; x, t) \partial_x f_p(\lambda'; x, t)$, and $\mathcal{D}(\lambda, \lambda'; x, t)$ is the diffusion kernel, calculable from elements already present in the Euler scale theory [S20, S25]. In Ref. [S22] it was demonstrated that, when an external trapping potential breaks the integrability of a gas, diffusive dynamics inevitably lead the system to thermalize at late times [S11]. Dynamics are soluble through Crank-Nicolson methods available within the *iFluid* package [S14].

II. DYNAMICS OF A LOCALIZED DENSITY DIP

In this section, we demonstrate the outcomes of finite resolution averaging applied to the localized density dip results presented in the main text. Additionally, we provide a description of the finite resolution averaging procedure applied.

Finite resolution averaging.—The finite resolution averaging procedure implemented in Fig. 1(c) of the main text,

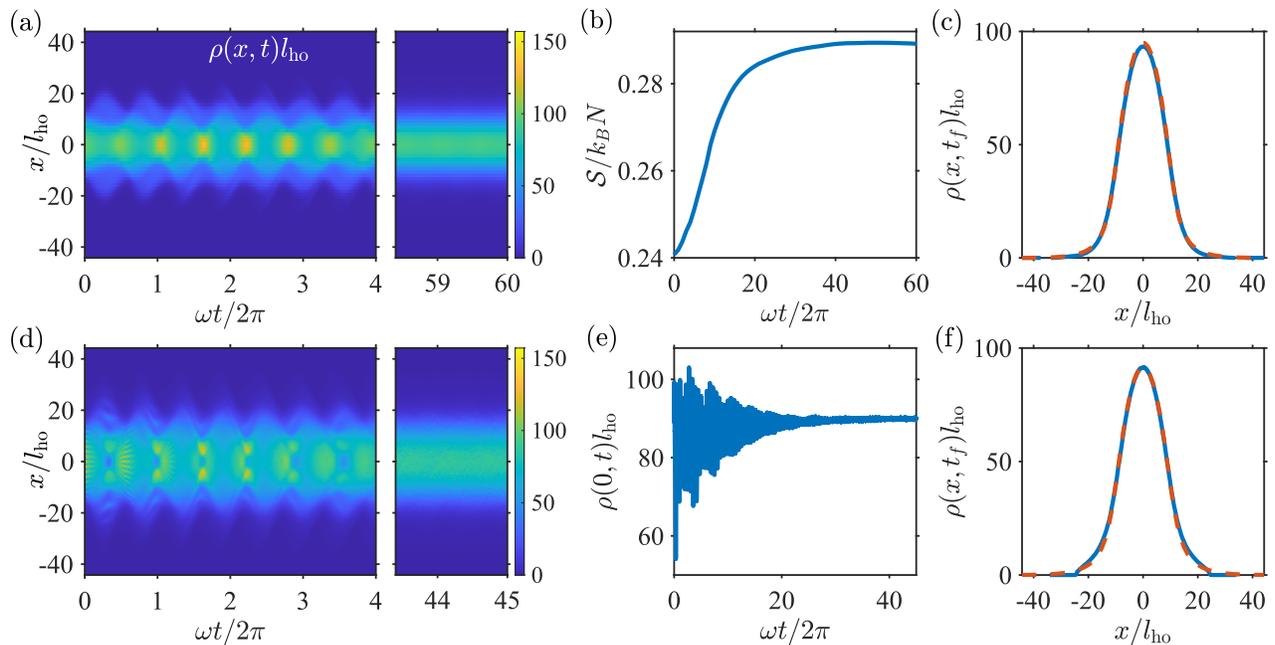


FIG. S2. Dynamics of thermalization of a harmonically trapped quasicondensate for GHD and SPGPE evolution in a quantum Newton’s cradle setup under a Bragg pulse of momentum $q_0 = k_0 l_{\text{ho}} = 1$, where $l_{\text{ho}} = \sqrt{\hbar/m\omega}$ is the harmonic oscillator length. The corresponding value of Bragg quasimomentum for this system is $\bar{\lambda}_{\text{Bragg}} = 3.1$. Initial thermal states are characterised by a dimensionless interaction parameter at the trap centre of $\gamma_0 = 0.01$, and dimensionless temperature $\bar{T} = T/(\hbar\omega/k_B) = 55.1$, with a total atom number of $N = 1834$ [S26]. Panel (a) shows diffusive GHD evolution of the density profile, $\rho(x,t)l_{\text{ho}}$; (b) shows the respective total entropy per particle of diffusive GHD simulation, which plateaus upon reaching the final thermal state; in (c) we show the density profile of the relaxed state after dynamics at time $t_f = 60/(\omega/2\pi)$ (blue solid line), compared with the thermal distribution of temperature $\bar{T} \simeq 148$ (red dashed line). Density profile evolution under SPGPE simulation is shown in (d); panel (e) demonstrates the evolution of the respective peak density (averaged over the region $x/l_{\text{ho}} \in [-2, 2]$) as it approaches thermalization and plateaus; in (d) we show the relaxed density profile after dynamics at time $t_f = 45/(\omega/2\pi)$ (blue solid line) and its matching Yang-Yang thermodynamic density profile of temperature $\bar{T} = 171$ (red dashed line).

and Fig. S1(c-f) of the supplementary, emulates the finite spatial resolution of experimental absorption imaging systems. Following the method detailed in Ref. [S27], we denote the impulse response function of the imaging system by \mathcal{A} , which we here assume to be a normalized Gaussian of width $1\mu\text{m}$. The impulse response for a pixel of width Δ centered at x_p is then,

$$\mathcal{F}(x_0) = \int_{x_p - \Delta/2}^{x_p + \Delta/2} dx \mathcal{A}(x - x_0). \quad (\text{S10})$$

Thus, the measured atom number in the given pixel is,

$$N_m = \mathcal{N} \int_{-\infty}^{+\infty} dx_0 \mathcal{F}(x_0) \rho(x_0), \quad (\text{S11})$$

where \mathcal{N} provides the correct normalization for the total particle number in the limit of zero pixel width.

Coarse-grained density dip dynamics.—Results of coarse-graining through a finite resolution averaging procedure is demonstrated for a localized density bump in Fig. 1(c) of the main text. Here, we demonstrate that same finite resolution averaging applied to the localized density dip dynamics presented in Figs. 3(a) and 3(b) of the main text.

The results of finite resolution averaging on the localized density dip dynamics are presented in Fig. S1, where panels

(a) and (b) repeat the results presented in Figs. 3(a) and (b) of the main text. The results of finite resolution averaging (where the bin offset has been optimized) for the time evolved ($\tau = 0.0005$ and $\tau = 0.002$) GPE and TWA data are shown in Figs. S1(c) and Figs. S1(d). In panel (c), the pixel width is $\Delta = 4.5\mu\text{m}$ – the same as used in Fig. 1(c) of the main text. In panel (d), due to the presence of grey solitons whose width is on the order of $(2-4)l_h$, a twice broader bin or pixel width is required in order to mimic the large-scale framework of GHD.

III. QUANTUM NEWTON’S CRADLE

Here, we provide additional comparison between GHD and SPGPE dynamics for a quantum Newton’s cradle under a Bragg pulse protocol, thermalization of which was recently studied in Ref. [S28]. We also provide further analysis for the thermalization of the double-well to single-well quantum Newton’s cradle discussed in the main text.

Bragg pulse quantum Newton’s cradle.—Collisional dynamics and thermalization of a quantum Newton’s cradle under a Bragg pulse protocol in the finite-temperature quasicondensate regime has recently been studied in Ref. [S28]. Here, we provide an additional point of comparison between

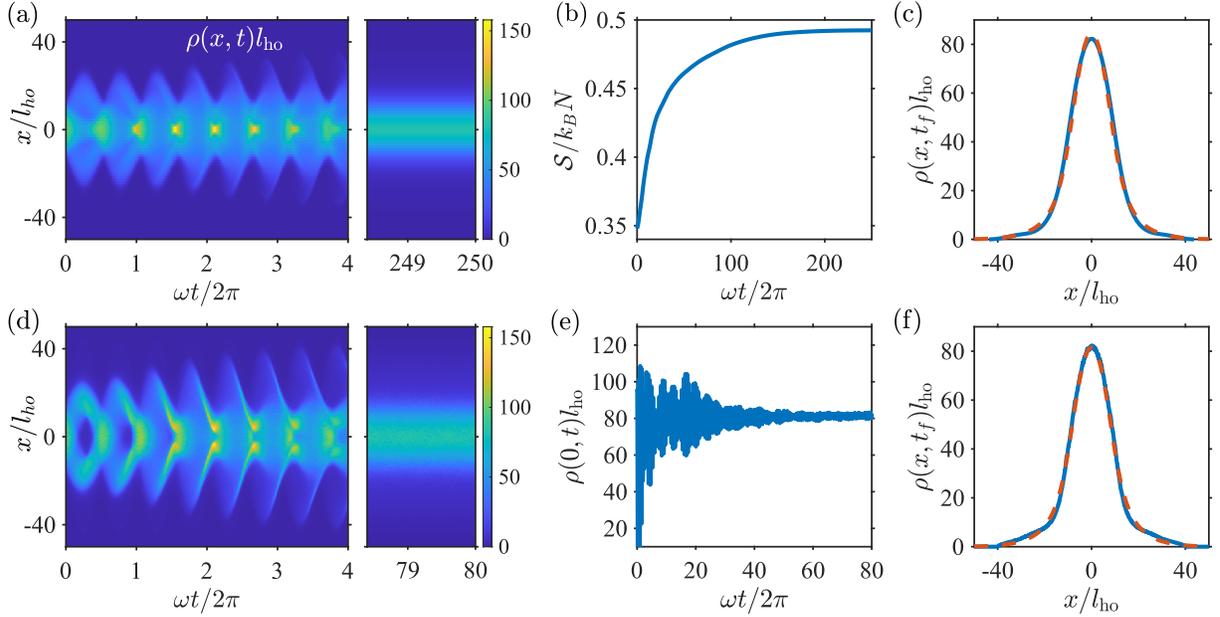


FIG. S3. Same as in Fig. S2, but for $q_0 = 5$. The corresponding Bragg quasimomentum for this system is $\bar{\lambda}_{\text{Bragg}} = 5.78$, whereas the total atom number, and the initial values of γ_0 and \bar{T} are the same. The final relaxed density profile from GHD is shown in (c) at time $t_f = 250/(\omega/2\pi)$, with the temperature of its matching thermal distribution being $\bar{T} = 276$. The relaxed SPGPE state in (f) is given at a time $t_f = 80/(\omega/2\pi)$, with the matching Yang-Yang thermodynamic density profile of temperature $\bar{T} = 277$.

GHD and SPGPE dynamics through simulation of this system, studying the rate of convergence to thermalization within the two simulation methods. Dynamics are instigated from a thermal state in a harmonic trap of frequency ω , characterized by dimensionless interaction parameter $\gamma_0 = 0.01$ in the trap centre, with temperature $\bar{T} = T/(\hbar\omega/k_B) = 55.1$, and total atom number $N = 1834$. A Bragg pulse is then applied to initiate Newton's cradle dynamics; this splits the initial atomic wavepacket at rest into two counterpropagating halves corresponding to $\pm 2\hbar k_0$ diffraction orders of Bragg scattering [S28, S29], where k_0 is the Bragg momentum in wave-number units.

Simulating a Bragg pulse protocol in GHD is done through the quasiparticle density distribution, $f_p(\lambda; x, t)$, which does not directly depend on the bare atomic momenta, instead relying on the rapidity of quasiparticles, and is generally not equivalent to the momentum distribution, except in free models [S15, S30, S31]. Implementing a Bragg pulse in GHD requires one to work with the quasiparticle density distribution of the initial thermal state, $f_p^{t=0^-}(\lambda; x, t)$, adding positive and negative momenta to the quasiparticles with equal probability. Correspondingly, the post-pulse quasiparticle distribution may be modelled as $f_p^{t=0^+}(\lambda; x, t) = [f_p^{t=0^-}(\lambda + 2\lambda_{\text{Bragg}}; x, t) + f_p^{t=0^-}(\lambda - 2\lambda_{\text{Bragg}}; x, t)]/2$ [S31]. One typically chooses the dimensionless quasimomentum of the Bragg pulse, $\bar{\lambda}_{\text{Bragg}} = (m/\hbar)\lambda_{\text{Bragg}}l_{\text{ho}}$ (given here in harmonic oscillator units) to be equal to the Bragg momentum, $q_0 \equiv k_0l_{\text{ho}}$, however this assumes a large momentum difference between the clouds [S31]. Here, we instead choose the Bragg pulse quasimomentum such that the total energy imparted in the GHD simulation is equal to the the energy difference be-

tween the initial thermal state and the post-Bragg pulse state calculated in SPGPE. Using this method, we observe a large difference between the physical Bragg momentum and the corresponding GHD quasimomentum under a low momentum Bragg pulse ($\sim 100\%$ difference at a Bragg momentum of $q_0 = 1$), with this difference decreasing for larger momentum Bragg pulses ($\sim 14\%$ difference at a Bragg momentum of $q_0 = 5$).

We first investigate a low momentum Bragg pulse of $q_0 = 1$, demonstrating dynamics of the density profile for GHD and SPGPE simulations in Fig. S2(a) and (d) respectively. At short times we see good agreement between the two simulation methods, with regular oscillations at a matching frequency. The rate of thermalization of these two simulation methods, however, shows a minor difference, with GHD simulations thermalizing at a time of $t_f \simeq 60/(\omega/2\pi)$, after approximately 120 oscillation periods. In comparison, the SPGPE simulation of the $q_0 = 1$ Bragg pulse Newton's cradle system thermalizes at a slightly earlier time of $t_f \simeq 45/(\omega/2\pi)$, after approximately 90 oscillation periods.

To quantify this difference, we analyse the approach to thermalization through two methods. First, in GHD we have access to the thermodynamic entropy density [S3]

$$s/k_B = -\frac{m}{\hbar} \int d\lambda [f \ln f - f_p \ln f_p - f_h \ln f_h], \quad (\text{S12})$$

which may be integrated for the total entropy per particle, $S/k_B N = N^{-1} \int dx s/k_B$, this quantity being known to plateau upon the system reaching a thermalized state [S22]. The approach to thermalization of the $q_0 = 1$ system in GHD is demonstrated in Fig. S2(b), showing thermalization at a

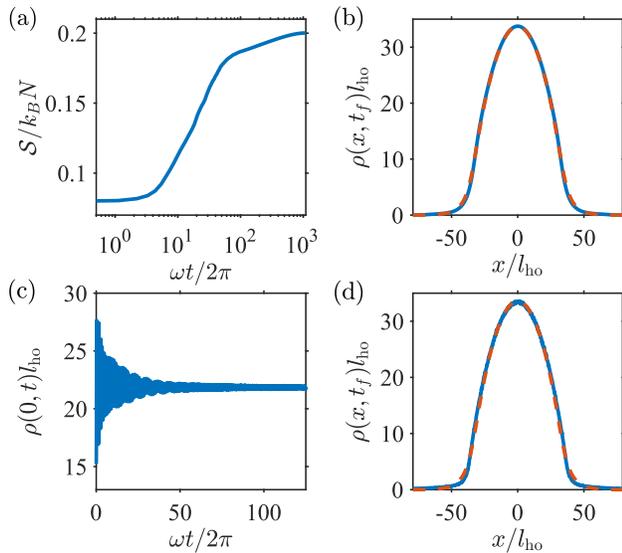


FIG. S4. Same as in Fig. S2(b)-(c) and (e)-(f), except for evolution of a double-well to single well quench. (The counterparts of panels (a) and (d) from Fig. S2, i.e., evolution of the full density profile $\rho(x, t)l_{\text{ho}}$, is shown in Fig. 4 of the main text.) The relaxed GHD density profile in (b) is given at time $t_f = 1100/(\omega/2\pi)$ with its matching thermal distribution of temperature $\bar{T} \simeq 46$, whereas the relaxed SPGPE state in (f) is given at $t_f = 125/(\omega/2\pi)$, with its matching Yang-Yang thermodynamic density profile of temperature $\bar{T} \simeq 49$.

time of $t_f \simeq 60/(\omega/2\pi)$. The density of the final thermal state after relaxation is compared directly with its matching thermal distribution, of temperature $\bar{T} \simeq 148$, in Fig. S2(c), where the final temperature, \bar{T} , and global chemical potential, μ_0 , are fixed by the initial total energy and number of particles [S22].

For SPGPE simulations, we estimate the rate of thermalization through monitoring the peak density averaged over the region $x/l_{\text{ho}} \in [-2, 2]$, which plateaus upon the system reaching equilibrium. Under non-equilibrium Newton's cradle dynamics, this quantity undergoes oscillations at twice the longitudinal trap frequency, relaxing to a final steady state upon thermalization. Results of this analysis are shown for the $q_0 = 1$ Newton's cradle in Fig. S2(e), demonstrating thermalization at a time of $t_f \simeq 45/(\omega/2\pi)$, with the density profile of the relaxed state shown in Fig. S2(f). Temperature estimation of the thermal state is achieved via Yang-Yang thermometry of the relaxed density profile [S32, S33], and is shown as the dashed lines in Fig. S2(f) at a temperature of $\bar{T} = 171$.

The slight difference in the temperatures extracted for the relaxed states in GHD and SPGPE simulations comes from a small difference between the respective final density profiles, so that the best-fit density profiles from Yang-Yang thermodynamics return slightly different values of the respective temperatures. The difference in extracted temperatures diminishes for a larger value of q_0 , as can be seen in the next example.

Simulation of the same system with a Bragg momentum of $q_0 = 5$ is demonstrated in Fig. S3(a)-(c) for GHD, and in Fig. S3(d)-(f) for SPGPE, respectively, where we again see good agreement between GHD and SPGPE at short and intermediate times. However, the disparity in thermalization rates is significantly greater, with SPGPE simulations thermalizing at around $t_f \simeq 80/(\omega/2\pi)$ (approximately 160 oscillation periods), and GHD at $t_f \simeq 250/(\omega/2\pi)$ (approximately 500 oscillation periods). This disparity is confirmed when comparing the total entropy per particle for GHD and the average peak density for SPGPE, shown in Fig. S3(b) and (e), respectively. The density profile of the final relaxed state for the GHD simulation is shown in Fig. S3(c), with its matching thermal density profile of temperature $\bar{T} = 276$. Likewise, the relaxed density profile for the SPGPE simulation is shown in Fig. S3(f), with the density profile of its matching Yang-Yang thermodynamic state of temperature $\bar{T} = 277$. The temperatures extracted for the relaxed states in GHD and SPGPE simulations are much closer to each other than in the example of Fig. S2.

Additional results for double-well to harmonic trap quench.—Dynamical thermalization of a double-well to single-well trap quench, demonstrated in Fig. 4 of the main text, is further investigated here using the methods introduced above. Total entropy per particle of GHD simulations is demonstrated in Fig. S4(a), showing thermalization at time $t_f \simeq 1100/(\omega/2\pi)$. The relaxed GHD density profile after dynamics is shown in Fig. S4(b) with its matching thermal density profile of temperature $\bar{T} \simeq 46$. Likewise, for SPGPE simulation of this system, evolution of the average peak density is shown in Fig. S4(c), with thermalization occurring at time $t_f \simeq 125/(\omega/2\pi)$. The relaxed density profile of the SPGPE simulation is shown in Fig. S4(d), with its matching Yang-Yang thermodynamic density profile of temperature $\bar{T} \simeq 49$. In general, we see that systems further from equilibrium demonstrate a greater difference in thermalization rates when comparing GHD with SPGPE. However, short and intermediate time dynamics, along with the final relaxed state of both GHD and SPGPE simulation, are seen to agree remarkably.

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