Exercise: A particle moves on the inside surface of a cone of half angle α. The axis of the cone is vertical with the vertex downwards. Determine the condition on the angular velocity ω such that the particle can describe a horizontal circle h above the vertex. Show that the period of small oscillations about this circular path is

$$\frac{2\pi}{\cos\alpha}\sqrt{\frac{h}{3g}}.$$

Solution: The kinetic and potential energies are given by

$$T = \frac{1}{2}m(\frac{\dot{z}^2}{\cos^2\alpha} + z^2\tan^2\alpha\dot{\theta}^2), \qquad V = mgz$$

where z is the vertical distance of the particle from the apex and m the mass of the particle. Then

$$L = \frac{1}{2}m(\frac{\dot{z}^2}{\cos^2\alpha} + z^2\tan^2\alpha\dot{\theta}^2) - mgz.$$

The equations of motion are:

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = m\tan^2\alpha\frac{d}{dt}(z^2\dot{\theta}) \\ \Rightarrow \quad & z^2\dot{\theta} = c, \qquad c \text{ a constant,} \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{z}}\right) - \frac{\partial L}{\partial z} \\ &= \frac{m\ddot{z}}{\cos^2\alpha} - mz\tan^2\alpha\dot{\theta}^2 + mg \\ \Rightarrow \quad \ddot{z} &= z\sin^2\alpha\dot{\theta}^2 - g\cos^2\alpha. \end{aligned}$$

If the particle is to describe a horizontal circle h above the vertex then $\dot{z} = \ddot{z} = 0$. Solving the second equation of motion, we find

$$0 = h\sin^2\alpha\omega^2 - g\cos^2\alpha \quad \Rightarrow \quad \omega = \frac{1}{\tan\alpha}\sqrt{\frac{g}{h}}.$$

Now consider a small deviation around this circular path, so $z = h + \delta z$ where δz is small. Then $\ddot{z} = \ddot{\delta z}$. Now from the first equation of motion we have

$$z^2\dot{\theta} = c = h^2\omega \quad \Rightarrow \quad \theta = \frac{h^2}{z^2}\omega.$$

Substituting into the second equation of motion, we find

$$
\begin{aligned}
\ddot{z} = \ddot{\delta z} &= z \sin^2 \alpha \frac{h^4 \omega^2}{z^4} - g \cos^2 \alpha \\
&= \cos^2 \alpha \frac{g h^3}{(h + \delta z)^3} - g \cos^2 \alpha \\
&\approx g h^3 \cos^2 \alpha \left(\frac{1}{h^3} - \frac{3 \delta z}{h^4} \right) - g \cos^2 \alpha \\
&=\approx -3 \cos^2 \alpha \frac{g}{h} z.
\end{aligned}
$$

This describes simple harmonc motion with angular frequency $\omega_0 = \cos \alpha \sqrt{\frac{3g}{h}}$ and period $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\cos \alpha} \sqrt{\frac{h}{3g}}$.