Quantum Electronics **25** (6) 573 – 578 (1995)

PACS numbers: 42.65.Ky; 42.65.Hw

Parametric oscillation in four-wave mixing with nondegenerate pumping

G Yu Kryuchkyan, K G Petrosyan†, K V Kheruntsyan

Abstract. A theoretical investigation is made of parametric generation of light by four-wave mixing in an optical cavity with two pump fields of different frequencies. Mixing configurations resulting in the excitation of one and three cavity modes are considered taking into account the effects of phase modulation and of depletion of the pump fields. An analysis is made of stable steady-state oscillation regimes. Calculations are reported of the intensities of the generated radiation modes at the exit from the cavity above the threshold. An analysis is made of the conditions for the appearance of optical bistability at nonzero values of the cavity detuning.

1. Introduction

Four-wave mixing (FWM) under the influence of a pump with one carrier frequency is one of the main nonlinear optical processes. Such mixing is governed by the third-order nonlinear susceptibility $\chi^{(3)}$ in a semiclassical theory and it generally describes the conversion of two pump photons into two other photons. Parametric oscillation based on FWM in an optical cavity has been observed on a num-ber of occasions (see, for example, Refs [1 – 3]). Less is known about FWM under the influence of two pump fields with different frequencies. The latter process has been investigated experimentally [4] and oscillation at the frequency $\omega_0 = (\omega_1 + \omega_2)/2$, equal to the half-sum of the frequencies ω_1 and ω_2 of the two pump fields, has been observed.

The present paper provides a theoretical analysis of the problem of parametric oscillation based on intracavity FWM under the action of nondegenerative pumping ($\omega_1 \neq \omega_2$). Two configurations of such a parametric oscillator will be considered.

In one configuration a nonlinear $\chi^{(3)}$ medium is inside a ring cavity where one radiation field mode is excited (Fig. 1a). Two pump beams with frequencies ω_1 and ω_2 propagate in the same direction at a small angle to the cavity axis. Intracavity FWM excites spontaneously a signal mode of frequency $\omega_0 = (\omega_1 + \omega_2)/2$. In this configuration the condition of phase-matching between the wave vectors $\boldsymbol{k}_m$ ($m = 0, 1, 2$) of the ω_m modes is satisfied approximately ($\boldsymbol{k}_1 + \boldsymbol{k}_2 \approx 2\boldsymbol{k}_0$). This configuration is relatively easy to describe because the pump fields cross the nonlinear medium only once and can be regarded as undepletable (Section 2). In the experiments reported by Grandclement et al. [4] a similar parametric oscillator configuration has been used with a linear cavity in which two pump fields of different frequencies propagate in opposite directions.

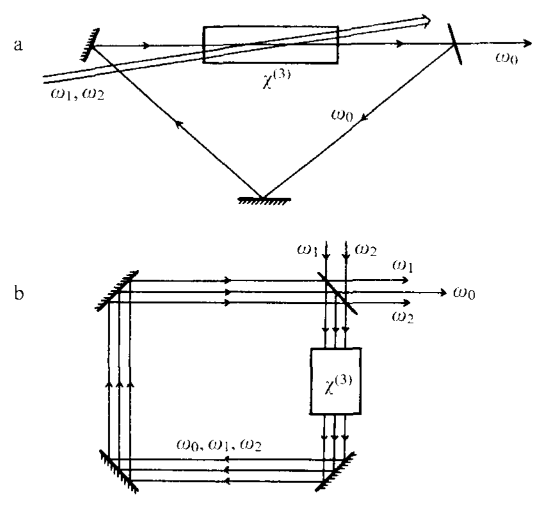

Figure 1. Schematic diagrams illustrating parametric four-wave mixing under the influence of two pump fields with the frequencies ω_1 and ω_2: (a) configuration with excitation of one cavity mode at the frequency $\omega_0 = (\omega_1 + \omega_2)/2$; (b) configuration with excitation of three cavity modes at the frequencies ω_1, ω_2, and $\omega_0 = (\omega_1 + \omega_2)/2$.

In another configuration (Fig. 1b) the directions of propagation of the two pump beams coincide with the cavity axis. When the phase-matching conditions are satisfied, three modes are excited in the cavity: they have the pump field frequencies ω_1 and ω_2, and the frequency $\omega_0 = (\omega_1 + \omega_2)/2$. In this configuration the effects of depletion of the pump fields and of the mutual influence of the modes are important (see Section 3).

A theoretical analysis will be made of these two nonlinear optical configurations and the conditions will be found under which stable above-threshold generation is possible. The intensities of the radiation field modes at the exit from the cavity will be calculated. The results obtained indicate that FWM parametric oscillation of nondegenerate pumping has a number of special features. The results obtained for one configuration (Fig. 1b) in the special case of zero cavity detuning have already been reported by us [5]. We shall now consider the case of nonzero values of the cavity detuning, so as to

†The surname of K G Petrosyan is spelt Petrossian in some Western publications.

G Yu Kryuchkyan, K G Petrosyan, K V Kheruntsyan Institute of Physics Research, Academy of Sciences of Armenia, Ashtarak-2, Armenia

Received 22 April 1994
Kvantovaya Elektronika **22** (6) 599 – 604 (1995)
Translated by A Tybulewicz

obtain information on real conditions in parametric oscillation and optical bistability for FWM with nondegenerate pumping.

2. Single-mode oscillation in the presence of phase modulation

We shall now consider the FWM configuration shown in Fig. 1a in the absence of optical resonances when the nonlinear medium is described phenomenologically by the third-order nonlinear susceptibility $\chi^{(3)}$. In this case we can ignore wave dispersion and describe the Fourier component of the nonlinear polarisability at the frequency ω_0 of the signal mode as follows [6]:

$$P_{nl}(\omega_0) = \tfrac{3}{4}\chi^{(3)}(2E_1E_2E_0^* + |E_0|^2E_0 + 2|E_1|^2E_0 + 2|E_2|^2E_0) , \tag{1}$$

where E_m ($m = 0, 1, 2$) are the slowly varying complex amplitudes of the fields with the frequencies ω_m. Expression (1) is derived subject to the phase-matching condition $\boldsymbol{k}_1 + \boldsymbol{k}_2 \approx 2\boldsymbol{k}_0$ and subject to $\omega_1 + \omega_2 = 2\omega_0$. The first term in expression (1) describes the FWM interaction, the second represents the self-phase-modulation of the signal wave ω_0, and the last two terms correspond to cross-phase-modulation caused by the pump fields.

As pointed out, in this FWM configuration the pump fields can be regarded as undepletable (E_1 and E_2 are constant). However, the phase modulation effects will be included.

2.1 Equation for the signal mode in the cavity and its linearisation

Standard methods readily yield the following equation for the slowly varying amplitude E_0 of the signal wave in the cavity:

$$\frac{dE_0}{dt} = -\gamma E_0 + i[\Delta - \chi(|E_1|^2 + |E_2|^2)]E_0 - i\tfrac{1}{2}\chi|E_0|^2E_0 - i\chi E_1E_2E_0^* . \tag{2}$$

Here, γ is the damping constant of the cavity mode ω_0; $\Delta = \omega_0 - \omega_c$ is the cavity detuning; ω_c is the cavity eigenfrequency; $\chi = 6\pi i\omega_0\chi^{(3)}l/nL$ is the net nonlinear coupling constant; l is the length of the nonlinear medium; L is the cavity length; n is the refractive index.

We shall now analyse steady-state oscillation in the investigated nonlinear configuration. The steady-state solutions E_0^0 of Eqn (2) can be found by an analysis of their stability in the presence of small perturbations $\delta E_0(t) = E_0(t) - E_0^0$. An equation linearised in terms of these perturbations is readily obtained from Eqn (2) and from the corresponding complex-conjugate equation; the matrix form of the new equation is

$$\frac{d}{dt}\delta E = -A\delta E . \tag{3}$$

Here, δE denotes the column vector $\delta E = (\delta E_0, \delta E_0^*)^T$ and the matrix A has the following elements

$$A_{11} = \gamma - i[\Delta - \chi(|E_1|^2 + |E_2|^2) - \chi|E_0^0|^2] , \quad A_{22} = A_{11}^* , \quad A_{12} = i\tfrac{1}{2}\chi[(E_0^0)^2 + 2E_1E_2] , \quad A_{21} = A_{12}^* . \tag{4}$$

We can easily see that one of the steady-state solutions of Eqn (2) is trivial: $E_0^0 = 0$. It corresponds to oscillation at the spontaneous noise level. The stability of the steady-state solution requires that the real parts of the eigenvalues of the matrix A of the linearised equation should be positive. The use of a characteristic equation and of the Routh–Hurwitz criterion [7] makes it possible to write down the stability condition of the steady-state solution $E_0^0 = 0$ in the form of the following inequality

$$\left(\frac{\chi}{\gamma}\right)^2(|E_1|^4 + |E_2|^4 + |E_1|^2|E_2|^2) - \frac{2\chi\Delta}{\gamma^2}(|E_1|^2 + |E_2|^2) + \left(\frac{\Delta}{\gamma}\right)^2 + 1 > 0 . \tag{5}$$

It follows from this inequality that if $(\chi/\gamma)^2|E_1|^2|E_2|^2 < 1$, the steady-state solution $E_0^0 = 0$ is stable for any value of the parameter Δ/γ. If $(\chi/\gamma)^2|E_1|^2|E_2|^2 > 1$, this solution ceases to be stable in the range $d^{(-)} < \Delta/\gamma < d^{(+)}$, where

$$d^{(\pm)} = \frac{\chi}{\gamma}(|E_1|^2 + |E_2|^2) \pm \left[\left(\frac{\chi}{\gamma}\right)^2|E_1|^2|E_2|^2 - 1\right]^{1/2} . \tag{6}$$

In the latter case, when the conditions for changes in the intensities of the pumps $|E_1|^2$ and $|E_2|^2$ are specified, the inequality (5) defines also the oscillation threshold (which is discussed below).

Other steady-state solutions with a nonzero amplitude of the signal wave can be obtained more conveniently by considering the intensities I_m and the phases φ_m of the fields $E_m = (I_m)^{1/2}\exp(i\varphi_m)$ ($m = 0, 1, 2$). Then, instead of Eqn (2), we obtain the following equations of motion:

$$\frac{dI_0}{dt} = -2\gamma I_0 + 2\chi(I_1I_2)^{1/2}I_0\sin\psi , \tag{7}$$

$$\frac{d\varphi_0}{dt} = \Delta - \chi(I_1 + I_2) - \frac{\chi}{2}I_0 - \chi(I_1I_2)^{1/2}\cos\psi , \tag{8}$$

where $\psi = \varphi_1 + \varphi_2 - 2\varphi_0$. An analysis of the stability of the steady-state solutions I_0^0 and φ_0^0 of Eqns (7) and (8) also requires linearisation of the equations of motion. This is done by substituting $I_0(t) = I_0^0 + \delta I_0(t)$, $\varphi_0(t) = \varphi_0^0 + \delta\varphi_0(t)$ into Eqns (7) and (8). The linearised equations are

$$\frac{d}{dt}\begin{pmatrix}\delta I_0 \\ \delta\varphi_0\end{pmatrix} = -\begin{pmatrix}0 & 4\chi(I_1I_2)^{1/2}I_0\cos\psi^0 \\ \chi/2 & 2\chi(I_1I_2)^{1/2}I_0\sin\psi^0\end{pmatrix}\begin{pmatrix}\delta I_0 \\ \delta\varphi_0\end{pmatrix} , \tag{9}$$

where $\psi^0 = \varphi_1 + \varphi_2 - 2\varphi_0^0$.

2.2 Above-threshold oscillation. Bistability

We shall now consider the conditions for stable above-threshold oscillation and find the signal mode intensity at the exit from the cavity. The steady-state solutions of Eqns (7) and (8) are

$$I_0^{(\pm)} = \frac{2\gamma}{\chi}\left\{\frac{\Delta}{\gamma} - \frac{\chi}{\gamma}(I_1 + I_2) \pm \left[\left(\frac{\chi}{\gamma}\right)^2 I_1I_2 - 1\right]^{1/2}\right\} , \tag{10}$$

$$\varphi_0^0 = \frac{1}{2}\left[\varphi_1 + \varphi_2 - \arcsin\frac{\gamma}{\chi(I_1I_2)^{1/2}}\right] . \tag{11}$$

They are physically meaningful if

$$\left(\frac{\chi}{\gamma}\right)^2 I_1I_2 > 1 . \tag{12}$$

In an analysis of the stability of these solutions we must turn to the appropriate characteristic equation for the eigenvalues of the matrix of the linearised system of equations (9). If the real parts of these eigenvalues are positive, the Routh–Hurwitz criterion can be used to show that the stable steady-state solution $I^{(+)}$ satisfies

$$I_0^{(+)} > \frac{2\gamma}{\chi}\left[\frac{\Delta}{\gamma} - \frac{\chi}{\gamma}(I_1 + I_2)\right] . \tag{13}$$

This inequality and inequality (12) define together the ranges of the dimensionless parameters $(\chi/\gamma)I_{1,2}$ and Δ/γ in which stable generation of the field with the frequency ω_0 is possible above the threshold. As pointed out earlier, the oscillation threshold is found from inequality (5).

A more detailed analysis of these problems is given below for two cases: equal and different intensities of the pump fields I_1 and I_2.

Equal intensities of the pump fields. If $I_1 = I_2 = I$, the threshold intensity I, defined by inequality (5), is

$$I_A = \frac{\gamma}{3\chi}\left\{\frac{2\Delta}{\gamma} - \left[\left(\frac{\Delta}{\gamma}\right)^2 - 3\right]^{1/2}\right\} . \tag{14}$$

Above the threshold the signal field intensity at the cavity exit is

$$I^{\text{out}} = 2\gamma I_0^{(+)} = \frac{4\gamma^2}{\chi}\left\{\frac{\Delta}{\gamma} - \frac{2\chi}{\gamma}I + \left[\left(\frac{\chi}{\gamma}\right)^2 I^2 - 1\right]^{1/2}\right\} . \tag{15}$$

The steady-state solution is stable in the following ranges of the pump intensities:

$$I_A < I < I_B , \quad \sqrt{3} < \frac{\Delta}{\gamma} < 2 , \tag{16}$$

$$\frac{\gamma}{\chi} < I < I_B , \quad \frac{\Delta}{\gamma} > 2 , \tag{17}$$

where

$$I_B = \frac{\gamma}{3\chi}\left\{\frac{2\Delta}{\gamma} + \left[\left(\frac{\Delta}{\gamma}\right)^2 - 3\right]^{1/2}\right\} . \tag{18}$$

When the pump field intensities reach the value $I = I_B$, stable above-threshold oscillation disappears and the configuration again changes over to oscillation at the spontaneous noise level.

The dependence of the radiation field intensity, normalised to $4\gamma^2/\chi$, at the cavity exit on the dimensionless pump intensity parameter $(\chi/\gamma)I$ is shown in Fig. 2. The continuous curves represent stable steady-state oscillation regimes and the dashed curve represents the unstable steady-state solution $I_0^{(-)}$ described by expression (10). The curve representing the above-threshold oscillation regime is calculated for the range $\Delta/\gamma > 2$. In this case, it follows from the inequalities (17) and $\gamma/\chi < I_A$ that the nonlinear configuration under discussion exhibits optical bistability of the output radiation intensity, relative to the pump field intensities.

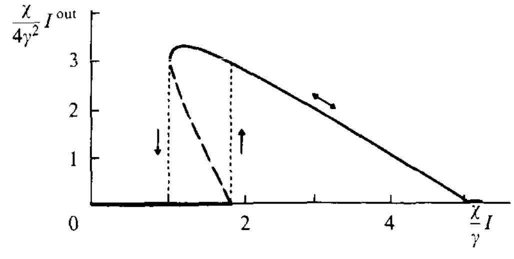

Figure 2. Dependence of the ratio $\chi I^{\text{out}}/4\gamma^2$ on $\chi I/\gamma$, calculated for $\Delta/\gamma = 5$

Different intensities of the pump fields. We shall now consider the situation in which the intensity of one of the pump fields (for example, I_2) is constant and the intensity of the other (I_1) varies (it describes exactly the situation in the experiments reported by Grandclement et al. [4]). We shall study the behaviour of the steady-state intensity of the output radiation $I^{\text{out}} = 2\gamma I_0^{(+)}$ as a function of I_1 for fixed values of I_2 and Δ/γ. In this case the threshold intensity I_1 is, according to inequality (5):

$$I_{1.A} = \frac{\gamma}{2\chi}\left[\frac{2\Delta}{\gamma} - \frac{\chi}{\gamma}I_2 - \left(\frac{4\Delta\chi}{\gamma^2}I_2 - \frac{3\chi^2}{\gamma^2}I_2^2 - 4\right)^{1/2}\right] , \tag{19}$$

and the corresponding stability regions are defined by the inequalities

$$I_{1.A} < I_1 < I_{1.B} , \quad \frac{3(\chi/\gamma)^2 I_2^2 + 4}{4(\chi/\gamma)I_2} < \frac{\Delta}{\gamma} < \frac{(\chi/\gamma)^2 I_2^2 + 1}{(\chi/\gamma)I_2} , \tag{20}$$

$$\frac{\gamma^2}{\chi^2 I_2} < I_1 < I_{1.B} , \quad \frac{\Delta}{\gamma} > \frac{(\chi/\gamma)^2 I_2^2 + 1}{(\chi/\gamma)I_2} , \tag{21}$$

Here,

$$I_{1.B} = \frac{\gamma}{2\chi}\left[\frac{2\Delta}{\gamma} - \frac{\chi}{\gamma}I_2 + \left(\frac{4\Delta\chi}{\gamma^2}I_2 - \frac{3\chi^2}{\gamma^2}I_2^2 - 4\right)^{1/2}\right] . \tag{22}$$

The behaviour of the output radiation intensity I^{out} as a function of I_1 for fixed values of I_2 and Δ/γ is demonstrated in Fig. 3. In accordance with the formulation of the problem and with relationships (19)–(22), the bistable behaviour of the intensity I^{out} is observed in the range $\gamma^2/\chi^2 I_2 < I_1 < I_{1.A}$. It should be pointed out that this result is in agreement with the experimental observations of the optical bistability in a similar FWM system with a linear cavity and counter-propagating pump beams [4].

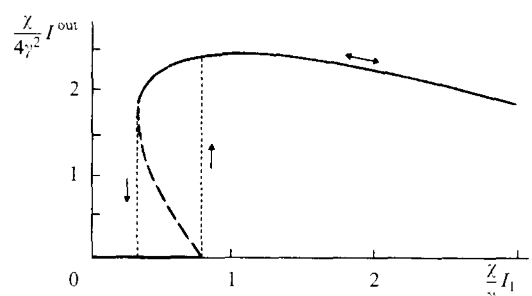

Figure 3. Dependence of $\chi I^{\text{out}}/4\gamma^2$ on $\chi I_1/\gamma$, calculated for $\chi I_2/\gamma = 3$ and $\Delta/\gamma = 5$.

We shall conclude this section with the following comment. If expression (1) for the nonlinear polarisability is simplified by ignoring the term corresponding to the self-phase-modulation effect, the equations of motion for the signal mode become analogous to the equations describing the processes of parametric frequency division of light in a cavity and of degenerate FWM under the influence of monochromatic undepletable pumping (see, for example, Refs [8, 9], as well as Refs [10, 11], where these equations and their analysis are given in the form best suited for comparison).†

†In this case, inclusion of the effect of cross-phase-modulation in FWM corresponds to the replacement of the resonator detuning Δ with the effective detuning $\Delta_{\text{eff}} = \Delta - \chi(|E_1|^2 + |E_2|^2)$.

The stable steady-state solution corresponding to these processes describes the behaviour below the oscillation threshold.

Our analysis of the investigated nonlinear system takes into account the influence of self-phase-modulation of the signal mode when the FWM interaction occurs effectively in a $\chi^{(3)}$ medium. This analysis yields the conditions for stable generation of the signal mode below and above the oscillation threshold. Inclusion of the phase modulation effects and an analysis of steady-state oscillation in another FWM configuration, with monochromatic pumping and two nondegenerate output modes, have been reported by Brambilla et al. [12].

3. Multimode oscillation with depletion of the pump fields

We shall now consider parametric oscillation in the other FWM configuration (Fig. 1b) when both the signal mode and the pump modes propagate in the cavity. In this case we must take account of the pump depletion and of the mutual influence of all the modes. A quantum-mechanical analysis of the process within the framework of the stochastic equations of motion of intracavity mode amplitudes is given in Ref. [5] in connection with the generation of squeezed states of light. However, the results of Ref. [5] apply to the specific case of zero detuning $\Delta_m = \omega_m - \omega_{c,m} = 0$ $(m = 0, 1, 2)$ of the frequencies of the pump modes $\omega_{1,2}$ and of the signal mode ω_0 from the cavity eigenfrequencies $\omega_{c,m}$. In this case we shall identify the real conditions for FWM parametric oscillation, including the influence of the cavity detuning on stable oscillation and on bistability, by considering the more general case when $\Delta_m \neq 0$.

3.1 Equations of motion for the cavity modes

This problem can be solved by using just the semiclassical equations for slowly varying complex amplitudes of the cavity fields E_m $(m = 0, 1, 2)$, which are (see also Ref. [5]):

$$\begin{aligned}
\frac{dE_0}{dt} &= -\bar{\gamma}_0 E_0 + \chi E_1 E_2 E_0^* , \\
\frac{dE_1}{dt} &= -\bar{\gamma}_1 E_1 - \frac{\chi}{2} E_0^2 E_2^* + (2\gamma_1)^{1/2} E_1^{\text{in}} , \\
\frac{dE_2}{dt} &= -\bar{\gamma}_2 E_2 - \frac{\chi}{2} E_0^2 E_1^* + (2\gamma_2)^{1/2} E_2^{\text{in}} .
\end{aligned} \tag{23}$$

Here, $\bar{\gamma}_m = \gamma_m - i\Delta_m$, γ_m are the damping constants of the ω_m modes in the cavity; Δ_m represents the values of the cavity detuning which, because the phase-matching condition is obeyed rigorously, are related by $2\Delta_0 = \Delta_1 + \Delta_2$; $E_{1,2}^{\text{in}}$ are the input amplitudes of the pump fields at the entry to the cavity. In the case under consideration, when the coupling in and out occur at one of the ring-cavity mirrors, the relationships between the amplitudes of the input and output (E_m^{out}) cavity fields and the intracavity amplitudes are given by

$$E_{1,2}^{\text{out}} = (2\gamma_{1,2})^{1/2} E_{1,2} - E_{1,2}^{\text{in}} , \quad E_0^{\text{out}} = (2\gamma_0)^{1/2} E_0 . \tag{24}$$

These relationships take into account that the signal mode ω_0 is generated spontaneously. Moreover, the system of equations (23), describing the FWM interaction, ignores the phase modulation effects.

Our aim is to find the steady-state solution E_m^0 of the system of equations (23) and to study their stability. Once again, we need linearised equations for small deviations $\delta E_m(t) = E_m(t) - E_m^0$ from the steady-state solutions. These equations follow from the system of equations (23) and from the corresponding conjugate equations. In the matrix form, the new equations can be written in the same form as Eqn (3), where $\delta E = (\delta E_0, \delta E_0^*, \delta E_1, \delta E_1^*, \delta E_2, \delta E_2^*)^{\text{T}}$ denotes a column vector and the matrix is

$$A = \begin{pmatrix}
\bar{\gamma}_0 & -\chi E_1^0 E_2^0 & -\chi (E_0^0)^* E_2^0 & 0 & -\chi (E_0^0)^* E_1^0 & 0 \\
-\chi (E_1^0 E_2^0)^* & \bar{\gamma}_0^* & 0 & -\chi E_0^0 (E_2^0)^* & 0 & -\chi E_0^0 (E_1^0)^* \\
\chi E_0^0 (E_2^0)^* & 0 & \bar{\gamma}_1 & 0 & 0 & \frac{1}{2}\chi (E_0^0)^2 \\
0 & \chi (E_0^0)^* E_2^0 & 0 & \bar{\gamma}_1^* & \frac{1}{2}\chi (E_0^0)^{*2} & 0 \\
\chi E_0^0 (E_1^0)^* & 0 & 0 & \frac{1}{2}\chi (E_0^0)^2 & \bar{\gamma}_2 & 0 \\
0 & \chi (E_0^0)^* E_1^0 & \frac{1}{2}\chi (E_0^0)^{*2} & 0 & 0 & \bar{\gamma}_2^*
\end{pmatrix} .$$

3.2 Oscillation regimes

We shall consider the case of equal damping constants $\gamma_1 = \gamma_2 \equiv \gamma$, equal values of the resonator detuning $\Delta_1 = \Delta_2 \equiv \Delta$ $(\bar{\gamma}_1 = \bar{\gamma}_2 \equiv \bar{\gamma})$, and equal amplitudes of the pump fields at the entry to the resonator $|E_1^{\text{in}}| = |E_2^{\text{in}}| \equiv E^{\text{in}}$. Omitting details of the calculations, we shall give directly the steady-state intensities I_m^0 and phases ψ_m^0, corresponding to the amplitudes $E_m^0 = (I_m^0)^{1/2} \exp(i\psi_m^0)$ of the fields with the frequencies ω_m.

It is easy to show that the threshold intensity of the input pump fields at the entry to the cavity $I^{\text{in}} = |E^{\text{in}}|^2$ is

$$I_{\text{t}}^{\text{in}} = \frac{|\bar{\gamma}|^2 |\bar{\gamma}_0|}{2\gamma\chi} . \tag{25}$$

Below the oscillation threshold $(I^{\text{in}} < I_{\text{t}}^{\text{in}})$ the stable steady-state solution is

$$I_0^0 = 0 , \quad I_1^0 = I_2^0 = \frac{2\gamma I^{\text{in}}}{|\bar{\gamma}|^2} , \tag{26}$$

$$\psi_{1,2}^0 = \phi_{1,2} + \arccos\left(\frac{\gamma}{|\bar{\gamma}|}\right) , \tag{27}$$

where $\phi_{1,2}$ are the phases of the pump fields $E_{1,2}^{\text{in}} = E^{\text{in}} \exp(i\phi_{1,2})$.

Above the threshold $(I^{\text{in}} > I_{\text{t}}^{\text{in}})$ as well as for zero cavity detuning [5] there are two oscillation regimes. In one of them, the steady-state intensities of the pump modes I_1^0 and I_2^0 are equal and the expressions for I_m^0 and ψ_m^0 $(m = 0, 1, 2)$ are

$$I_1^0 = I_2^0 = \frac{|\bar{\gamma}_0|}{\chi} , \tag{28}$$

$$I_0^0 = \frac{2|\bar{\gamma}|}{\chi} \left[-c \pm (\mu + c^2 - 1)^{1/2} \right] , \tag{29}$$

$$\psi_k^0 = \phi_k + \arccos\left\{ \frac{1}{\sqrt{\mu}} \left[\frac{\gamma}{|\bar{\gamma}|} + \frac{\gamma_0}{|\bar{\gamma}_0|} \times \left[-c \pm (\mu + c^2 - 1)^{1/2} \right] \right] \right\} ,$$

$$k = 1, 2 , \tag{30}$$

$$\psi_0^0 = \frac{1}{2}\left(\psi_1^0 + \psi_2^0 + \arctan\frac{\Delta_0}{\gamma_0} \right) . \tag{31}$$

The notation used above is

$$\mu = \frac{I^{\text{in}}}{I_{\text{t}}^{\text{in}}} , \quad c = \frac{\gamma\gamma_0 - \Delta\Delta_0}{|\bar{\gamma}||\bar{\gamma}_0|} ,$$

and it should be noted that $|c| \leqslant 1$.

Expressions (29) and (30) [with the minus sign in front of $(\mu + c^2 - 1)^{1/2}$] corresponds to the unstable steady-state solution. The solution with the plus sign is stable in the following ranges of the input intensity I^{in}:

$$I_t^{\text{in}} < I^{\text{in}} < 2I_t^{\text{in}}(1+c), \quad c > 0\,, \tag{32}$$

$$I_t^{\text{in}}(1-c^2) < I^{\text{in}} < 2I_t^{\text{in}}(1+c), \quad c < 0\,. \tag{33}$$

In the other regime, which occurs at higher input intensities of the pump fields at the entry to the cavity, the intensities I_1^0 and I_2^0 are different and the steady-state solutions are

$$I_0^0 = \frac{2|\bar{\gamma}|}{\chi}\,, \tag{34}$$

$$\begin{aligned} I_1^0 &= \frac{|\bar{\gamma}_0|}{2\chi}\left\{\mu - 2c + \left[(\mu-2c)^2 - 4\right]^{1/2}\right\}, \\ I_2^0 &= \frac{|\bar{\gamma}_0|}{2\chi}\left\{\mu - 2c - \left[(\mu-2c)^2 - 4\right]^{1/2}\right\} \end{aligned} \tag{35}$$

or

$$\begin{aligned} I_1^0 &= \frac{|\bar{\gamma}_0|}{2\chi}\left\{\mu - 2c - \left[(\mu-2c)^2 - 4\right]^{1/2}\right\}, \\ I_2^0 &= \frac{|\bar{\gamma}_0|}{2\chi}\left\{\mu - 2c + \left[(\mu-2c)^2 - 4\right]^{1/2}\right\}. \end{aligned} \tag{36}$$

The steady-state phases are then

$$\psi_{1,2}^0 = \phi_{1,2} + \arccos\frac{\chi\gamma I_{1,2}^0 + \gamma_0|\bar{\gamma}|}{|\bar{\gamma}|(\chi|\bar{\gamma}_0|\mu I_{1,2}^0)^{1/2}}\,, \tag{37}$$

$$\psi_0^0 = \frac{1}{2}\left(\psi_1^0 + \psi_2^0 + \arctan\frac{\varDelta_0}{\gamma_0}\right). \tag{38}$$

An analysis shows that these solutions are stable in the range

$$I^{\text{in}} > 2I_t^{\text{in}}(1+c)\,. \tag{39}$$

3.3 Bistability in the presence of two above-threshold oscillation regimes

We shall now give the results of calculations of the output intensities $I_m^{\text{out}} = |E^{\text{out}}|^2$ of three interacting modes at the cavity exit. It follows from expressions (24), (26)–(29), and (34)–(36) that below the oscillation threshold ($I^{\text{in}} < I_t^{\text{in}}$), we have

$$I_0^{\text{out}} = 0\,, \quad I_{1,2}^{\text{out}} = I^{\text{in}}\,. \tag{40}$$

Above the threshold in the ranges $I_t^{\text{in}} < I^{\text{in}} < 2I_t^{\text{in}}(1+c)$ if $c > 0$ and $I_t^{\text{in}}(1-c^2) < I^{\text{in}} < 2I_t^{\text{in}}(1+c)$ if $c < 0$, we find that

$$\begin{aligned} I_0^{\text{out}} &= \frac{4\gamma_0|\bar{\gamma}|}{\chi}\left[(\mu + c^2 - 1)^{1/2} - c\right], \\ I_1^{\text{out}} = I_2^{\text{out}} &= \frac{|\bar{\gamma}|^2|\bar{\gamma}_0|}{2\chi\gamma}\left\{\mu - \frac{4\gamma_0\gamma}{|\bar{\gamma}_0||\bar{\gamma}|}\left[(\mu + c^2 - 1)^{1/2} - c\right]\right\}. \end{aligned} \tag{41}$$

Finally, in the range $I^{\text{in}} > 2I_t^{\text{in}}(1+c)$

$$\begin{aligned} I_0^{\text{out}} &= \frac{4\gamma_0|\bar{\gamma}|}{\chi}\,, \\ I_1^{\text{out}} = I_2^{\text{out}} &= \frac{|\bar{\gamma}|^2|\bar{\gamma}_0|}{2\chi\gamma}\left[\mu - \frac{4\gamma_0\gamma}{|\bar{\gamma}_0||\bar{\gamma}|}\right]. \end{aligned} \tag{42}$$

It should be pointed out that the pump mode intensities I_1^{out} and I_2^{out} at the cavity exit are equal throughout the above-threshold range, although inside the cavity, when $I^{\text{in}} > 2I_t^{\text{in}}(1+c)$, the intensities I_1^0 and I_2^0 are different [see expressions (35) and (36)]. This circumstance is related, in accordance with expression (24), to the interference of the amplitudes $E_{1,2}$ and $E_{1,2}^{\text{in}}$.

The dependences of the intensity of the signal mode I_0^{out} on the ratio $\mu = I^{\text{in}}/I_t^{\text{in}}$ are plotted in Fig. 4, where the continuous curves represent the stable steady-state solutions and the dashed curves represent the unstable solution described by expression (29). We can easily note that if the parameter c is negative, i.e. if $\varDelta\varDelta_0 > \gamma\gamma_0$, the intensity I_0^{out} demonstrates bistability behaviour. The existence of two above-threshold oscillation regimes in the case of bistability of the investigated nonlinear system manifests itself as follows. In the range of the parameter $-1 < c < -0.5$ a transition can take place from oscillation below the threshold directly to the second above-threshold regime. In this case two stable states are possible in the range $2I_t^{\text{in}}(1+c) < I^{\text{in}} < I_t^{\text{in}}$ with constant intensities I_0^{out}, independent of the pump intensities.

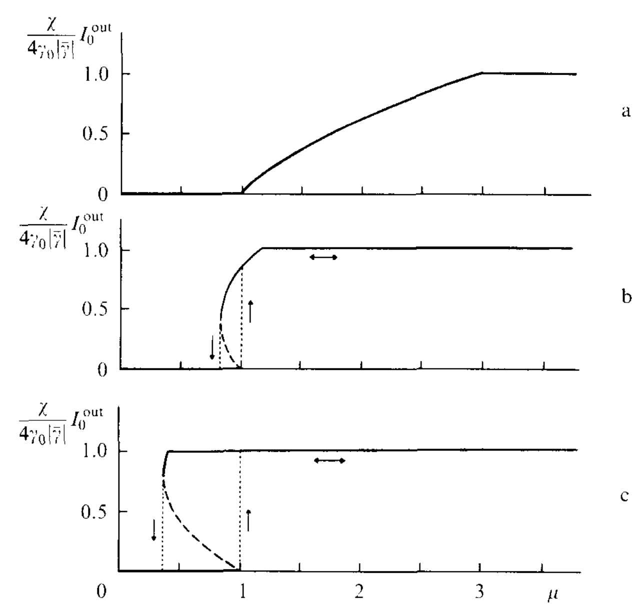

Figure 4. Dependences of the intensities of the signal mode $\chi I_0^{\text{out}}/4\gamma_0|\bar{\gamma}|$ on the ratio $\mu = I^{\text{in}}/I_t^{\text{in}}$, calculated for $c = 0.5$ (a), -0.4 (b), and -0.8 (c).

Acknowledgements. This work was partly supported by a grant from the Meyer Foundation, awarded by the American Physical Society.

References

1. Pinard M, Grandclement D, Grynberg G *Europhys. Lett.* **2** 755 (1986)
2. Grandclement D, Grynberg G, Pinard M *Phys. Rev. Lett.* **59** 44 (1987)
3. Khitrova G, Valley J F, Gibbs H M *Phys. Rev. Lett.* **60** 1126 (1988)
4. Grandclement D, Pinard M, Grynberg G *IEEE J. Quantum Electron.* **25** 580 (1989)
5. Kryuchkyan G Yu, Kheruntsyan K V *Zh. Eksp. Teor. Fiz.* **103** 18 (1993) [*J. Exp. Theor. Phys.* **76** 9 (1993)]
6. Reintjes J F *Nonlinear Optical Parametric Processes in Liquids and Gases* (New York: Academic Press, 1984)

7. Korn G A, Korn T M *Mathematical Handbook for Scientists and Engineers* 2nd edition (New York: McGraw-Hill, 1967)
8. Akhmanov S A, Khokhlov R V *Problemy Nelineĭnoĭ Optiki* (Problems in Nonlinear Optics) (Moscow: Academy of Sciences of the USSR, 1964)
9. Klyshko D N *Fotony i Nelineĭnaya Optika* (Photons and Nonlinear Optics) (Moscow: Nauka, 1980)
10. Drummond P D, McNeil K J, Walls D F *Opt. Acta* **27** 321 (1980)
11. Savage C M, Walls D F *J. Opt. Soc. Am. B* **4** 1514 (1987)
12. Brambilla M, Castelli F, Lugiato L A, Prati F, Strini G *Opt. Commun.* **83** 367 (1991)