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Four-wave mixing with non-degenerate pumps: steady states and squeezing in the presence of phase modulation

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Abstract. We present a quantum theory of an optical four-wave mixing oscillator under pumping by two laser fields of different frequencies. Our analysis includes all the physical processes that are relevant in the parametric limit of interaction in a medium with cubic non-linearity: four-wave mixing, self- and cross-phase modulation. Dynamical aspects of stable above-threshold generation of an intracavity signal mode, the bistability and squeezing phenomena are studied.

1. Introduction

Four-wave mixing (FWM) in a cavity is one of the basic processes in quantum optics, resulting in the generation of non-classical light. In general, this process is caused by the third-order susceptibility $\chi^{(3)}$ and consists of the transformation of two photons of the pump modes with frequencies ω_1, ω_2 into a pair of photons of two signal modes with frequencies ω_3, ω_4, such that $\omega_1+\omega_2=\omega_3+\omega_4$. In most theoretical and experimental works (see [1–4] and references therein) the case of degenerate pump modes ($\omega_1=\omega_2$) and non-degenerate signal modes ($\omega_3\neq\omega_4$) has been considered in order to obtain light possessing quantum features.

A scheme of intracavity FWM with non-degenerate pumps ($\omega_1\neq\omega_2$) and degenerate signals ($\omega_3=\omega_4\equiv\omega_0$) has been analysed theoretically in [5–7]. The interest in such a FWM scheme is connected with the possibility of stable above-threshold generation of intense one-mode squeezed-light beams (at all three intracavity mode frequencies ω_0, ω_1, ω_2) with non-zero mean amplitudes. The advantages and peculiarities of this FWM scheme are caused by the absence of the phase diffusion effect. We recall that phase diffusion in the above-threshold FWM with degenerate pumps and non-degenerate signals leads to some difficulties in quantum fluctuation analysis itself and, in particular, to the vanishing of the mean amplitudes of generated signal modes. The analysis presented in [5–7] ignores, however, the phase modulation effects which would normally arise in a parametric limit of interaction in a $\chi^{(3)}$ medium.

In the present paper we consider another scheme of FWM with non-degenerate pumps and degenerate signals. We include in our model all the physical processes that are relevant in the limit: four-wave mixing, self-phase modulation and cross-phase modulation. The model is simplified however in that we consider a single-mode cavity tuned to the signal field frequency $\omega_0=(\omega_1+\omega_2)/2$, while the frequencies of the pump fields ω_1 and ω_2 are considered to be away from the cavity resonance, i.e. the cavity is

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transparent for the pump beams. This makes the single-pass effects in the pumps irrelevant, and we can thus neglect the pump depletion. The undepleted pump approximation can be employed also if we consider nearly collinear wavefactor matching conditions in which the propagation directions of the pump beams are slightly tilted with respect to the cavity axis. Note that our analysis is related to the ring cavity configuration. The parametric generation in a similar FWM scheme, however, with counterpropagating pumps and linear cavity has been realized experimentally in [4].

The analysis presented here concerns both semiclassical and quantum-statistical features of the generated signal mode. The semiclassical analysis includes the dynamical aspects of stable above-threshold generation and the bistability phenomenon (section 2). The quantum analysis (section 3) is related to the phase and intensity fluctuations and to the study of the squeezing effect in the above-threshold regime of generation.

2. Equations of motion, steady states and stability analysis

The non-linear system under consideration consists of a $\chi^{(3)}$ medium placed in a single-mode ring cavity with resonant frequency ω_c. The medium is pumped by two pump fields of different frequencies ω_1 and ω_2 which lie away from the cavity resonance. The four-wave mixing (FWM) interaction results in the excitation of the intracavity signal mode whose rotating frame frequency $\omega_0=(\omega_1+\omega_2)/2$ (see below) is determined by exact consideration of the phase matching and is assumed to be close to ω_c. We also assume that the pump fields travel through the medium only once, which makes the single-pass effects in the pumps irrelevant. In this case we can neglect the pump depletion and treat the amplitudes E_1 and E_2 of the pump fields as fixed constants. Including the phase modulation effects into the consideration and accounting for the decay of the cavity mode we write the model Hamiltonian in the rotating wave approximation as follows:

$$H=\hbar\omega_c a^+a+\tfrac{1}{2}\hbar\chi(a^2E_1^*E_2^*\,e^{i(\omega_1+\omega_2)t}+a^{+2}E_1E_2\,e^{-i(\omega_1+\omega_2)t}) \\ +\tfrac{1}{4}\hbar\chi a^{+2}a^2+\hbar\chi(|E_1|^2+|E_2|^2)a^+a+(a^+\Gamma+a\Gamma^+). \qquad (1)$$

Here a^+ and a are the boson creation annihilation operators for the signal mode, χ is the coupling constant proportional to the third-order susceptibility $\chi^{(3)}$. The first term in equation (1) is the free part of the Hamiltonian, the second term is responsible for the FWM interaction, the third and the fourth terms describe the self-phase modulation and the cross-phase modulation respectively. The fifth term accounts for the coupling of the cavity mode with the reservoir, where Γ and Γ^+ are the reservoir operators which will give rise to the cavity damping constant γ.

Following the standard procedures (see, e.g., [8, 9]) and transforming to the frame rotating at the $\omega_0=(\omega_1+\omega_2)/2$ frequency

$$a(t)\to a\exp(-i\omega_0 t) \qquad a^+(t)\to a^+\exp(i\omega_0 t), \qquad (2)$$

we obtain from equation (1) the following interaction picture master equation for the density operator ρ of the signal mode

$$\frac{\partial\rho}{\partial t}=i\Delta[a^+a,\rho]-\frac{i\chi}{2}(E_1^*E_2^*[a^2,\rho]+E_1E_2[a^{+2},\rho])-\frac{i\chi}{4}[a^{+2}a^2,\rho] \\ -i\chi(|E_1|^2+|E_2|^2)[a^+a,\rho]+\gamma(2a\rho a^+-\rho a^+a-a^+a\rho). \qquad (3)$$

Here $\Delta=\omega_0-\omega_c$ is the cavity detuning and we have neglected the thermal fluctuations.

Then we may transform this operator master equation into a *c*-number Fokker–Planck equation in a generalized *P*-representation [9, 10] with independent complex field variables α and α^+, which correspond to slowly varying operators a and a^+. The Fokker–Planck equation derived is equivalent to the following set of stochastic differential equations:

$$\frac{d\alpha}{dt}=-\gamma\alpha+i[\Delta-\chi(|E_1|^2+|E_2|^2)]\alpha-i\frac{\chi}{2}\alpha^+\alpha^2-i\chi E_1E_2\alpha^++R(t) \tag{4a}$$

$$\frac{d\alpha^+}{dt}=-\gamma\alpha^+-i[\Delta-\chi(|E_1|^2+|E_2|^2)]\alpha^++i\frac{\chi}{2}\alpha\alpha^{+2}+i\chi E_1^*E_2^*\alpha+R^+(t) \tag{4b}$$

where R and R^+ are Gaussian noise terms which have the following non-zero correlators

$$\begin{aligned}\langle R(t)R(t')\rangle&=-i\chi(E_1E_2+\tfrac{1}{2}\alpha^2)\delta(t-t')\\ \langle R^+(t)R^+(t')\rangle&=i\chi(E_1^*E_2^*+\tfrac{1}{2}\alpha^{+2})\delta(t-t').\end{aligned} \tag{5}$$

We note that the Ito form (see, e.g., [9, 11]) of the equations has been utilized here. However the difference between the Ito and Stratonovich calculus [9] gives an additional term $i(\chi/4)\alpha$ in equation (4*a*) (and a term $-i(\chi/4)\alpha^+$ in equation (4*b*) respectively) which can be neglected in the approximation of strong pumps $|E_{1,2}|^2\gg1$.

The further analysis of non-linear system described by equation (4) is based on the linearized treatment of quantum fluctuations. The linearization is carried out about the steady-state semiclassical solutions α_0 and α_0^+ ($\alpha_0^+=\alpha_0^*$) obtained from equation (4) by setting the time derivatives to zero and by ignoring the noise terms. For validity of this method it is necessary that the steady states be stable with respect to small fluctuations

$$\delta\alpha(t)=\alpha(t)-\alpha_0 \qquad \delta\alpha^+(t)=\alpha^+(t)-\alpha_0^+.$$

The linearization of equation (4) leads to the following equations for $\delta\alpha$ and $\delta\alpha^+$:

$$\frac{d}{dt}\begin{pmatrix}\delta\alpha\\ \delta\alpha^+\end{pmatrix}=-A\begin{pmatrix}\delta\alpha\\ \delta\alpha^+\end{pmatrix}+\begin{pmatrix}R_0(t)\\ R_0^+(t)\end{pmatrix}, \tag{6}$$

where the drift matrix A is

$$A=\begin{pmatrix}\gamma_0-i\Delta_{ef}+i\chi|\alpha_0|^2 & i\chi(E_1E_2+\alpha_0^2/2)\\ -i\chi(E_1^*E_2^*+\alpha_0^{+2}/2) & \gamma_0+i\Delta_{ef}-i\chi|\alpha_0|^2\end{pmatrix} \tag{7}$$

$\Delta_{ef}=\Delta-\chi(|E_1|^2+|E_2|^2)$, and the non-zero correlators of the noise terms R_0 and R_0^+ are determined by equation (5) with zero subscripts indicating that α and α^+ are replaced by α_0 and α_0^+ respectively.

The steady-state solutions of equation (4) are stable if the real parts of the eigenvalues of the drift matrix A are positive. This condition can easily be checked by the Hurwitz criterion [12]. Solving equation (4) for the steady states and carrying out the stability analysis we obtain the following results.

One of the semiclassical steady-state solutions of equation (4) is a trivial solution $\alpha_0=0$ which describes, in particular, the below-threshold regime of spontaneous generation of the signal mode. The stability condition of this solution is

$$\left(\frac{\chi}{\gamma}\right)^2(|E_1|^4+|E_2|^4+|E_1|^2|E_2|^2)-\frac{2\chi\Delta}{\gamma^2}(|E_1|^2+|E_2|^2)+\left(\frac{\Delta}{\gamma}\right)^2+1>0. \qquad (8)$$

The inequality (8) determines the stability domains and the boundaries (like, for instance, the generation threshold) for dimensionless parameters $(\chi/\gamma)|E_{1,2}|^2$ and Δ/γ, at which the solution $\alpha_0=0$ becomes unstable (see below). In particular, this solution is stable for arbitrary values of detuning parameter Δ/γ in the case $(\chi/\gamma)^2|E_1|^2|E_2|^2<1$. In the reverse case $(\chi/\gamma)^2|E_1|^2|E_2|^2>1$ it is stable in the regions

$$\frac{\Delta}{\gamma}>\frac{\chi}{\gamma}(|E_1|^2+|E_2|^2)+\sqrt{(\chi/\gamma)^2|E_1|^2|E_2|^2-1}$$

and

$$\frac{\Delta}{\gamma}<\frac{\chi}{\gamma}(|E_1|^2+|E_2|^2)-\sqrt{(\chi/\gamma)^2|E_1|^2|E_2|^2-1}.$$

In order to analyse the steady states with non-zero amplitude $\alpha_0\neq 0$ it is more convenient to transform to the intensity (in photon number units) and phase variables

$$n=\alpha^+\alpha \qquad \varphi=\frac{1}{2\mathrm{i}}\ln(\alpha/\alpha^+).$$

The corresponding equations of motion become

$$\frac{\mathrm{d}n}{\mathrm{d}t}=-2\gamma n+2\chi\sqrt{I_1I_2}\,n\sin\psi+f_n \qquad (9a)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\Delta-\chi(I_1+I_2)-\frac{\chi}{2}n-\chi\sqrt{I_1I_2}\cos\psi+f_\varphi \qquad (9b)$$

where we introduce the intensities I_k $(k=1,2)$ and phases φ_k of the pump fields $E_k=\sqrt{I_k}\exp(\mathrm{i}\varphi_k)$ and use the following notation: $\psi=\varphi_1+\varphi_2-2\varphi$.

In equation (9) the noise terms are

$$f_n=R\alpha^++R^+\alpha \qquad (10a)$$

$$f_\varphi=\frac{R}{2\,\mathrm{i}\alpha}-\frac{R^+}{2\,\mathrm{i}\alpha^+}. \qquad (10b)$$

The semiclassical steady-state solutions n_0 and φ_0 ($\alpha_0=\sqrt{n_0}\exp(\mathrm{i}\varphi_0)$) of equation (9) are determined by the following expressions

$$n_0^{(\pm)}=\frac{2\gamma}{\chi}\left[\frac{\Delta}{\gamma}-\frac{\chi}{\gamma}(I_1+I_2)\pm\sqrt{(\chi/\gamma)^2I_1I_2-1}\right], \qquad (11a)$$

$$\sin(\varphi_1+\varphi_2-2\varphi_0)=\frac{\gamma}{\chi\sqrt{I_1I_2}}. \qquad (11b)$$

The stability analysis, carried out on the basis of corresponding linearized equations (see equation (19)), shows that the solution with the minus sign in equation (11a) is unstable. The stable above-threshold generation of the signal mode is described by the solution with the plus sign in equation (11a).

The intensity $I^{\rm out}=\langle a_{\rm out}^{+}a_{\rm out}\rangle$ of the signal mode at the output of a single-ported cavity can be calculated with use of the well known input–output formalism of Collett and Gardiner [13], in which the cavity output field operator $a_{\rm out}$ is connected with the cavity input field $a_{\rm in}$ and with the intracavity operator a by the following relation $a_{\rm out}=\sqrt{2\gamma}a-a_{\rm in}$. Taking into account that the signal mode is initially in a vacuum state ($\langle a_{\rm in}\rangle=\langle a_{\rm in}^{+}a_{\rm in}\rangle=0$) and using the correspondence between the normally ordered operator averages and the averages for c-number stochastic amplitudes α^{+} and α [10], we obtain for the case of the steady-state solution $n_0^{(+)}$:

$$I^{\rm out}=2\gamma\langle\alpha^{+}\alpha\rangle=\frac{4\gamma^2}{\chi}\left[\frac{\Delta}{\gamma}-\frac{\chi}{\gamma}(I_1+I_2)+\sqrt{(\chi/\gamma)^2I_1I_2-1}\right]. \tag{12}$$

In accordance with the linear approximation the contribution of the spontaneous noise has been ignored here. Thus the result (12) corresponds to the semiclassical theory. In the case of the steady-state solution $\alpha_0=0$ the corresponding semiclassical cavity output intensity is $I^{\rm out}=0$.

Now we shall give the results of analysis of the stability conditions in the above-threshold regime.

2.1. *Case of equal pump intensities*

In the case of equal intensities of the pump fields $I_1=I_2\equiv I$ the stability domains for the steady-state solution $n_0^{(+)}$ are determined in explicit form by the following inequalities:

$$I_{\rm A}<I<I_{\rm B} \qquad \text{for } \sqrt{3}<\Delta/\gamma<2 \tag{13a}$$

and

$$\gamma/\chi<I<I_{\rm B} \qquad \text{for } \Delta/\gamma>2. \tag{13b}$$

Here

$$I_{\rm A}=\frac{\gamma}{3\chi}\left(\frac{2\Delta}{\gamma}-\sqrt{(\Delta/\gamma)^2-3}\right) \tag{14}$$

is the threshold value of I, obtained with use of equation (8). The quantity

$$I_{\rm B}=\frac{\gamma}{3\chi}\left(\frac{2\Delta}{\gamma}+\sqrt{(\Delta/\gamma)^2-3}\right) \tag{15}$$

corresponds to the value of the intensity I, beyond which the stability condition is fulfilled for the zero-amplitude solution $\alpha_0=0$. At this point the system returns from the above-threshold generation with non-zero mean amplitude to the stable generation at the spontaneous noise level.

The behaviour of the cavity output intensity $I^{\rm out}$ depending on the pump field intensity I is represented in figure 1(a). The full curves correspond to the stable steady-state solutions, while the broken curve describes the unstable solution $n_0^{(-)}$ (see equation (11a)). We see, in particular, that in accordance with equation (13b) and as a consequence of the inequality $\chi/\gamma<I_{\rm A}$ the phenomenon of optical bistability occurs in our non-linear system for $\Delta/\gamma>2$.

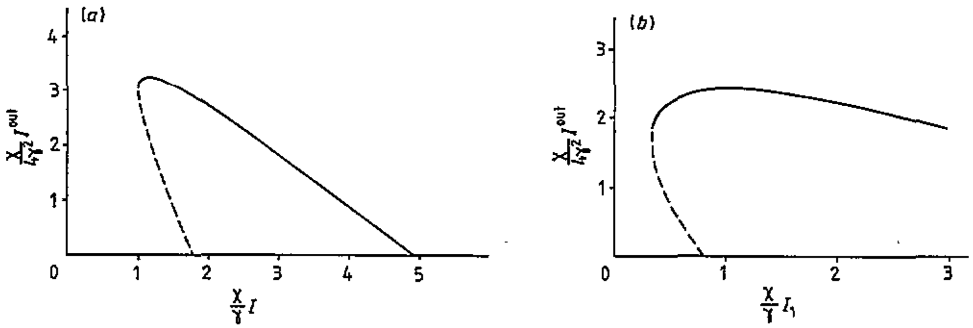

Figure 1. Cavity output intensity versus the intensities of the pump fields. (*a*) case $I_1 = I_2 \equiv I$ with $\Delta/\gamma = 5$; (*b*) case $I_1 \neq I_2$ with $(\chi/\gamma)I_2 = 3$ and $\Delta/\gamma = 5$.

2.2. Case of unequal pump intensities

Consider also the case of unequal pump field intensities $I_1 \neq I_2$ within the conditions when the intensity I_2 (for definiteness) is kept constant, while the intensity I_1 is continuously changed. In this case the threshold value of I_1 is

$$I_{1,\mathrm{A}} = \frac{\gamma}{2\chi}\left[\frac{2\Delta}{\gamma} - \frac{\chi}{\gamma}I_2 - \sqrt{\frac{4\Delta\chi}{\gamma^2}I_2 - \frac{3\chi^2}{\gamma^2}I_2^2 - 4}\right] \tag{16}$$

and the stability domains are

$$I_{1,\mathrm{A}} < I_1 < I_{1,\mathrm{B}} \qquad \text{for} \quad \frac{4 + 3(\chi/\gamma)^2 I_2^2}{4(\chi/\gamma)I_2} < \frac{\Delta}{\gamma} < \frac{1 + (\chi/\gamma)^2 I_2^2}{(\chi/\gamma)I_2} \tag{17a}$$

$$\frac{\gamma^2}{\chi^2 I_2} < I_1 < I_{1,\mathrm{B}} \qquad \text{for} \quad \frac{\Delta}{\gamma} > \frac{1 + (\chi/\gamma)^2 I_2^2}{(\chi/\gamma)I_2} \tag{17b}$$

where

$$I_{1,\mathrm{B}} = \frac{\gamma}{2\chi}\left[\frac{2\Delta}{\gamma} - \frac{\chi}{\gamma}I_2 + \sqrt{\frac{4\Delta\chi}{\gamma^2}I_2 - \frac{3\chi^2}{\gamma^2}I_2^2 - 4}\right]. \tag{18}$$

The dependence of the cavity output intensity I^{out} on the pump intensity I_1 for fixed values of I_2 and Δ/γ is represented in figure 1(*b*). Note that the bistable behaviour of I^{out} on I_1 is in a qualitative accord with the experimental result obtained in [4].

In conclusion of this section we point out the role which the phase modulation effects play in the behaviour of the system. Without the incorporation of the self-phase modulation effect into our model the equation of motion for the signal mode amplitude would be the same as that of the pure degenerate FWM and degenerate parametric oscillator in the undepleted pump approximation (see, e.g., [1, 14]). Here the usual cavity detuning parameter Δ should be formally replaced by an effective cavity detuning $\Delta_{\mathrm{ef}} = \Delta - \chi(|E_1|^2 + |E_2|^2)$, which reflects the influence of the cross-phase modulation effect. In these non-linear systems the only stable steady-state solution is a zero-amplitude solution $\alpha_0 = 0$, which describes the below-threshold regime of generation. The stable above-threshold regime with $\alpha_0 \neq 0$, occuring in the behaviour of our non-linear system, becomes possible due to the inclusion of the self-phase modulation effect into the consideration. The influence of the cross-phase modulation on the non-linear dynamics of the system is reflected in the vanishing of the signal mode intensity well above threshold. Really, since the usual cavity detuning Δ is replaced by Δ_{ef} in the equations of motion, the increase of the pump field intensities $I_{1,2}=|E_{1,2}|^2$ leads to the decrease of the effective cavity detuning. This implies that with increase of $I_{1,2}$ the system is carried to the off-resonance operation regime (with respect to the cavity resonance ω_c) and the signal mode generation vanishes.

We note also that the influence of the cross- and self-phase modulation effects for parametric FWM with degenerate pumps and non-degenerate signals has been analysed in [2].

3. Quantum fluctuations and squeezing above threshold

Now let us turn to the analysis of quantum fluctuations of the signal mode and to the problem of squeezing in the above-threshold regime. Introducing small deviations $\delta n(t)=n(t)-n_0^{(+)}$ and $\delta\varphi(t)=\varphi(t)-\varphi_0$ from the stable steady states and using equation (9) we obtain the following linearized equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta n\\ \delta\varphi\end{pmatrix}=-\begin{pmatrix}0 & 4\chi\sqrt{I_1I_2}\,n_0^{(+)}\cos\psi_0\\ \chi/2 & 2\chi\sqrt{I_1I_2}\sin\psi_0\end{pmatrix}\begin{pmatrix}\delta n\\ \delta\varphi\end{pmatrix}+\begin{pmatrix}f_n^0(t)\\ f_\varphi^0(t)\end{pmatrix} \tag{19}$$

where $\psi_0=\varphi_1+\varphi_2-2\varphi_0$ and the value of $\cos\psi_0$, which corresponds to the stable solution $n_0^{(+)}$, is

$$\cos\psi_0=-\frac{\gamma\sqrt{(\chi/\gamma)^2I_1I_2-1}}{\chi\sqrt{I_1I_2}}.$$

The correlators of the noise terms f_n^0 and f_φ^0, obtained with the use of equations (5) and (10), are

$$\langle f_n^0(t)f_n^0(t')\rangle=2\gamma n_0^{(+)}\delta(t-t') \tag{20a}$$

$$\langle f_\varphi^0(t)f_\varphi^0(t')\rangle=-\frac{\gamma}{2n_0^{(+)}}\delta(t-t') \tag{20b}$$

$$\langle f_n^0(t)f_\varphi^0(t')\rangle=[\chi(I_1+I_2)-\Delta]\delta(t-t'). \tag{20c}$$

In order to reveal the possibility of reduction of quantum fluctuations below the shot-noise level in the quadrature component operator $X^\vartheta=a\exp(-i\vartheta)+a^+\exp(i\vartheta)$, we shall calculate first the corresponding variance

$$V(\vartheta)=1+\langle:(\Delta X^\vartheta)^2:\rangle. \tag{21}$$

Here $\Delta X^\vartheta=X^\vartheta-\langle X^\vartheta\rangle$, ϑ is an arbitrary phase angle defined by the phase of the local oscillator, :: denotes the normal ordering of operators, and the squeezed noise reduction occurs if $V(\vartheta)<1$.

Using the correspondence between normally ordered averages for the operators a^+, a and averages for the c-number stochastic amplitudes α^+, α we obtain, in the lowest order in small fluctuations,

$$V(\vartheta)=1+4n_0^{(+)}\langle\delta\varphi^2\rangle\sin^2(\vartheta-\varphi_0)+\frac{1}{n_0^{(+)}}\langle\delta n^2\rangle\cos^2(\vartheta-\varphi_0)+2\langle\delta n\delta\varphi\rangle\sin(2\vartheta-2\varphi_0). \tag{22}$$

From equations (19), (20) one can obtain the following results for the intensity and phase fluctuation correlators

$$n_0^{(+)}\langle\delta\varphi^2\rangle=\frac{d-J_1-J_2}{8\sqrt{J_1J_2-1}} \tag{23}$$

$$\frac{\langle\delta n^2\rangle}{n_0^{(+)}}=\frac{1+(J_1+J_2-d)\sqrt{J_1J_2-1}}{2\sqrt{J_1J_2-1}\,[d-J_1-J_2+\sqrt{J_1J_2-1}]} \tag{24}$$

$$\langle\delta n\delta\varphi\rangle=-\frac{1}{4\sqrt{J_1J_2-1}}. \tag{25}$$

Here and below we use the following notations

$$d=\frac{\Delta}{\gamma} \qquad J_{1,2}=\frac{\chi}{\gamma}I_{1,2}.$$

The fact that the intensity and phase fluctuations of the signal mode are mutually correlated (equation (25)) is another peculiar property of our non-linear system. We recall that such a correlation does not occur in the previously formulated models of FWM, which do not take into account the phase modulation effects (see, e.g., [1, 3, 5]). This fact leads, in particular, to the variance $V(\vartheta)$ being minimized at a phase angle ϑ which differs in general from the case of pure phase fluctuations ($\vartheta=\varphi_0+\pi/2$) or of pure amplitude fluctuations ($\vartheta=\varphi_0$). Nevertheless the results (22)–(24) show that in the cases of pure phase or pure amplitude fluctuations the squeezed noise reduction is sufficiently high and the intracavity squeezing may reach ~50% for appropriate values of the parameters Δ/γ and $(\chi/\gamma)I_{1,2}$.

Really, for the case of phase fluctuations, which corresponds to the choice $\vartheta=\varphi_0+\pi/2$, we have

$$V(\vartheta=\varphi_0+\pi/2)=1+4n_0^{(+)}\langle\delta\varphi^2\rangle=1-\frac{J_1+J_2-d}{2\sqrt{J_1J_2-1}} \tag{26}$$

and the phase squeezing occurs for $J_1+J_2-d>0$. The maximal 50% squeezing is reached in the limit where $n_0^{(+)}\to 0$.

In the case $\vartheta=\varphi_0$, i.e. in the case of amplitude fluctuations, the corresponding variance is

$$V(\vartheta=\varphi_0)=1+\frac{\langle\delta n^2\rangle}{n_0^{(+)}}=1-\frac{(d-J_1-J_2)\sqrt{J_1J_2-1}-1}{2\sqrt{J_1J_2-1}\,(d-J_1-J_2+\sqrt{J_1J_2-1})}. \tag{27}$$

The amplitude squeezing occurs for $d-J_1-J_2>1/\sqrt{J_1J_2-1}$ and may reach ~50% for large values of the detuning parameter $d=\Delta/\gamma\gg 1$ and for $J_{1,2}\sim 1$.

It should be pointed out that the negativeness of the correlation function of intensity fluctuations $\langle\delta n^2\rangle$ is manifested also in the non-classical effect of photon antibunching via the second-order correlation function $g^{(2)}(0)=\langle a^+a^+aa\rangle/\langle a^+a\rangle^2$. The correlation function $g^{(2)}(0)$ in the lowest order in quantum fluctuations is

$$g^{(2)}(0)=1+\frac{\langle\delta n^2\rangle}{(n_0^{(+)})^2}. \tag{28}$$

However this antibunching effect ($g^{(2)}(0)<1$) is small in accordance with the condition of validity of the linearization procedure, i.e. with the condition of small fluctuations $\langle \delta n^2 \rangle \ll (n_0^{(+)})^2$.

Now we shall present the results on squeezing in the spectrum of quadrature component fluctuations for the cavity output signal field:

$$S(\vartheta, \omega) = 1 + 2\gamma \int_{-\infty}^{+\infty} \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}\omega\tau} \langle : \Delta X^{\vartheta}(t) \Delta X^{\vartheta}(t+\tau) : \rangle. \tag{29}$$

The unity in the right-hand side of equation (29) corresponds to the shot-noise level. As the output field is a broadband continuum around the carrier frequency ω_0, the squeezed noise reduction ($S(\omega, \vartheta) < 1$) may occur in a certain frequency band.

The result for the squeezing spectrum in the case of phase fuctuations ($\vartheta = \varphi_0 + \pi/2$) is

$$S(\varphi_0 + \pi/2, \omega) = 1 + 8\gamma n_0^{(+)} \langle \delta\varphi(-\omega) \delta\varphi(\omega) \rangle$$

$$= 1 - \frac{4(\omega/\gamma)^2 - 16(d - J_1 - J_2 + \sqrt{J_1 J_2 - 1})^2}{[4(d - J_1 - J_2 + \sqrt{J_1 J_2 - 1})\sqrt{J_1 J_2 - 1} - (\omega/\gamma)^2]^2 + 4(\omega/\gamma)^2}. \tag{30}$$

Here the correlator $\langle \delta\varphi(-\omega) \delta\varphi(\omega) \rangle$ has been calculated with use of the Fourier-transformed version of equation (19).

A graphical representation of the phase squeezing spectrum $S(\varphi_0 + \pi/2, \omega)$ in the case of equal pump intensities $I_1 = I_2 \equiv I$ and for particular values of the parameters $d = \Delta/\gamma$ and $J = (\chi/\gamma)I$ is given in figure 2(a). We note that the squeezing occurs at the sideband frequencies located symmetrically about the zero frequency. The dependence of the spectrum value $S(\varphi_0 + \pi/2, \omega_{\text{opt}})$ (ω_{opt} is the optimal frequency, for which the phase squeezing is maximal) on the intensity parameter of the pump field J is plotted in figure 2(b). We see that a nearly perfect (100%) phase squeezing is realized for $I \simeq I_B$.

At the phase angle $\vartheta = \varphi_0$, determining the amplitude fluctuations, we obtain for

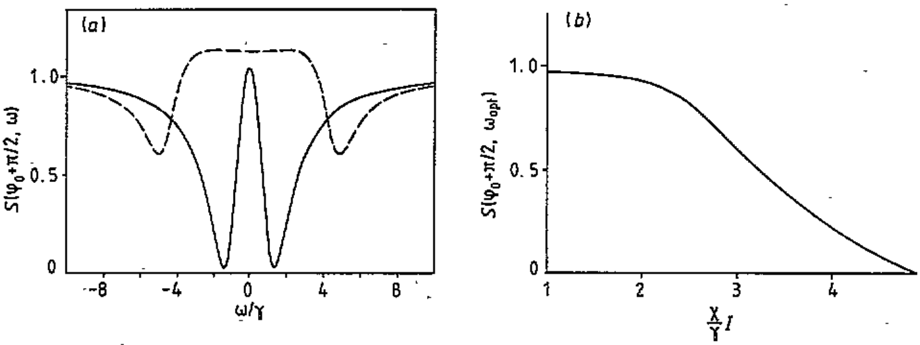

Figure 2. (a) Squeezing spectrum $S(\varphi_0 + \pi/2, \omega)$ versus ω/γ for $\Delta/\gamma = 5$, $(\chi/\gamma)I = 3$ (broken curve), $(\chi/\gamma)I = 4.8$ (full curve). (b) Dependence of $S(\varphi_0 + \pi/2, \omega_{\text{opt}})$ on $(\chi/\gamma)I$ for $\Delta/\gamma = 5$.

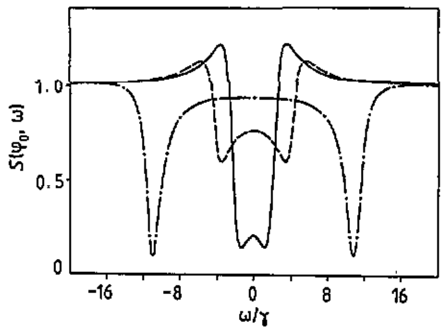

Figure 3. Squeezing spectrum $S(\varphi_0, \omega)$ versus ω/γ for $(\chi/\gamma)I=1.1$, $\Delta/\gamma=5$ (full curve); $(\chi/\gamma)I=2$, $\Delta/\gamma=5$ (broken curve); $(\chi/\gamma)I=2$, $\Delta/\gamma=20$ (dotted curve).

the squeezing spectrum

$$S(\varphi_0, \omega)=1+\frac{2\gamma}{n_0^{(+)}}\langle\delta n(-\omega)\delta n(\omega)\rangle$$

$$=1-\frac{16J_1J_2+32(d-J_1-J_2)\sqrt{J_1J_2-1}-32-4(\omega/\gamma)^2}{[4(d-J_1-J_2+\sqrt{J_1J_2-1})\sqrt{J_1J_2-1}-(\omega/\gamma)^2]^2+4(\omega/\gamma)^2}. \quad (31)$$

Examples of the curves for the squeezing spectrum (31) are given in figure 3 for the case $I_1=I_2\equiv I$. It is seen that a nearly perfect amplitude squeezing occurs for values of the pump intensity parameter J close to the unity. To obtain large amplitude squeezing for higher pump intensities it is necesary to increase the detuning parameter d.

It should be noted that the results on experimental measurements of the noise on the quadrature component in a similar FWM configuration have been presented in [4]. However, the squeezed noise reduction has not been achieved in this experiment. This is not so surprising because, as follows from our analysis, the squeezing depends strongly on a proper choice of the values of the parameters ω/γ, Δ/γ and $(\chi/\gamma)I_{1,2}$.

Finally we turn to the behaviour of the quadrature component fluctuations for the case of the steady-state solution $\alpha_0=0$ in the corresponding stability domains, i.e. in the regimes below threshold and well above threshold. In this case, in accordance with the linearized approximation, the self-phase modulation terms in equations (4a) and (4b) (the third terms) give no contribution in the final linearized equations for the signal-mode amplitude fluctuations $\delta\alpha$ and $\delta\alpha^+$ (see equations (6), (7)). These linearized equations turn out to be similar to those of the pure degenerate FWM and of the degenerate parametric oscillator in the below-threshold regime [1, 14]. The calculation of the corresponding squeezing spectrum gives a result coinciding with equation (47) of [1], in which the parameters ν, γ_2 and $\bar{\gamma}_2$ must be replaced by $\nu\to\chi E_1E_2$, $\gamma_2\to\gamma$, $\bar{\gamma}_2\to\gamma-i\Delta_{ef}$, where $\Delta_{ef}=\Delta-\chi(|E_1|^2+|E_2|^2)$. It should be borne in mind, however, that in our non-linear system the stability domain for the solution $\alpha_0=0$ is not restricted by the below-threshold regime. This solution, under appropriate values of the parameters $(\chi/\gamma)I_{1,2}$ and Δ/γ, becomes stable also well above threshold. Therefore equation (47) of [1], being applied to our non-linear system, describes the squeezing effect not only below the generation threshold, but also above threshold. The particular behaviour of the maximally squeezed quadrature component fluctuations in dependence on the parameters of the system can be analysed numerically.

4. Conclusions

In conclusion we have presented a quantum analysis of a model of intracavity four-wave mixing, for which the effects of self-phase modulation and cross-phase modulation were taken into account. These effects are important in the parametric limit of the interaction in the $\chi^{(3)}$ medium, however they are usually neglected. We have shown that they have an essential influence on the non-linear dynamics and on the quantum statistical properties of generated signal field.

In particular, the semiclassical steady-state solutions and the stability analysis show that the self-phase modulation is responsible for the appearance of the stable above-threshold regime of generation with non-zero amplitude of the signal field. The stabilization of the zero amplitude solution well above threshold is the result of the cross-phase modulation which determines the effective cavity detuning $\Delta_{ef}=\Delta-\chi(|E_1|^2+|E_2|^2)$ and carries the system away from the resonance operation. A bistable behaviour of the signal intensity versus the pump intensities has also been predicted for appropriate values of the detuning parameter Δ/γ.

The analysis of quantum statistical properties of the signal field has been carried out for the above-threshold generation regime within the framework of a linear treatment of quantum fluctuations. This analysis shows that the signal field posseses non-classical features. In particular, the possibility of phase squeezing as well as of amplitude squeezing is predicted for appropriate values of the parameters Δ/γ and $(\chi/\gamma)I_{1,2}$. The photon antibunching effect for the signal field is shown as well.

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