Critical effects in photon correlation for parametric generation

G. Yu. Kryuchkyan, K. G. Petrosyan, and K. V. Kheruntsyan
Institute of Physics Research, Armenian National Academy of Sciences, 378410 Ashtarak-2, Armenia[a)]

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An exact quantum theory of nondegenerate parametric generation in a cavity is developed with allowance for quantum noise of arbitrary intensity. Critical behavior of the second-order correlation functions which describe photon correlation effects is found in the threshold region in the bistable generation regime. © *1996 American Institute of Physics.* [S0021-3640(96)00507-5]

1. Photon correlation in nonlinear optical processes was first investigated for parametric scattering of light in a $\chi^{(2)}$ medium,[1] where a photon is split into a pair of photons. Many consequences of this phenomenon in both parametric scattering and parametric generation of light in a cavity are now known. Among the consequences in quantum optics we mention the generation of squeezed light[2] and the suppression of quantum fluctuations in the difference of the intensities of two correlated modes of a radiation field.[3]

The phenomenon of intermode correlation has practically escaped study for parametric generation in the threshold region, where the level and role of the quantum fluctuations of light increase substantially. This question for degenerate and nondegenerate parametric generation in the absence of a bistable generation regime[7,8] is mentioned in passing in a few works.[4–6] It follows from the results that, depending on the intensity of the pump field, the normalized second-order correlation functions have no features in the threshold region, in contrast to the Fano factor $F=\langle(\Delta n)^2\rangle/\langle n\rangle$ ($\langle(\Delta n)^2\rangle$ is the variance of the photon number fluctuations and $\langle n\rangle$ is the average number of photons in a mode), which possesses a sharp peak in this region.

In the present paper an exact quantum-statistical analysis of the process of nondegenerate parametric generation in a cavity is performed on the basis of a nonlinear treatment of the quantum fluctuations. The results obtained are applicable for arbitrary intensity of the quantum noise in the entire region of generation, including the threshold regime. In contrast to the results of Ref. 6, our results also describe the case of nonzero detunings of the cavity, which, as is well known,[7,8] result in optical bistability. As is shown below, in the latter case the correlation of the photons of the generated modes can increase substantially near the threshold.

2. The process of nondegenerate parametric generation in a cavity is based on the splitting of a pump-mode photon with frequency ω_3 into two photons ω_1, ω_2 ($\omega_3=\omega_1+\omega_2$) of the generated modes. In the resonance approximation it is described by the following Hamiltonian:[9]

$$H=\sum_{i=1}^{3}\hbar\omega_i a_i^+ a_i + i\hbar k(a_1^+ a_2^+ a_3 - a_3^+ a_1 a_2) + i\hbar(\epsilon e^{-i\omega_L t}a_3^+ - \epsilon^* e^{i\omega_L t}a_3) + \sum_{i=1}^{3}(a_i\Gamma_i^+ + a_i^+\Gamma_i). \tag{1}$$

Here a_i^+ and a_i are creation and annihilation operators of the cavity modes ω_i $(i=1,2,3)$; k is the coupling coefficient between the modes, which is proportional to the second-order nonlinear susceptibility $\chi^{(2)}$; ϵ is a complex amplitude of the pump field at the frequency ω_L $(\omega_L \simeq \omega_3)$; and, Γ_i are thermostat operators which determine the damping constants γ_i of the cavity modes.

We take into account the detuning of the cavity $\Delta_3=\omega_L-\omega_3$, $\Delta_{1,2}=\omega_L/2-\omega_{1,2}$ and we also assume that inside the cavity the pump mode decays much more rapidly than the generated modes ($\gamma_3 \gg \gamma_1,\gamma_2$), as a result of which the pump can be adiabatically eliminated from the analysis. Thus we obtain by the standard procedures (see, for example, Ref. 10) the following Fokker–Planck equation in the so-called complex P-representation[11] of the density matrix:

$$\begin{aligned}\frac{\partial}{\partial t}P(\overline{\alpha}) = \Bigg\{ &-\frac{\partial}{\partial\alpha_1}\left[-\overline{\gamma_1}\alpha_1 + \left(\frac{k\epsilon}{\overline{\overline{\gamma_3}}} - \frac{k^2}{\overline{\overline{\gamma_3}}}\alpha_1\alpha_2\right)\alpha_2^+\right] - \frac{\partial}{\partial\alpha_1^+} \\ &\times\left[-\overline{\gamma_1^*}\alpha_1^+ + \left(\frac{k\epsilon^*}{\overline{\overline{\gamma_3^*}}} - \frac{k^2}{\overline{\overline{\gamma_3^*}}}\alpha_1^+\alpha_2^+\right)\alpha_2\right] - \frac{\partial}{\partial\alpha_2}\left[-\overline{\gamma_2}\alpha_2 + \left(\frac{k\epsilon}{\overline{\overline{\gamma_3}}} - \frac{k^2}{\overline{\overline{\gamma_3}}}\alpha_1\alpha_2\right)\alpha_1^+\right] \\ &-\frac{\partial}{\partial\alpha_2^+}\left[-\overline{\gamma_2^*}\alpha_2^+ + \left(\frac{k\epsilon^*}{\overline{\overline{\gamma_3^*}}} - \frac{k^2}{\overline{\overline{\gamma_3^*}}}\alpha_1^+\alpha_2^+\right)\alpha_1\right] + \frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\left[\frac{k\epsilon}{\overline{\overline{\gamma_3}}} - \frac{k^2}{\overline{\overline{\gamma_3}}}\alpha_1\alpha_2\right] \\ &+ \frac{\partial^2}{\partial\alpha_1^+\partial\alpha_2^+}\left[\frac{k\epsilon^*}{\overline{\overline{\gamma_3^*}}} - \frac{k^2}{\overline{\overline{\gamma_3^*}}}\alpha_1^+\alpha_2^+\right]\Bigg\} P(\overline{\alpha}). \end{aligned} \tag{2}$$

Here $\overline{\alpha}=(\alpha_1,\alpha_1^+,\alpha_2,\alpha_2^+)$, α_i,α_i^+ $(i=1,2)$ are the independent complex variables corresponding to the operators a_i, a_i^+ and $\overline{\gamma_i}=\gamma_i - i\Delta_i$. To find the stationary solution of Eq. (2), in what follows we examine the case of equal damping constants $\gamma_1=\gamma_2\equiv\gamma$ and detunings $\Delta_1=\Delta_2\equiv\Delta$ $(\overline{\gamma_1}=\overline{\gamma_2}\equiv\overline{\gamma})$ and we employ the method of potential equations.[10] Then the potential conditions hold, and we obtain for the indicated solution

$$P_S(\overline{\alpha}) = N(\epsilon - k\alpha_1\alpha_2)^{\lambda}(\epsilon^* - k\alpha_1^+\alpha_2^+)^{\lambda} e^{2(\alpha_1^+\alpha_1 + \alpha_2^+\alpha_2)}, \tag{3}$$

where $\lambda = -1 + 2\overline{\gamma\gamma_3}/k^2$ and N is a normalization constant. Using the expression for the moments of the normal-ordered products of the operators a_1, a_2 in terms of the P-representation,[10] we obtain with the aid of the expression (3), after integrations in the complex plane, which are performed similarly to Ref. 6,

$$\langle a_1^{+m} a_1^m a_2^{+n} a_2^n\rangle = N_0\left(\frac{\epsilon}{k}\right)^n\left(\frac{\epsilon^*}{k}\right)^m \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+\lambda+2)\Gamma(m+\lambda^*+2)}$$

$$\times \sum_{l=0}^{\infty} \frac{(n+1)_l (m+1)_{l+n-m}}{(n+\lambda+2)_l (m+\lambda^*+2)_{l+n-m}} \frac{(2\epsilon/k)^l (2\epsilon^*/k)^{l+n-m}}{l!(l+n-m)!}, \tag{4}$$

where

$$N_0 = \frac{1}{|\Gamma(\lambda+2)|^2} \sum_{l=0}^{\infty} \frac{1}{|(\lambda+2)_l|^2} \left(\frac{2|\epsilon|}{k} \right)^{2l},$$

$\Gamma(z)$ is the gamma function and $(n)_l = n(n+1)\ldots(n+l-1)$.

3. First we present the stable stationary solutions for the amplitudes of the modes ω_1 and ω_2 in the semiclassical approximation[7] neglecting quantum noise. For $|\epsilon| < \epsilon_{th}$ ($\epsilon_{th} = |\overline{\gamma\gamma_3}|/k$ is the threshold value of the pump field) the amplitudes of the modes equal zero; this corresponds to the excitation of modes at the level of spontaneous noise. For $|\epsilon| > \epsilon_{th}$ the above-threshold generation regime is obtained. The intensities of the modes ω_1 and ω_2 at the cavity exit (in units of average photon number per unit time) are equal to one another and have the form

$$I^{\text{out}} = \frac{2\gamma^2 \gamma_3}{k^2} \left(-1 + \frac{\Delta\Delta_3}{\gamma\gamma_3} + \sqrt{\left(\frac{k|\epsilon|}{\gamma\gamma_3} \right)^2 - \left(\frac{\Delta}{\gamma} + \frac{\Delta_3}{\gamma_3} \right)^2} \right). \tag{5}$$

The corresponding stationary phases φ_1 and φ_2 of the generated modes satisfy the expression

$$\cos(\varphi_1 + \varphi_2 - \varphi) = \frac{\gamma\Delta_3 + \gamma_3\Delta}{k|\epsilon|}, \tag{6}$$

where φ of the phase of the pump field.

For $\gamma\gamma_3 - \Delta\Delta_3 < 0$ the regions of stability of the zeroth and above-threshold solutions overlap and therefore optical bistability is realized in the system.[7,8] The result for the intensity I^{out} in the semiclassical approximation is presented in Fig. 1 (curves *1* and *2*).

The quantum-mechanical result for the average intensity of the modes,

$$I^{\text{out}} \equiv 2\gamma\langle n_1 \rangle = 2\gamma\langle n_2 \rangle,$$

where $n_i = a_i^+ a_i$, $(\langle n_1 \rangle = \langle n_2 \rangle \equiv \langle n \rangle)$, is obtained from the general formula (4) by setting $m=1$, $n=0$ $(n=1,\ m=0)$ and it is also presented in Fig. 1 (curves *3* and *4*). It is evident that the quantum-mechanical average intensity is different from the semiclassical result in the threshold region and, moreover, it does not exhibit hysteresis for $\gamma\gamma_3 - \Delta\Delta_3 < 0$. This is a general result for bistable systems (see, for example, Ref. 12).

4. The effects arising from the photon correlations in each mode as well as between two modes are investigated with the aid of the normalized second-order correlation functions

$$g_{ii}^{(2)} = \frac{\langle a_i^{+2} a_i^2 \rangle}{\langle a_i^+ a_i \rangle^2}, \quad g_{12}^{(2)} = \frac{\langle a_1^+ a_1 a_2^+ a_2 \rangle}{\langle a_1^+ a_1 \rangle \langle a_2^+ a_2 \rangle} \quad (i=1,2).$$

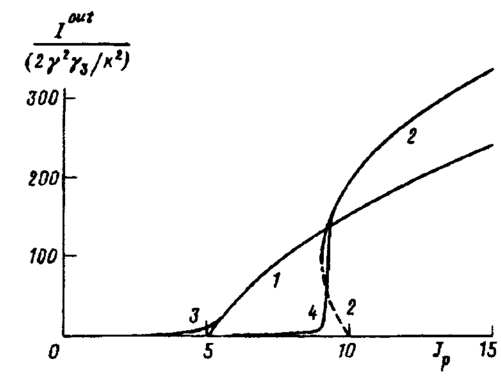

FIG. 1. Normalized intensity of the generated modes $I^{\text{out}}/(2\gamma^2\gamma_3/k^2)$ as a function of the dimensionless parameter of the pumping intensity $J_p=(k|\epsilon|/\gamma\gamma_3)^2$ with $\Delta_3/\gamma_3=2$. The curves *1* and *2* describe the semiclassical result obtained with $\Delta/\gamma=0.1$ and $\Delta/\gamma=1$, respectively; the dashed extension of the curve *2* describes the unstable semiclassical solution. The curves *3* and *4* represent the quantum-mechanical average intensity of the modes for $k^2/\gamma\gamma_3=0.01$ and are presented for the same values of Δ/γ as the curves *1* and *2*, respectively.

These functions are calculated using the general result (4). This gives, specifically, $g_{11}^{(2)}=g_{22}^{(2)}\equiv g^{(2)}$. The numerical results are plotted in Figs. 2 and 3. They show new qualitative effects for $\gamma\gamma_3-\Delta\Delta_3<0$, where according to the semiclassical results optical bistability obtains: Peaks appear in the correlation functions in the critical or threshold regions of generation. For $\gamma\gamma_3-\Delta\Delta_3>0$, when there is no bistability, the correlation functions do not exhibit critical behavior (peaks) in the threshold region. As is well known, the correlation functions $g^{(2)}$ and $g_{12}^{(2)}$ describe the ratio of the number of photon pairs emitted simultaneously to the product of the number of photons emitted independently of one another. Therefore critical growth of the correlation functions means that the number of emitted photon pairs in the region of bistable behavior of the intensity increases substantially because the system, being in two different energy states between which quantum tunneling occurs, can radiate.

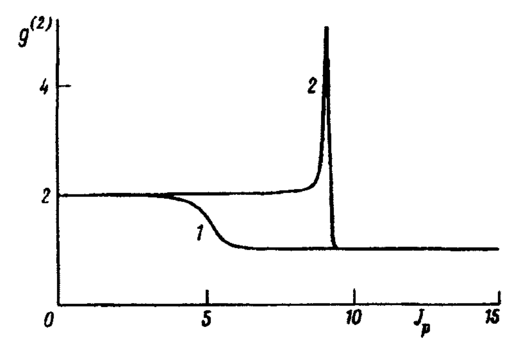

FIG. 2. Autocorrelation function $g^{(2)}$ as a function of J_p for $k^2/\gamma\gamma_3=0.01$ and $\Delta_3/\gamma_3=2$. Curve *1* — $\Delta/\gamma=0.1$; curve *2* — $\Delta/\gamma=1$.

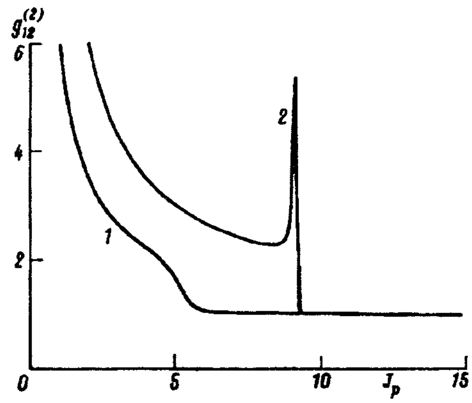

FIG. 3. Cross correlation function $g_{12}^{(2)}$ as a function of J_p for the same values of the parameters as in Fig. 2.

Below the generation threshold $|\epsilon|<\epsilon_{\rm th}$ the correlation function $g^{(2)}$ describes photon bunching ($g^{(2)}\simeq 2$ for $|\epsilon|\ll\epsilon_{\rm th}$). This is, of course, a reflection of the Gaussian statistics of the photons in each mode. The behavior of the correlation function $g_{12}^{(2)}$ for $|\epsilon|<\epsilon_{th}$ describes photon superbunching, whose nature corresponds to the creation of a photon pair $\omega_1,\ \omega_2$ in the regime of spontaneous parametric excitation of the modes. In the region above threshold $|\epsilon|\gg\epsilon_{\rm th}$ the correlation functions describe the quantum statistics of coherent states ($g^{(2)},g_{12}^{(2)}\rightarrow 1$).

As can be verified by a direct calculation with the aid of Eq. (4) the following relation between the correlation functions holds for all values of the parameters:

$$g_{12}^{(2)}=g^{(2)}+\frac{1}{2\langle n\rangle}. \tag{7}$$

This relation demonstrates the breakdown of the classical Cauchy–Schwarz inequality (see, for example, Ref. 6) in the entire region of analysis and leads to the following interesting effect. As is well known,[3] the correlation of the instantaneous fluctuations of the number of photons in the two modes is reflected in the variance of the fluctuations in the photon number difference. For the system under consideration this variance, normalized to the level of the fluctuations for the coherent states, is given by

$$V\equiv\frac{\langle\Delta(n_1-n_2)^2\rangle}{\langle n_1\rangle+\langle n_2\rangle}=1+\langle n\rangle(g^{(2)}-g_{12}^{(2)}). \tag{8}$$

As a consequence of the result (7), we obtain $V=1/2$. Therefore, for the system under study the quantum fluctuations in the photon number difference is suppressed by 50% below the coherent level for the entire region of generation and for arbitrary values of the parameters. It is significant that this result was obtained in the exact quantum theory without resorting to a linear treatment of the quantum fluctuations.

In summary, one of the main consequences of the results obtained is, in our opinion, the possibility of obtaining strongly correlated light beams in the near-threshold region of generation, where the mode intensities grow rapidly. This behavior in the photon correlation is qualitatively different from the well-known superbunching effect far below threshold — in the mode excitation regime at the spontaneous noise level, where the mode intensities are low.

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[a] e-mail: ifi@arminco.com

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