PHYS2100:
Hamiltonian dynamics and chaos

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Chapter 1

Introduction

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Useful texts

• Rasband: Chaotic dynamics of nonlinear systems. Q172.5.C45 R37 1990.
• Percival and Richards: Introduction to dynamics. QA614.8 P47 1982.
• Baker and Gollub: Chaotic dynamics: an introduction. QA862 .P4 B35 1996.
• Gleick: Chaos: making a new science. Q172.5.C45 G54 1998.
• Abramowitz and Stegun, editors: Handbook of mathematical functions: with formulas, graphs, and mathematical tables. QA47.L8 1975

The lecture notes will be complete: However you can only improve your understanding by reading more. We will begin this section of the course with a brief reminder of a few essential conncepts from the first part of the course taught by Dr Karen Dancer.

1.1 Basics

A mechanical system is known as *conservative* if

$$\oint \mathbf{F} \cdot \mathbf{dr} = 0. \tag{1.1}$$

Frictional or dissipative systems do not satisfy Eq. (1.1).

Using vector analysis it can be shown that Eq. (1.1) implies that there exists a potential function such that

$$\mathbf{F} = -\nabla V(\mathbf{r}). \tag{1.2}$$

for some $V(\mathbf{r})$. We will assume that conservative systems have *time-independent* potentials.

A *holonomic constraint* is a constraint written in terms of an equality e.g.

$$|\mathbf{r}| = a, \qquad a > 0. \tag{1.3}$$

A non-holonomic constraint is written as an inequality e.g. $|\mathbf{r}| \geq a$.

1.2 Lagrangian mechanics

For a mechanical system of N particles with k holonomic constraints, there are a total of $3N - k$ degrees of freedom.

The system can be represented by $3N - k$ generalised coordinates $q_1, q_2, \ldots, q_{3N-k}$, such that

$$\mathbf{r} \ = \ \mathbf{r}(q_1, q_2, \ldots, q_{3N-k}, t).$$

For a *conservative* system with $V = V(q_1, q_2, \ldots, q_{3N-k})$, the Lagrangian is defined as

$$\mathcal{L} \ = \ T - V, \tag{1.4}$$

and the dynamics of the system can be found from the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} \ = \ 0 \tag{1.5}$$

[If V is a function of $\dot{q}_i$ then an extra term is required.]

1.3 Hamiltonian mechanics

The Hamiltonian formulation of mechanics does not add any new physics. However it provides a method that is more powerful and versatile than the Lagrangian approach. It is particular useful for extending the theory into other fields such as statistical mechanics and quantum mechanics: fundamental areas of physics that are covered in detail at 3rd year at UQ.

Lagrange's equations form a system of $n = 3N - k$ *second-order* differential equations requiring $2n$ initial conditions to obtain a unique solution. The Hamiltonian formulation is based upon Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{1.6}$$

which form a system of $2n$ *first-order* differential equations, again requiring $2n$ boundary conditions for a unique solution. We define the generalised momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{1.7}$$

and the Hamiltonian for the system is

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_i \dot{q}_i p_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{1.8}$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Note that $\mathbf{q}$ and $\mathbf{p}$ are *independent* variables.

For simple mechanical systems you will have shown that

$$H = T + V, \tag{1.9}$$

which says that H is the total mechanical energy of the system, and this will be the case for the majority of systems that we look at. It is also worth noting that the Hamiltonian of a conservative system has no explicit time dependence i.e. $H = H(\mathbf{q}, \mathbf{p})$.

Chapter 2

Phase space

The “space” of the $(\mathbf{q}, \mathbf{p})$ coordinates specifying a dynamical system is referred to as the “phase space”, and is a very important concept in physics. The complete specification of all phase space co-ordinates is sometimes called a “microstate” and contains all you can possibly know about the system.

In Hamiltonian mechanics, the dynamics is defined by the evolution of points in phase space. For a system with n degress of freedom, the phase space coordinates are made up of n generalised position coordinates $\mathbf{q}$ and n generalised momentum coordinates $\mathbf{p}$, and so phase space has a total of $2n$ dimensions.

2.1 Flow vector field

Example: Bead on a wire.

The trajectory of the bead is a curve in (q, p) space parameterised by time. It can be drawn out by following a point travelling with a certain “velocity”. It is not a true velocity as its components are time derivatives of generalised position and momentum coordinates.

If we consider a system with one degree of freedom, the velocity is

$$\mathbf{v} = (\dot{q}, \dot{p}), \tag{2.1}$$

and making use of Hamilton’s equations

$$\mathbf{v} = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right). \tag{2.2}$$

For a general system with n degrees of freedom

$$\mathbf{v}(\mathbf{q}, \mathbf{p}) \quad = \quad (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n), \tag{2.3}$$

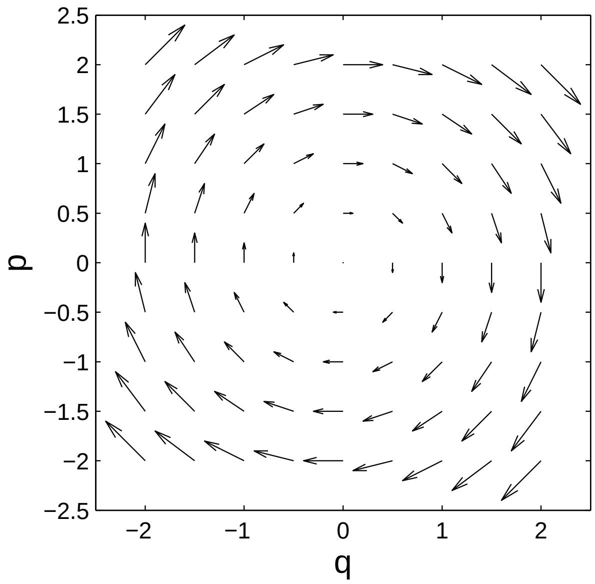

Figure 2.1: Vector flow field for the SHO with $m = \omega = 1$.

$$= \quad \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \ldots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \ldots, \frac{\partial H}{\partial p_n} \right). \tag{2.4}$$

$$\tag{2.5}$$

So Hamilton's equations are enough to define $\mathbf{v}(\mathbf{q}, \mathbf{p})$.

For every point of phase space there is a velocity vector — in other words there exists a velocity field $\mathbf{v}(\mathbf{q}, \mathbf{p})$, usually referred to as the "flow vector field".

In principle the flow vector field enables the dynamics of the system completely.

Example: Simple Harmonic Oscillator (SHO) in 1D

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2. \tag{2.6}$$

Hamilton's equations give

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q, \tag{2.7}$$

so the vector flow field is

$$\mathbf{v} = \left(\frac{p}{m}, -m\omega^2 q \right). \tag{2.8}$$

If we set $m = \omega = 1$ then we can represent this graphically: $\mathbf{v} = (p, -q)$ as in Fig. 2.1.

Note that we have been assuming that the flow vector field $\mathbf{v}$ is not a function of time. This is true when H is time independent. But if, for example we have

$$H = \frac{p^2}{2m} + q \cos \omega t \tag{2.9}$$

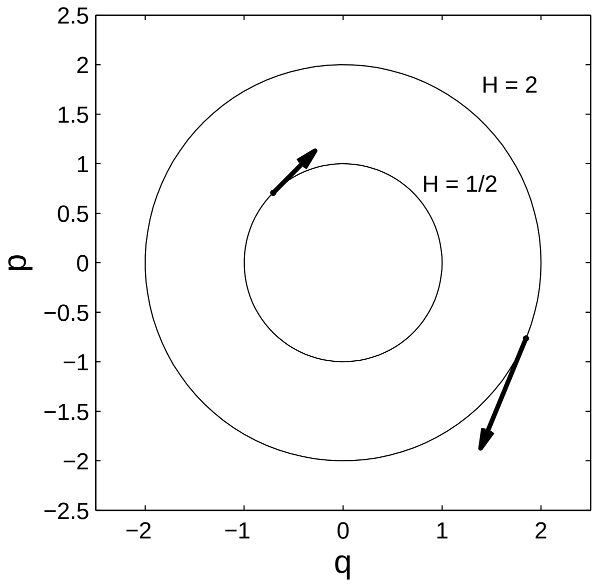

Figure 2.2: Lines of constant H for the SHO

then

$$\mathbf{v} = (p/m, -\cos\omega t), \tag{2.10}$$

which is a *time-varying* flow vector field.

2.2 Phase portraits

Most of our focus will be on systems with time independent Hamiltonians, for which there is the useful result that the velocity vectors $\mathbf{v}$ are always tangential to lines of constant H (energy). For the SHO example with $m = \omega = 1$ then the Hamiltonian is

$$H = \frac{1}{2}(p^2 + q^2), \tag{2.11}$$

which describes a circle of radius $\sqrt{2H}$ as in Fig. 2.2 To prove that $\mathbf{v}$ is tangential to the lines of constant H, you need to make use of the well known result from vector calculus that the gradient of a scalar function f is perpendicular to lines of constant f. Then we find that

$$\mathbf{v} \cdot \nabla H = 0 \tag{2.12}$$

so $\mathbf{v}$ is perpendicular to ∇H. But ∇H is also perpendicular to lines of constant H, so $\mathbf{v}$ must be *tangential* to lines of constant H.

Lines of constant H are extremely important for time independent Hamiltonians as they define *trajectories* (or paths) through phase space for the system. To demonstrate this we first shown that if H has no explicit time dependence ($\partial H/\partial t = 0$) then it has no implicit time dependent either ($dH/dt = 0$).

If $\partial H/\partial t = 0$ then

$$\begin{aligned} \frac{d}{dt}H(q,p) &= \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}, \\ &= \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \times -\frac{\partial H}{\partial q}, \\ &= 0. \end{aligned} \tag{2.13}$$

where we have used Hamilton's equations in the second line. Thus we must have that $H =$ constant, and that a particle will move along a line of constant H.

In systens with one degree of freedom, and hence a 2D phase space, lines of constant H are call *phase portraits*. They are simply paths in phase space.

Our analysis here has essentially been a proof that energy is conserved in a system with a time dependent Hamitonian. (Note that it was assumed that H was not a function of $\dot{p}$ or $\dot{q}$.)

Phase portraits are an excellent means of visualising the dynamics of a mechanical system.

2.3 Fixed points

For many systems there may be special points in phase space where the velocity vector $\mathbf{v}$ is zero. These are known as *fixed points*, and provide a starting point for the analysis of dynamical systems.

If $\mathbf{v} = (0,0)$ then $\dot{q} = 0$ and $\dot{p} = 0$ and hence $\nabla H = (\partial H/\partial q, \partial H/\partial p) = (0,0)$. When the system resides at a fixed point it is in *mechanical equilibrium*.

Fixed points only occur for $p = 0$ in simple mechanical systems where we have $T = p^2/2m$ and the potential $V = V(q)$ only.

2.4 Examples

Unless otherwise specified, we consider Hamiltonians of the form

$$H(q,p) = \frac{p^2}{2m} + V(q). \tag{2.14}$$

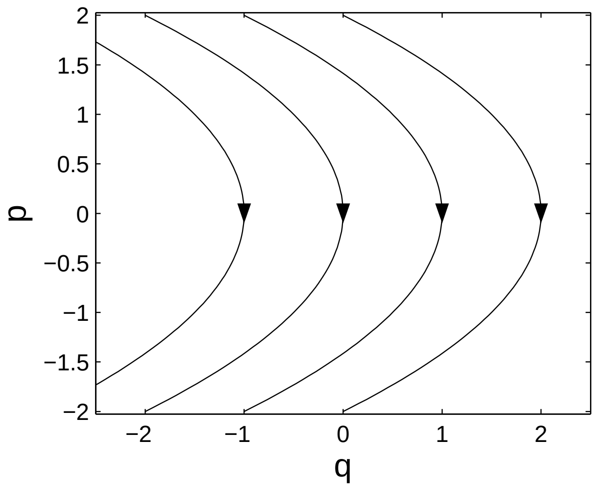

Figure 2.3: Phase portrait for a linear potential $(a = m = 1)$ and $H = -1, 0, 1, 2$. Note that all trajectories in a phase portrait must have an arrow indicating the direction.

2.4.1 Linear potential

$$V(q) = aq, \qquad a > 0. \tag{2.15}$$

The phase portrait can be found by fixing H and plotting p as a function of q (or vice versa). In general we have

$$p(q) = \pm\sqrt{2m}\sqrt{H - V(q)}. \tag{2.16}$$

For the current example we have

$$p = \pm\sqrt{2m(H - aq)}, \qquad \text{or} \qquad q = \frac{H}{a} - \frac{p^2}{2ma}. \tag{2.17}$$

Thus the trajectories are parabolas, which makes sense seeing as the potential is like the gravitational potential. The phase portrait is in Fig. 2.3.

Note also that this potential has no fixed points (the proof is in tutorial problems).

We now solve Hamilton's equations for this potential. We have

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -a. \tag{2.18}$$

We need boundary conditions to find a specific solution: let's say that at time $t = t_0$ we have $q = q(t_0)$ and $p = p(t_0)$. For this simple situation we can directly integrate the equation for $\dot{p}$ in Eqs. (2.18) to give

$$p(t) \;=\; p(t_0) + \int_{t_0}^{t}(-a)dt' \;=\; p(t_0) - a(t - t_0). \tag{2.19}$$

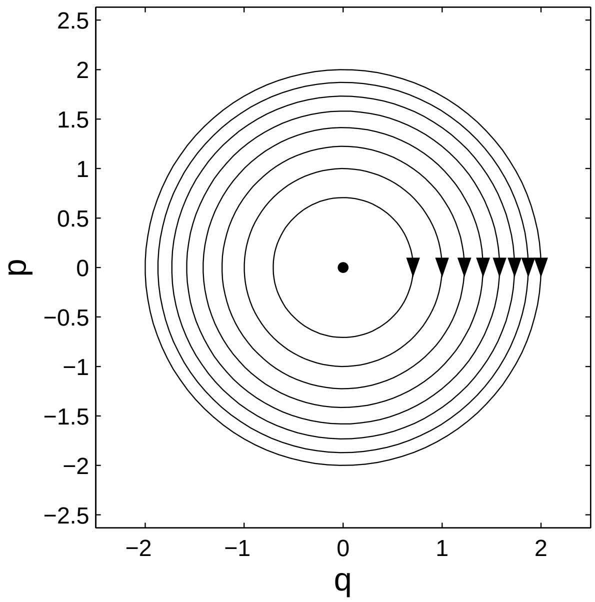

Figure 2.4: Phase portrait for a SHO $(m = \omega = 1)$, for $H = 0.25$ to $H = 2$ in steps of 0.25. Note that in general the trajectories are ellipses.

We can use Eq. (2.19) to solve for q(t)

$$\begin{aligned} q(t) &= q(t_0) + \int_{t_0}^{t} dt' \frac{p(t')}{m} \\ &= q(t_0) + \frac{1}{m}\int_{t_0}^{t} dt' \left[p(t_0) - a(t' - t_0)\right], \\ &= q(t_0) + \frac{p(t_0)}{m}(t - t_0) - \frac{a}{2m}(t - t_0)^2. \end{aligned} \tag{2.20}$$

It is not difficult to show that if the solutions Eqs. (2.19) and (2.19) are substituted back into the Hamiltonian that the result is time independent.

2.4.2 Quadratic potential (SHO)

The simple harmonic oscillator (or SHO) is an extremely important model in physics. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2, \tag{2.21}$$

and the phase portrait is shown in Fig. 2.4. We first determine the fixed points. Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q, \tag{2.22}$$

so that $\dot{q} = 0$ when $p = 0$ and $\dot{p} = 0$ when $q = 0$. Hence there is only one fixed point for this system at the origin: $(q, p) = (0, 0)$. This is classified as an *elliptic* fixed point — a fixed point that is encircled by a line of constant H. Elliptic fixed points are *stable* — any small perturbation away from equilibrium remains contained in a small region about the fixed point.

We can now solve for arbitrary trajectories by differentiating the equation for $\dot{q}$ with respect to t and substituting in for $\dot{p}$. This gives

$$\ddot{q} = -\omega^2 q, \tag{2.23}$$

which has the general solution

$$q(t) = A\cos(\omega t + \delta), \tag{2.24}$$

where the constants A and δ are determined by the boundary conditions.

The momentum is then determined from the equation for $\dot{q}$ as

$$p(t) \;=\; m\dot{q}(t) \;=\; -m\omega A\sin(\omega t + \delta). \tag{2.25}$$

The motion is obviously oscillatory with period $T = 2\pi/\omega$ which is independent of A. This may not seem like a big deal, but for a general potential the period of motion usually depends on the amplitude, and we will spend quite some time later developing a method to calculate the period of motion for confining potentials.

The energy H of the system is determined by the amplitude a. By substituting the solutions into the Hamiltonian we find (you should check this!)

$$H = \frac{1}{2}m\omega^2 A^2. \tag{2.26}$$

At the elliptic fixed point it is clear that $H = 0$.

The phase space trajectories are in general ellipses in phase space, with $p_{max} = \sqrt{2mH}$ and $q_{max} = \sqrt{2H/m\omega^2}$. The area of the ellipse in phase space we denote I and find

$$I = \pi \times \frac{2H}{m\omega^2} \times \sqrt{2mH} = \frac{2\pi}{\omega}H. \tag{2.27}$$

Thus we find that here we have

$$\omega = 2\pi\frac{\partial H}{\partial I}, \tag{2.28}$$

which is a specific example of a general result that we will derive later.

2.4.3 Linear Repulsive Force

In this situation we have

$$F(q) = -aq, \qquad a > 0. \tag{2.29}$$

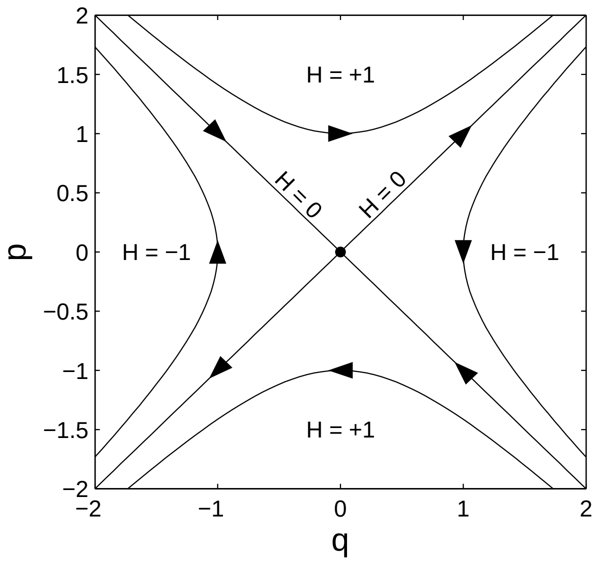

Figure 2.5: Phase portrait for a linearly repulsive force $(m = a = 1)$ for $H = 0, \pm 1$.

Since $F(q) = -\partial V/\partial q$ we can integrate to find

$$V(q) = -\frac{1}{2}aq^2. \tag{2.30}$$

which is an inverted parabola. (Question for you: what about the constant of integration?). Thus the Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{1}{2}aq^2. \tag{2.31}$$

To plot the phase portrait, we rearrange Eq. 2.31 to obtain

$$2mH = p^2 - m^2\gamma^2 q^2, \qquad \left(\gamma = \frac{a}{m}\right). \tag{2.32}$$

This is the equation of a hyperbola.

Hamilton's equations for the system are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = aq \tag{2.33}$$

Setting these to zero to finc the fixed points we find that there is only one and it is at the origin $(q, p) = (0, 0)$. This is a different type of fixed point compared to the one we found for the SHO — it is known as a *hyperbolic* fixed point. Hyperbolic fixed points are *unstable*, as any small perturbation from equilibrium will grow.

Any curve in phase space that meets a hyperbolic fixed point is known as a *separatrix*, as they "separate" different types of motion. The determination of seperatrices is an important part of determining global dynamics.

Solutions for this system can be found by the same method as for the SHO. We find

$$\ddot{q} = \gamma q, \tag{2.34}$$

which has the general solution

$$q(t) = A_1 e^{\gamma t} + A_2 e^{-\gamma t}, \tag{2.35}$$

with the constants A_1 and A_2 determined by the boundary conditions. Correspondingly the momentum is

$$p(t) = m\gamma(A_1 e^{\gamma t} + A_2 e^{-\gamma t}). \tag{2.36}$$

A tutorial problem asked you to show that

$$H = -2aA_1A_2. \tag{2.37}$$

Note that the motion is unbounded in general as $q(t)$ and $p(t) \to \pm\infty$.

2.4.4 Cubic potentials

$$V(q) = \frac{1}{2}m\omega^2 q^2 - \frac{1}{3}Amq^3, \qquad A > 0. \tag{2.38}$$

Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q + Amq^2, \tag{2.39}$$

Fixed points

$$\dot{q} = p/m = 0 \qquad \Rightarrow \qquad p = 0, \tag{2.40}$$
$$\dot{p} = mq(Aq - \omega^2) = 0 \qquad \Rightarrow \qquad q = 0, \omega^2/A. \tag{2.41}$$

So there are two fixed points $(q, p) = (0, 0)$, $(\omega^2/A, 0)$.

To determine if they are elliptic or hyperbolic it is sufficient to analyse their local region in phase space.

Fixed point $(0, 0)$

In the vicinity of $q = 0$ we have $|q| \ll 1$, and so to a good approximation

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 \tag{2.42}$$

This is the Hamiltonian for the SHO: and we have found already that the fixed point for this type of potential is a elliptic fixed point that is stable.

Fixed point $(\omega^2/A, 0)$

The analysis for this point is a little more complicated, and we will write $q_1 = \omega^2/A$ for short. We consider a Taylor series of $\dot{p}(q)$ about the point $q = q_1$.

Remember: a Taylor series of a function $f(x)$ about the point $x = a$ is

$$f(x) \approx f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots \tag{2.43}$$

So we have

$$\dot{p}(q) = \dot{p}(q_1) + (q-q_1)\frac{\partial \dot{p}}{\partial q}\Big|_{q=q_1} + \frac{(q-q_1)^2}{2}\frac{\partial^2 \dot{p}}{\partial q^2}\Big|_{q=q_1} + \dots \tag{2.44}$$

As we are carrying out the expansion about a fixed point, the first term is zero. Keeping the first non-zero term, we find that

$$\dot{p}(q) = (q-q_1)(-m\omega^2 + 2Amq_1). \tag{2.45}$$

Substituting in $q_1 = \omega^2/A$ we have

$$\dot{p}(q) = m\omega^2(q-q_1), \tag{2.46}$$

and combining this with the the equation for $\dot{q}$ we finc that in the vicinity of $q = q_1$ we have the equation of motion

$$\ddot{q} = \omega^2(q-q_1), \tag{2.47}$$

with the general solution

$$q(t) = q_1 + a_1 e^{\omega t} + A_2 e^{-\omega t}. \tag{2.48}$$

So in general $q(t)$ increases exponentially with time and hence the solutions are unstable. As this corresponds to the situation with the linear repulsive force, this means $(\omega^2/A, 0)$ is a hyperbolic fixed point.

This result could also have been found by linearising the Hamiltonian about the point $q = q_1$, and finding that in this region the system corresponded to the Hamiltonian for the linear repulsive force.

The phase portrait for the system is shown in Fig. 2.6 The energy at each fixed point can be found by substitution into the Hamiltonian. We find

$$\text{For} \quad (q,p) = (0,0): \qquad H = 0, \tag{2.49}$$

$$\text{For} \quad (q,p) = (\omega^2/A, 0): \qquad H = \frac{m\omega^2}{2}\frac{\omega^4}{A^2} - \frac{1}{3}Am\frac{\omega^6}{A^3} \;=\; \frac{m\omega^6}{6A^2}. \tag{2.50}$$

Since all paths in phase space for this system have fixed values of H, this means the separatrices are defined by the equation

$$\frac{m\omega^6}{6A^2} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 - \frac{1}{3}Amq^3. \tag{2.51}$$

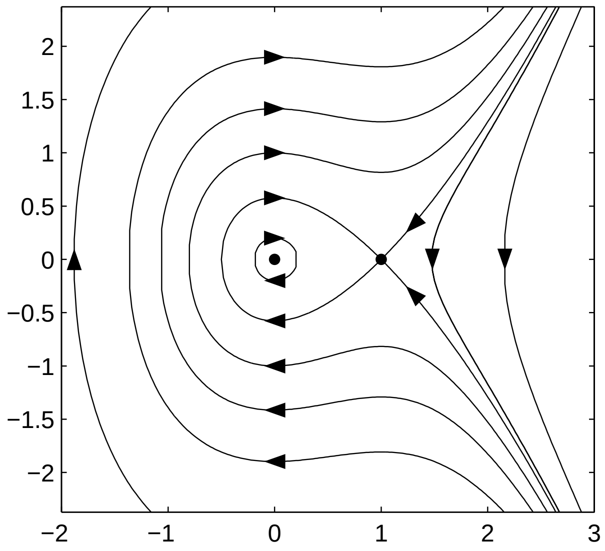

Figure 2.6: Phase portrait for a cubic potential with $m = \omega = A = 1$.

2.4.5 Summary

Through the previous four examples we have introduced the methods typically used in the analysis of a conservative Hamiltonian system with one degree of freedom. For this course a complete analysis can be summarised as follows:

1. Construct the Hamiltonian $H(q, p)$ where
$$H = T + V, \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{q}}.$$
2. Write down Hamilton's equations for the system
$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p},$$
3. Find all of the fixed points (q, p) such that $(\dot{q}, \dot{p}) = (0, 0)$.
4. Determine the stability of the dynamics in the vicinity of the fixed point. If the coordinate $q(t)$ does not, in general, become large with increasing time then it is stable, otherwise it is unstable. For systems we consider stable points are elliptic fixed points, and unstable points are hyperbolic fixed points.
5. For hyperbolic fixed points determine the equations of the separatrices.
6. Plot the phase portrait.
7. Solve Hamilton's equations.

8. If you manage to complete all these steps, you deserve a beer for a job well done.

Aside: In the study of dynamical systems other types of fixed points are possible, in particular stable or unstable "nodes", "stars" and "spirals". However, these do not occur for conservative Hamiltonian systems.

2.5 Periodic motion

There are two types of periodic motion that can occur in Hamiltonian dynamics. — *libration* and *rotation*.

2.5.1 Libration

Libration is closed motion, where the system retraces its steps periodically so that q and p are periodic functions of time with the same frequency. The name "libration" comes from astronomy. A pendulum in a clock is a classical example, and the trajectories are closed loops in a phase portrait.

2.5.2 Rotation

Here p is some periodic function of q with a period q_0, but q is not a periodic function of time. The most familiar example is rotation of a rigid body, with q as the angle of rotation and $q_0 = 2\pi$.

2.5.3 Free particle rotating in a plane

Imagine a particle of mass m attached to one end of a rigid, massless rod of length a that is able to pivot about the other end that is fixed. The configuration space of the system can be represented by the angle ϕ that it makes to the vertical axis as shown in Fig. 2.7. This is a simple system that can display rotational dynamics, for which we already know

$$\text{Moment of inertia:} \qquad J = ma^2, \tag{2.52}$$

$$\text{Angular momentum:} \qquad \ell = |\underline{\ell}| = J\dot{\phi}, \tag{2.53}$$

$$\text{Lagrangian:} \qquad \mathcal{L} = T = \frac{1}{2}J\dot{\phi}^2. \tag{2.54}$$

For the generalised coordinate ϕ the generalised momentum is $\partial\mathcal{L}/\partial\dot{\phi} = J\dot{\phi}$, which is simply the angular momentum ℓ as given above. Thus we have

$$H = \frac{\ell^2}{2J}. \tag{2.55}$$

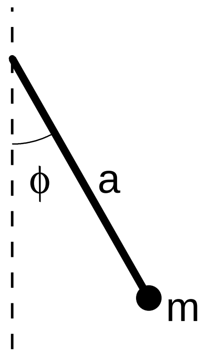

Figure 2.7: Configuration for a free particle rotating in a plane. The configuration is the same for a pendulum, with the addition of the gravitational force mg acting downwards.

Hamilton's equations

$$\dot{\ell} = -\frac{\partial H}{\partial \phi} = 0 \qquad \Rightarrow \qquad \ell \text{ is conserved.} \tag{2.56}$$

$$\dot{\phi} = \frac{\partial H}{\partial \ell} = \frac{\ell}{J}, \tag{2.57}$$

which is consistent with what we wrote down above. As ℓ is constant, Eq. (2.57) can be solved to give

$$\phi(t) = \phi(t_0) + \frac{\ell}{J}(t - t_0). \tag{2.58}$$

2.5.4 Pendulum

The configuration for the pendulum is again as in Fig. 2.7, but with the graviational force acting on the mass in the downwards direction. The gravitational potential is in this case given by

$$V(\phi) = -mga\cos\phi, \tag{2.59}$$

Thus the Lagrangian is

$$\mathcal{L} = \frac{1}{2}J\dot{\phi}^2 + mga\cos\phi \tag{2.60}$$

The conjugate momentum is the same as in the previous section and so the Hamiltonian is

$$H(\phi, \ell) = \frac{\ell^2}{2J} - mga\cos\phi. \tag{2.61}$$

Hamilton's equations

$$\dot{\ell} = -\frac{\partial H}{\partial \phi} = -mga\sin\phi, \tag{2.62}$$

$$\dot{\phi} = \frac{\partial H}{\partial \ell} = \frac{\ell}{J}. \tag{2.63}$$

The fixed points are where $(\dot{\phi}, \dot{\ell} = (0,0)$, and so we have

$$\dot{\phi} = 0 \qquad \text{when} \qquad \ell = 0, \tag{2.64}$$

$$\dot{\ell} = 0 \qquad \text{when} \qquad \phi = 0, \pm\pi, \pm 2\pi, \ldots, \tag{2.65}$$

however of course ϕ is periodic and so there are only two physically distinct fixed points $(\phi, \ell) = (0,0)$ and $(\pi, 0)$. These correspond to the pendulum hanging down and hanging up.

Intuitively you might guess that $(0,0)$ is a stable fixed point and that $(\pi, 0)$ is unstable. To show this mathematically we need only consider motion in the vicinity of the fixed point.

Fixed point $(0,0)$

For small ϕ we can approximate $\sin\phi \approx \phi$, and so Eq. (2.63) becomes

$$\dot{\ell} = -mga\phi. \tag{2.66}$$

Differentiating Eq. (2.63) with respect to time, and substituting in Eq. (2.66) gives

$$\ddot{\phi} = -\frac{mga}{J}\phi, \tag{2.67}$$

which has the general solution of the SHO

$$\phi(t) = A\cos(\omega_0 t + \delta), \qquad \omega_0 = \left(\frac{mga}{J}\right)^{1/2} = \sqrt{\frac{g}{a}}, \tag{2.68}$$

and again the constants A and δ are determined by the boundary conditions. As the solution is the same as for the SHO then this must be a stable elliptic fixed point.

Fixed point $(\pi, 0)$

In the vicinity of $\phi = \pi$, we have

$$\sin(\phi) \;=\; \sin(\pi - \phi) \;\approx\; \pi - \phi, \tag{2.69}$$

so Eq. (2.63) becomes

$$\dot{\ell} = mga(\pi - \phi). \tag{2.70}$$

Differentiating Eq. (2.63) with respect to time, and substituting in Eq. (2.70) gives

$$\ddot{\phi} = \frac{mga}{J}(\phi - \pi). \tag{2.71}$$

If we make the change of coordinate $\gamma = \phi - \pi$ then Eq. (2.71) becomes

$$\ddot{\gamma} = \omega_0 \gamma, \tag{2.72}$$

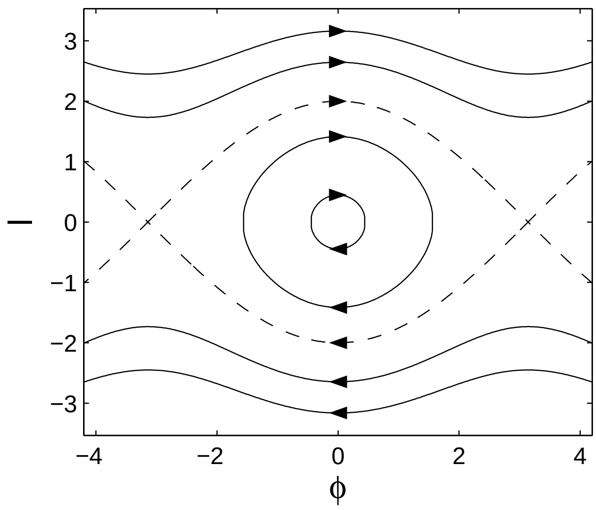

Figure 2.8: Phase portrait for the pendulum. The separatrices are indicated by the dashed lines.

that has the general solution

$$\gamma(t) = A_1 e^{\omega_0 t} + A_2 e^{-\omega_0 t}. \tag{2.73}$$

Thus we have

$$\phi(t) = \pi + A_1 e^{\omega_0 t} + A_2 e^{-\omega_0 t}. \tag{2.74}$$

This is the same situation as we had for the repulsive linear force: this is a hyperbolic fixed point and is unstable.

Phase portrait

The phase portrait for the pendulum is shown in Fig. 2.8. The separatrix divides the phase space into three types of motion

1. Above the upper separatrix the pendulum rotates about its pivot point in an anticlockwise direction.
2. Below the lower separatrix the pendulum rotates about its pivot point in an clockwise direction.
3. Between the separatrices the pendulum oscillates (librates) back and forth.

Note that the pendulum is NOT the simple harmonic oscillator except for the limiting case where the amplitude of oscilation is very small and hence $|\phi| \ll 1$ (you should check this!)

Separatrices

The energy of the system on the separatrices is given by substituting the coordinates of the hyperbolic fixed point $(\phi, \ell) = (\pi, 0)$ back into the expression for the Hamiltonian Eq. (2.61). We find that

$$H(\pi, 0) = mga, \tag{2.75}$$

which would have been expected: this is the gravitational potential energy at this point.

Thus, when the energy of the system exceeds mga the the pendulum rotates and the motion is of type (1) or (2) described earlier. If the energy is less that mga then the motion is of type (3).

The general solution of Hamilton's equations for the pendulum cannot be expressed in terms of simple functions. However, they can be for the special case of the separatrices, where from the Hamiltonian Eq. (2.61) we have

$$\begin{aligned} mga &= \frac{\ell^2}{2J} - mga\cos\phi, & (2.76)\\ \Rightarrow \qquad \ell &= \pm[2Jmga(1+\cos\phi)]^{1/2}. & (2.77)\end{aligned}$$

However, using the double angle formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we can have

$$(1+\cos\phi)^{1/2} = \sqrt{2}\cos(\phi/2), \tag{2.78}$$

and so

$$\ell = \pm 2(Jmga)^{1/2}\cos(\phi/2). \tag{2.79}$$

Substituting Eq. (2.79) into Eq. (2.64) for $\dot{\phi}$ gives the differential equation

$$\dot{\phi} = \pm 2\omega_0 \cos(\phi/2). \tag{2.80}$$

and made use of the definition of ω_0 from Eq. (2.68). Equation (2.80) is a seperable differential equation of first order and can be written

$$\frac{d\phi}{\cos(\phi/2)} = \pm 2\omega_0 dt. \tag{2.81}$$

Integrating both sides with the help of the result from tables that

$$\int \frac{dz}{\cos z} = \ln[\tan(\pi/4 + z/2)], \tag{2.82}$$

gives

$$2\ln[\tan(\pi/4 + \phi/4)] = \pm 2\omega_0 t + C, \tag{2.83}$$

where C is a constant that depends on the boundary conditions. If we choose $\phi(t=0) = 0$ then on rearranging Eq. (2.83) we have

$$\phi(t) = 4\tan^{-1}[\exp(\pm\omega_0 t)] - \pi. \tag{2.84}$$

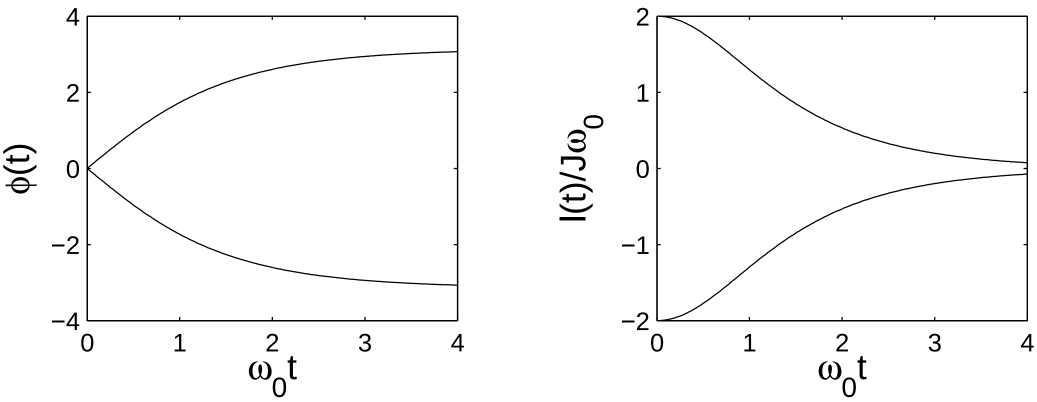

Figure 2.9: The solutions for $\phi(t)$ and $\ell(t)$ for the pendulum on the separatrix

Physically this corresponds to travelling from the bottom of the separatrix to the hyperbolic point

We can now find the solution for the momentum. From Eq. (2.63) we have

$$\ell(t) = J\dot{\phi}(t). \tag{2.85}$$

Using the chain rule on Eq. (2.84) and

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}, \tag{2.86}$$

we find that

$$\ell(t) = \pm\frac{2J\omega_0}{\cosh(\omega_0 t)}. \tag{2.87}$$

Note that as $t \to \infty$ then $\ell(t) \to 0$. These solutions are shown in Fig 2.9.

In the next section of the notes we will find the general solution of the pendulum using action-angle variables.