Chapter 3

Canonical transformations

3.1 Volume preserving flows

The concept of the conservation of volume in phase space is very important in Hamiltonian mechanics. For a system with n degrees of freedom, volume is defined as

$$V = \int_\Omega d\mathbf{q}\, d\mathbf{p}, \tag{3.1}$$

where Ω is the integration volume. Thus for a system with one degree of freedom, the volume is an "area" with units $kg\, m^2\, s^{-1} \equiv J\, s$.

Liouville's theorem: The volume of a phase space flow is conserved for Hamiltonian systems.

We can show this explicity for a 1D conservative Hamiltonian system.

Consider a small rectangular region bounded by two paths in phase space, with energies given by H and $H + \delta H$ respectively as depicted in Fig. 3.1. The magnitude of the flow vector field is given by

$$|\mathbf{v}| = \left| \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right) \right| = \left[\left(\frac{\partial H}{\partial p} \right)^2 + \left(\frac{\partial H}{\partial q} \right)^2 \right]^{1/2} = |\nabla H|, \tag{3.2}$$

So the length x of the area A is simply

$$x \approx |\mathbf{v}|\, \delta t = |\nabla H|\, \delta t. \tag{3.3}$$

As ∇H is perpendicular to curves of constant H then it follows that

$$|\nabla H| \approx \frac{\delta H}{y}. \tag{3.4}$$

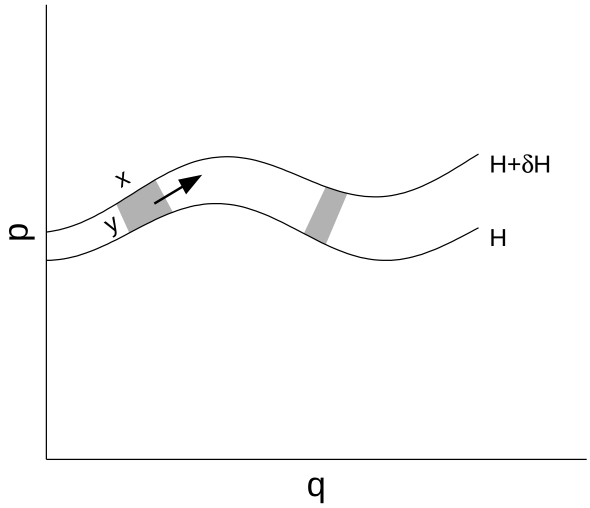

Figure 3.1: Volume preserving flows in phase space.

Thus the area A is given by

$$A = xy = |\nabla H|\,\delta t \times y = \frac{\delta H}{y}\delta t \times y = \delta H\,\delta t. \tag{3.5}$$

So what this means is that for *any* point along the two trajectories, the area will be given by $\delta H\,\delta t$, and these quantities are chosen by us. Thus the area of the element is conserved as it moves through phase space.

This result also holds of non-conservative Hamiltonian systems, and for any number of degrees of freedom.

3.2 Change of coordinates

There are many choices of coordindates to represent a particular Hamiltonian system. A good choice will

1. Reduce Hamilton's euqations to the simplest possible form.
2. Preserve the integrity of Hamilton's equations.

So given a system described by coordinates q and p with Hamiltonian $H(q,p)$ where

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \qquad \dot{p} = -\frac{\partial H(q,p)}{\partial q}, \tag{3.6}$$

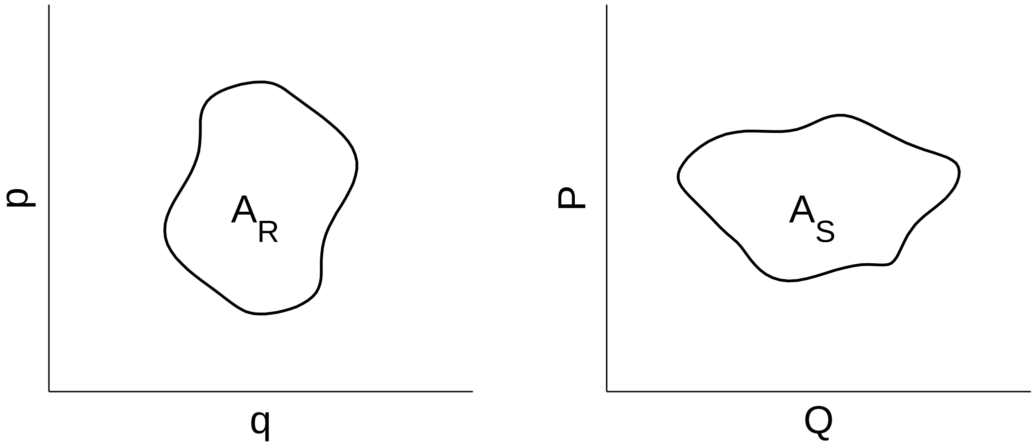

Figure 3.2: Coordinate transform of area A_R in (q, p) to area A_S in (Q, P).

we wish to find a transformation $(q, p) \rightarrow (Q, P)$ that simplifies the analysis of the system and also ensures that

$$\dot{Q} = \frac{\partial \bar{H}(Q, P)}{\partial P}, \qquad \dot{P} = -\frac{\partial \bar{H}(Q, P)}{\partial Q}, \qquad H(q, p) = \bar{H}(Q, P). \tag{3.7}$$

Such a transformation is called a *canonical transformation*.

To determine if a transformation is canonical we exploit the area preserving nature of a Hamiltonian system. Consider a region R with area A_R in (q, p) space. Suppose it is transformed to a region S with area A_S in the (Q, P) system, as depicted in Fig. 3.2. The area of the region R is

$$A_R = \int_R dq\, dp, \tag{3.8}$$

and the area of region S is

$$A_S = \int_S dQ\, dP, \tag{3.9}$$

In general there is no reason to assume that A_R and A_S will be the same. In fact, it can be shown that

$$\int_S dQ\, dP = \int_R dq\, dp\, \frac{\partial(Q, P)}{\partial(q, p)}. \tag{3.10}$$

where $\frac{\partial(Q,P)}{\partial(q,p)}$ is known as the Jacobian of the transformation. It is defined by

$$\frac{\partial(Q, P)}{\partial(q, p)} = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}. \tag{3.11}$$

Example

We again consider a freely rotating mass. We may choose our set of generalised coordinates as (q, p) where q is the distance around the circumference of the pivot from a fixed point, and $p = m\dot{q}$. An alternative set of coordinates is (ϕ, ℓ) as considered in Section 2.5.3. The transformation between the coordinates is

$$\phi = \frac{q}{a}, \qquad \ell = ma^2\dot{\phi} = ma\dot{q} = ap. \tag{3.12}$$

The Jacobian is

$$\frac{\partial(\phi,\ell)}{\partial(q,p)} \;=\; \begin{vmatrix} \frac{\partial\phi}{\partial q} & \frac{\partial\phi}{\partial p} \\ \frac{\partial\ell}{\partial q} & \frac{\partial\ell}{\partial p} \end{vmatrix} \;=\; \begin{vmatrix} 1/a & 0 \\ 0 & a \end{vmatrix} \;=\; \frac{1}{a}\times a - 0\times 0 \;=\; 1. \tag{3.13}$$

Thus the areas in phase space are identical in the two coordinate systems.

An important result of classical mechanics is that for a flow to be area preserving then the area must be independent of the coordinate system. This can be compactly expressed as

$$\{Q, P\}_{(q,p)} = 1. \tag{3.14}$$

where we have introduced the *Poisson bracket* notation.

Poisson brackets

For any two functions $f(q, p)$ and $g(q, p)$ the Poisson bracket is defined as

$$\{f, g\}_{(q,p)} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{3.15}$$

Another important result is that if the Jacobian of a transformation is unity then the Poisson bracket of any two functions f and g is also independent of the representation. That is

$$\frac{\partial(Q,P)}{\partial(q,p)} \;=\; 1 \qquad \Rightarrow \qquad \{f,g\}_{(q,p)} \;=\; \{f,g\}_{(Q,P)}. \tag{3.16}$$

Proof of this fact is left to an exercise.

The Poisson bracket also provides a means of determing time dependence. For example, we see that

$$\{q, H\}_{(q,p)} = \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q} \;=\; 1\times\frac{\partial H}{\partial p} - 0\times\frac{\partial H}{\partial q} \;=\; \frac{\partial H}{\partial q} \;=\; \dot{q}, \tag{3.17}$$

where we have used the fact that q and p are independent variables and Hamilton's equations. Similarly it can be shown that

$$\dot{p} = \{p, H\}. \tag{3.18}$$

For any function $f(q,p,t)$ it can be shown that

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial t}, \tag{3.19}$$

and you are asked to prove this in your tutorial sheet.

So far we have shown that a flow is area preserving if

$$\{Q, P\}_{(q,p)} = 1. \tag{3.20}$$

We now show that if Eq. (3.20) is satisfied then the transformation $(q,p) \to (Q,P)$ is canonical i.e. Hamilton's equations remain valid under the transformation according to Eq. (3.7).

Beginning with $\dot{Q}$, we have

$$\begin{aligned} \dot{Q}(q,p) &= \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p}, &(3.21)\\ &= \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial q} + \frac{\partial Q}{\partial p}\left(-\frac{\partial H}{\partial q}\right). &(3.22)\end{aligned}$$

As we have $H = \bar{H}$, then it follows that

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{\partial \bar{H}}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial \bar{H}}{\partial P}\frac{\partial P}{\partial p}, &(3.23)\\ \frac{\partial H}{\partial q} &= \frac{\partial \bar{H}}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial \bar{H}}{\partial P}\frac{\partial P}{\partial q}, &(3.24)\end{aligned}$$

Subsituting Eqs. (3.23) and (3.24) back into Eq. (3.22) gives

$$\begin{aligned} \dot{Q}(q,p) &= \frac{\partial Q}{\partial q}\left(\frac{\partial \bar{H}}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial \bar{H}}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial Q}{\partial p}\left(\frac{\partial \bar{H}}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial \bar{H}}{\partial P}\frac{\partial P}{\partial q}\right), &(3.25)\\ &= \frac{\partial \bar{H}}{\partial Q}\left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right) + \frac{\partial \bar{H}}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right), &(3.26)\\ &= \frac{\partial \bar{H}}{\partial P}\{Q, P\}_{(q,p)} \quad = \quad \frac{\partial \bar{H}}{\partial P}, &(3.27)\end{aligned}$$

where in the final line we have made use of Eq. (3.20). Similarly it can be shown that (see tutorial sheet)

$$\dot{P}(q,p) = -\frac{\partial \bar{H}}{\partial Q}. \tag{3.28}$$

Thus in summary, we have shown that if $\{Q, P\}_{(q,p)} = 1$ then the transformation $(q,p) \to (Q,P)$ preserves Hamilton's equations and is thus known as a canonical transformation.

Aside: A lot of the formalism of classical Hamiltonian mechanics carries through to quantum mechanics. In particular, the Poisson bracket is replaced by the commutator. When it comes to considering the dynamics of an operator $\hat{A}$ (in what is called the Heisenberg picture), we find that

$$\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] + \frac{\partial \hat{A}}{\partial t}, \tag{3.29}$$

where $\hat{H}$ is the Hamiltonian operator for the system, and we have the definition of the commutator is

$$[\hat{A}, \hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A}. \tag{3.30}$$

This is equivalent to the classical expression

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}. \tag{3.31}$$

where now A and H are functions on phase space and H is the Hamiltonian of the system.

3.3 Action-angle variables

One of the main uses of a canonical transformation is to transform to a set of coordinates for which the equations of motion take on a particularly simple form.

For example, suppose we find a transformation $(q, p) \to (Q, P)$ such that

$$\frac{\partial \bar{H}}{\partial Q} = 0. \tag{3.32}$$

This has two important consequences:

1. Since $\dot{P} = -\partial \bar{H}/\partial Q$ then P is a constant of the motion, and $\bar{H} = \bar{H}(P)$ only.
2. Since $\dot{Q} = \partial \bar{H}/\partial P$, then $\dot{Q} =$ constant as $\partial \bar{H}/\partial P$ is a function of P alone. Thus the solution for Q has the simple form

   $$Q(t) = a_1 t + a_2. \tag{3.33}$$

   for some constants a_1 and a_2 that are determined by the boundary conditions.

The *action-angle* variables (I, θ) are designed specifically to be a transformation of this type. They are particularly powerful for application to systems displaying periodic motion.

Consider a conservative system with Hamiltonian

$$H \;=\; H(q, p) = \frac{p^2}{2m} + V(q). \tag{3.34}$$

We have already shown that H is a constant of the motion. This can be rearranged to give

$$p(q, H) = \pm[2m(H - V(q)]^{1/2} \tag{3.35}$$

where you should think of H as a constant value that defined a particular trajectory. The sign of p depends on the region of phase space the solution is in.

We now define the action variable

$$I = \frac{1}{2\pi}\oint p\,dq \tag{3.36}$$

where the integral is over a complete period of motion. From Eqs. (3.35) and (3.36) we can see that I is a function of H alone i.e. $I = I(H)$. Since H is a constant of motion, then so too will I.

Inverting the relationship, we can also see that H will be a function of I alone i.e. $H = H(I)$. If we defin the generalised coordinate conjugate to I as θ, then from Hamilton's equations we have

$$\dot{\theta} = \frac{\partial H}{\partial I} = \omega \qquad \Rightarrow \qquad \theta(t) = \omega t + \delta, \tag{3.37}$$

for some (as yet undetermined) constants ω and δ.

Note that the action I is simply the area A enclosed by the trajectory with the particular value of H divided by 2π. Thus, we can equivalently write

$$I \;=\; \frac{1}{2\pi}\int_A dq\,dp \;=\; \frac{1}{2\pi}\oint p(q,H)\,dq \;=\; \frac{1}{2\pi}\oint q(p,H)\,dp. \tag{3.38}$$

To determine θ and its connection to the parameters of periodic motion consider the function W known as Hamilton's characteristic function

$$W = \int_0^q p(I,q)\,dq. \tag{3.39}$$

To determine θ we need only impose the area preserving condition of a canonical transformation

$$\text{Area} \quad \delta W = \int_0^q \delta p\,dq \qquad = \qquad \delta I\,\theta. \tag{3.40}$$

As the area is preserved (see Fig. 3.3)

$$\delta W = \delta I\,\theta \qquad \Rightarrow \qquad \theta = \frac{\partial W}{\partial I}, \tag{3.41}$$

i.e.

$$\theta = \frac{\partial}{\partial I}\int_0^q p(I,q)\,dq. \tag{3.42}$$

To obtain a physical interpretation of $\theta(t)$, consider the change in θ over one period of motion

$$\Delta\theta \;=\; \oint \frac{\partial \theta}{\partial q}\,dq \;=\; \oint \frac{\partial^2 W}{\partial q \partial I}\,dq \;=\; \oint \frac{\partial W}{\partial q}\,dq, \tag{3.43}$$

where we have used Eq. (3.41) and the fact that I is constant during the motion. From the definition of the characteristic function Eq. (3.39) we can see that

$$\oint \frac{\partial W}{\partial q}\,dq \;=\; \oint p\,dq \;=\; 2\pi I. \tag{3.44}$$

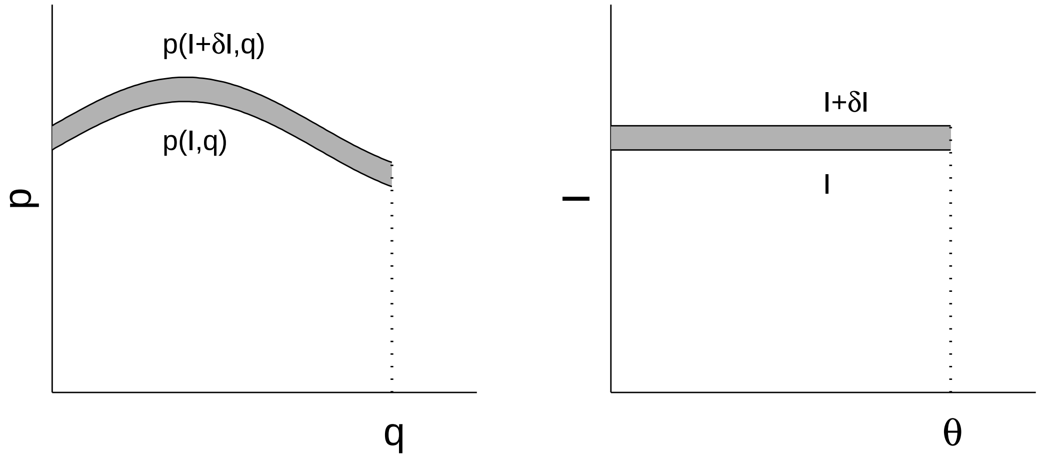

Figure 3.3: Determination of the variable θ in action-angle coordinates

So this means that over the period of motion θ changes by 2π. But from Eq. (3.37) this means that ω must be 2π multiplied by the inverse of the period T. That is, ω is the frequency of periodic motion

$$\omega = \frac{2\pi}{T} = \frac{\partial H}{\partial I}. \tag{3.45}$$

Thus, one of the advantages of the action-angle coordinate system becomes clear: *without solving the equations of motion we can find the angular frequency of the motion.*

Furthermore, since θ changes by 2π through one period of motion then for any other set of coordinates that we may wish to use to describe the system (e.g. ϕ, ℓ it follows that

$$\phi(\theta + 2\pi, I) = \phi(\theta, \ell), \qquad \ell(\theta + 2\pi, I) = \ell(\theta, \ell). \tag{3.46}$$

Example: simple harmonic oscillator

Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \tag{3.47}$$

and so

$$p = \pm(2mH - m^2\omega^2 q^2)^{1/2}. \tag{3.48}$$

The action variable is

$$I = \frac{1}{2\pi}\oint p\, dq, \tag{3.49}$$

$$= \frac{1}{2\pi}\left[\int_{-q_1}^{q_1} (2mH - m^2\omega^2 q^2)^{1/2} dq + \int_{q_1}^{-q_1} -(2mH - m^2\omega^2 q^2)^{1/2} dq\right]. \tag{3.50}$$

However, because

$$\int_a^{-a} f(x)dx = -\int_{-a}^{a} f(x)dx, \tag{3.51}$$

then Eq. (3.50) becomes

$$I = \frac{1}{\pi}\int_{-q_1}^{q_1}(2mH - m^2\omega^2 q^2)^{1/2}dq. \tag{3.52}$$

The limit of integration q_1 is a turning point of the motion where $p = 0$. So from the Hamiltonian Eq. (3.48) we see that

$$q_1 = \sqrt{\frac{2H}{m\omega^2}} \tag{3.53}$$

If we make the variable substitution

$$q = \sqrt{\frac{2H}{m\omega^2}}\sin\phi \qquad \Rightarrow \qquad dq = \sqrt{\frac{2H}{m\omega^2}}\cos\phi d\phi, \tag{3.54}$$

then Eq. (3.52) becomes

$$\begin{aligned} I &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(2mH - m^2\omega^2\frac{2H}{m\omega^2}\sin^2\phi\right)^{1/2}\sqrt{\frac{2H}{m\omega^2}}\cos\phi\, d\phi, && (3.55)\\ &= \frac{2H}{\pi}\int_{-\pi/2}^{\pi/2}\cos^2\phi\, d\phi, && (3.56)\\ &= \frac{H}{\omega}, && (3.57)\end{aligned}$$

where we have used the result that

$$\int_{-\pi/2}^{\pi/2}\cos^2\phi = \frac{\pi}{2}.$$

The frequency of oscillation is given by $\partial H/\partial I$ and thus it is clear that ω represents the frequency of oscillation as before. Also

$$\theta t = \omega t + \delta, \tag{3.58}$$

where δ is determined by the boundary conditions. Thus we have solved the SHO in the action-angle coordinate system.

3.4 Application to the pendulum

We now use action-angle coordinates to find the general solutions for the dynamics of the pendulum. Remember the Hamiltonian is

$$H(\phi, \ell) = \frac{\ell^2}{2J} - mga\cos\phi. \tag{3.59}$$

We define a new parameter

$$\Lambda = \left(\frac{\omega_1^2 + H}{2\omega_1^2}\right)^{1/2}, \qquad \omega_1 = \sqrt{J}\omega_0 = \sqrt{mga}. \tag{3.60}$$

where H represents the energy of a particular trajectory. Note that from Eq. (2.75) ω_1^2 is the energy of the system on the separatrix. It follows that

$$\begin{array}{lllll} \Lambda & = & 1, & \Rightarrow & \text{on the separatrices,} \\ \Lambda & < & 1, & \Rightarrow & \text{libration (within separatrices),} \\ \Lambda & > & 1, & \Rightarrow & \text{rotation (outside separatrices).} \end{array}$$

We have already found the solution for motion on the separatrices. We now consider motion inside and outside.

Librations: $\Lambda < 1$

We begin by calculating the action

$$I(H) = \frac{1}{2\pi}\oint p\,dq. \tag{3.61}$$

For the pendulum the generalised coordinate and momentum are ϕ and ℓ and therefore

$$I(H) = \frac{1}{2\pi}\left(\int_{-\phi_1}^{\phi_1} \ell\,d\phi + \int_{\phi_1}^{-\phi_1} \ell\,d\phi\right), \tag{3.62}$$

$$= \frac{1}{2\pi}\left(\int_{-\phi_1}^{\phi_1} \sqrt{2J(H + \omega_1^2\cos\phi)}\,d\phi + \int_{\phi_1}^{-\phi_1} -\sqrt{2J(H + \omega_1^2\cos\phi)}\,d\phi\right), \tag{3.63}$$

where we have divided the integral over a complete period into two integrals for which $\ell > 0$ and $\ell < 0$. Due to the symmetry of the integrals this reduces to

$$I(H) = \frac{1}{\pi}\int_{-\phi_1}^{\phi_1} \sqrt{2J(H + \omega_1^2\cos\phi)}\,d\phi. \tag{3.64}$$

The limit of integration ϕ_1 is the angle at the turning point of the motion, for which $\ell = 0$. By making this substituion into the Hamiltonian we find that

$$\phi_1 = \cos^{-1}\left(\frac{-H}{\omega_1^2}\right) = 2\sin^{-1}\Lambda. \tag{3.65}$$

The evaluation of the integral is rather lengthy, and instead we simply quote the result

$$I(H) = \frac{8J\omega_0}{\pi}\left[E(\Lambda^2) - (1 - \Lambda^2)K(\Lambda^2)\right], \qquad \Lambda < 1, \tag{3.66}$$

where

$$K(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2} d\theta, \tag{3.67}$$

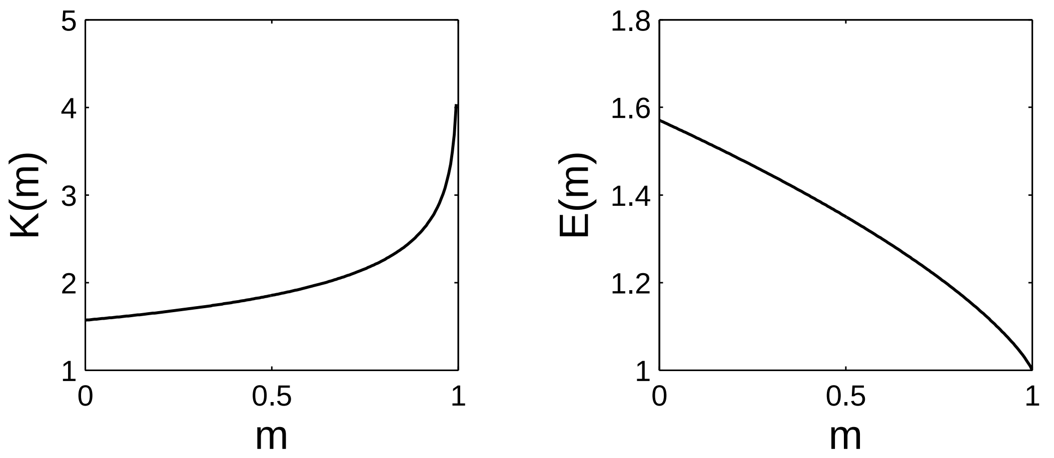

Figure 3.4: Complete elliptic integrals of the first $K(m)$ and second $E(m)$ kind.

is a complete elliptic integral of the first kind, and

$$E(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{1/2} d\theta, \tag{3.68}$$

is a complete elliptic integral of the second kind. (See http://mathworld.wolfram.com for a handy reference to special functions such as these.) These functions are plotted in Fig. 3.4.

Earlier we showed that the angular frequency of motion ω is given by

$$\omega \ = \ \frac{\partial H}{\partial I} \ = \ \left[\frac{\partial I(H)}{\partial H}\right]^{-1}. \tag{3.69}$$

Using the properties of elliptic integrals we can differentiate Eq. (3.66) with respect to H to find

$$\omega(H) = \frac{\pi}{2}\frac{\omega_0}{K(\Lambda^2)}, \qquad \Lambda < 1. \tag{3.70}$$

This result is plotted in Fig. 3.5.

Rotations: $\Lambda > 1$

As before we have

$$I(H) \ = \ \frac{1}{2\pi}\oint p\, dq \ = \ \frac{1}{2\pi}\int_{-\pi}^{\pi} \ell\, d\phi. \tag{3.71}$$

where now the pendulum explores all values of ϕ. We shall assume that the motion is in the anticlockwise direction, and so $\ell > 0$. Thus

$$I(H) \ = \ \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{2J(H + \omega_1^2\cos\phi)}\, d\phi. \tag{3.72}$$

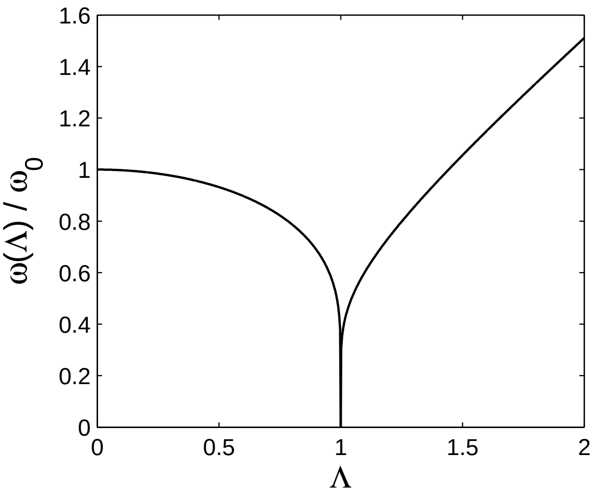

Figure 3.5: Angular frequency of a pendulum as a function of $\Lambda(H)$.

Using the fact that cosine is an even function $\cos\phi = \cos(-\phi)$ and the result $\cos\phi = 1 - 2\sin^2(\phi/2)$, we can write Eq. (3.72) as

$$I(H) = \frac{1}{\pi}\int_0^{\pi}\left[2J(H+\omega_1^2 - 2\omega_1^2\sin^2\phi/2)\right]^{1/2} d\phi, \tag{3.73}$$

$$= \frac{\sqrt{2J}}{\pi}(H+\omega_1^2)^{1/2}\int_0^{\pi}\left[1 - \frac{2\omega_1^2}{H+\omega_1^2}\sin^2\frac{\phi}{2}\right]^{1/2} d\phi \tag{3.74}$$

Making the change of variable $\phi' = \phi/2$ we find

$$I(H) = \frac{\sqrt{2J\times 2\omega_1^2}}{\pi}\left(\frac{H+\omega_1^2}{2\omega_1^2}\right)^{1/2}\times 2\int_0^{\pi/2}\left[1 - \frac{1}{\Lambda^2}\sin^2\phi'\right]^{1/2} d\phi', \tag{3.75}$$

$$= \frac{4\omega_1\sqrt{J}}{\pi}\Lambda E(1/\Lambda^2). \tag{3.76}$$

We find the angular frequency again according to Eq. (3.69) and find

$$\omega(H) = \frac{\pi\omega_1}{\sqrt{J}}\frac{\Lambda}{K(1/\Lambda^2)}, \qquad \Lambda > 1. \tag{3.77}$$

Combining this result with the frequency for $\Lambda < 1$ from Eq. (3.70) we get Fig. 3.5. As well as producing this plot on a computer, we can get it from the series expansion of $K(k)$

$$K(m) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3 + \ldots\right]. \tag{3.78}$$

and so we see $K(m) \to \pi/2$ as $m \to 0$. Thus we have the three limiting cases for the frequency

$$\begin{array}{lll} \omega\Lambda \to \omega_0 & \text{as} & \Lambda \to 0, \\ \omega\Lambda \to 0 & \text{as} & \Lambda \to 1, \\ \omega\Lambda \to 2\Lambda\omega_0 & \text{as} & \Lambda \to \infty. \end{array} \tag{3.79}$$

You should try to derive these results yourself.

Thus, we have exploited one of the advantages of action-angle coordinates: we have found the angular frequency of the pendulum without actually solving the equations of motion.

Full solution: $\phi(t)$

We can now work out the full solution to Hamilton's equations for $\ell(t)$ and $\phi(t)$. We apply Eq. (3.42) to the pendulum to find

$$\theta(\phi) = \frac{\partial}{\partial I} \int_0^{\phi} \ell(\phi', H)\, d\phi'. \tag{3.80}$$

If we solve this for $\phi(\theta, H)$ and substitute in

$$\theta(t) = \omega(H) t + \theta(0), \tag{3.81}$$

then the solutions are

$$\phi(t) \;=\; \begin{cases} 2\cos^{-1}[\mathrm{dn}(\omega_0 t, \Lambda^2)], & \Lambda \leq 1, \\ 2\sin^{-1}[\mathrm{sn}(\Lambda\omega_0 t, 1/\Lambda^2)], & \Lambda > 1, \end{cases} \tag{3.82}$$

where we have taken the boundary condition to be $\phi(0) = 0$. Here we have introduced the *Jacobi elliptic functions* dn and sn.

Aside: Jacobi elliptic functions

Adapted from http://mathworld.wolfram.com/JacobiEllipticFunctions.html (11/10/2006):

The Jacobi elliptic functions are standard forms of elliptic functions. The three basic functions are denoted cn(u, m), dn(u, m), and sn(u, m), where $m = k^2$ and k is known as the elliptic modulus. They arise from the inversion of the elliptic integral of the first kind

$$u = K(\phi, m) = \int_0^{\phi} \frac{dt}{\sqrt{1 - m\sin^2 t}}, \tag{3.83}$$

where $0 < m < 1$, and $\phi = \mathrm{am}(u, m) = \mathrm{am}(u)$ is the Jacobi amplitude, giving $\phi = K^{-1}(u, m) = \mathrm{am}(u, m)$.

From this, it follows that

$$\sin\phi \;=\; \sin(\mathrm{am}(u, m)) \;\equiv\; \mathrm{sn}(u, m), \tag{3.84}$$
$$\cos\phi \;=\; \cos(\mathrm{am}(u, m)) \;\equiv\; \mathrm{cn}(u, m), \tag{3.85}$$
$$\sqrt{1 - k^2\sin^2\phi} \;=\; \sqrt{1 - m\sin^2(\mathrm{am}(u, m))} \;\equiv=\; \mathrm{dn}(u, m). \tag{3.86}$$

These functions are doubly periodic generalizations of the trigonometric functions satisfying

$$\mathrm{sn}(u, 0) = \sin u, \qquad \mathrm{cn}(u, 0) = \cos u, \qquad \mathrm{dn}(u, 0) = 1. \tag{3.87}$$

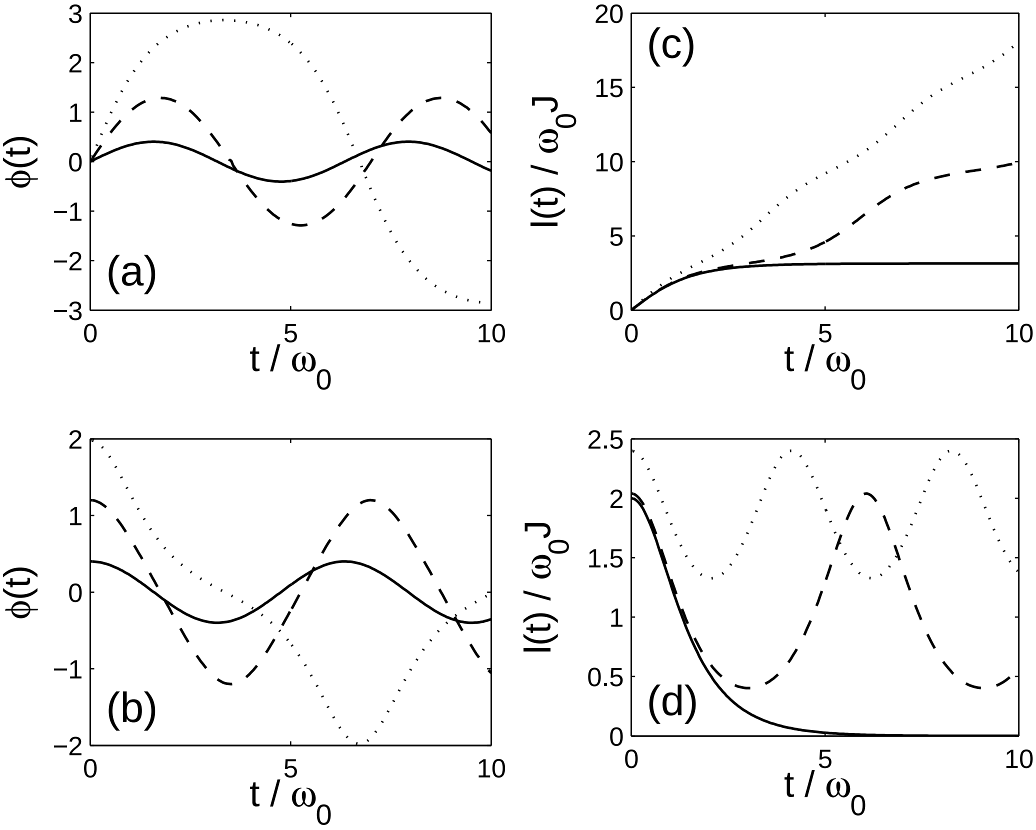

Figure 3.6: Solutions for the pendulum. (a) $\phi(t)$ and (b) $\ell(t)$ for $\Lambda = 0.2$ (solid), $\Lambda = 0.6$ (dashed), and $\Lambda = 0.99$ (dotted). (c) $\phi(t)$ and (d) $\ell(t)$ for $\Lambda = 1$ (solid), $\Lambda = 1.02$ (dashed), and $\Lambda = 1.2$ (dotted).

Full solution: $\ell(t)$

With the boundary condition $\phi(0) = 0$, then we can see that at $t = 0$

$$H = \frac{\ell^2(0)}{2J} - mga = \frac{\ell(0)^2}{2J} - \omega_1^2. \tag{3.88}$$

Inserting this into the definition of Λ from Eq. (3.60) we have

$$\Lambda^2 = \frac{\omega_1^2 + \ell(0)^2/2J - \omega_1^2}{2\omega_1^2}, \tag{3.89}$$

$$\Rightarrow \quad \ell(0) = \pm 2\sqrt{J}\Lambda\omega_1. \tag{3.90}$$

Using this as the boundary condition we can then show that the angular momentum is

$$\ell(t) = \begin{cases} 2\omega_0 J\Lambda\,\mathrm{cn}(\omega_0 t, \Lambda^2)], & \Lambda \leq 1, \\ 2\omega_0 J\Lambda\,\mathrm{dn}(\omega_0 t, 1/\Lambda^2)], & \Lambda > 1. \end{cases} \tag{3.91}$$

This can be derived by, for example, substituting the solution Eq. (3.82) into the Hamiltonian and solving for $\ell(t)$. Examples of the solutions are shown in Fig. 3.6.

Limiting cases

The Jacobi elliptic functions can be written in terms of infinite expansions which allow us to find simple expressions for $\phi(t)$ and $\ell(t)$ in limiting cases.

$$\mathrm{sn}(u,m) = u - (1+m)\frac{u^3}{3!} + (1 + 14m + m^2)\frac{u^5}{5!} + \ldots, \tag{3.92}$$

$$\mathrm{cn}(u,m) = 1 - \frac{u^2}{2!} + (1+4m)\frac{u^4}{4!} - (1 + 44m + 16m^2)\frac{u^6}{6!} + \ldots, \tag{3.93}$$

$$\mathrm{dn}(u,m) = 1 - m\frac{u^2}{2!} + m(4+m)\frac{u^4}{4!} - m(16 + 44m + m^2)\frac{u^6}{6!} + \ldots. \tag{3.94}$$

Case: $\Lambda \ll 1$

In this limit we find that

$$\mathrm{cn}(u, \Lambda \to 0) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \ldots = \cos u, \tag{3.95}$$

and so we find for the angular momentum

$$\ell(t) = 2\omega_0 J\Lambda\cos(\omega_0 t), \tag{3.96}$$

which is the same as the SHO solution in the limit of small oscillations.

Case: $\Lambda = 1$

Here we find

$$\mathrm{cn}(u, \Lambda = 1) \;=\; 1 - \frac{u^2}{2!} + 5\frac{u^4}{4!} - 61\frac{u^6}{6!} + \ldots \;=\; \frac{1}{\cosh u}, \tag{3.97}$$

and so we have

$$\ell(t) = \frac{2\omega_0 J}{\cosh(\omega_0 t)}, \tag{3.98}$$

as we have already derived for the separatrix.

Case: $\Lambda \gg 1$

Here we have

$$\mathrm{dn}(u, 1/\Lambda^2 \to 0) \;\approx\; 1 - \frac{1}{\Lambda^2}\frac{u^2}{2!} + \frac{1}{\Lambda^4}\frac{u^4}{4!} - \frac{1}{\Lambda^6}\frac{u^6}{6!} + \ldots \approx 1. \tag{3.99}$$

Ignoring the terms of O($1/\Lambda^2$) and higher we have

$$\ell(t) = \frac{2\omega_0 J}{\Lambda}, \tag{3.100}$$

which is constant rotation as we might have expected.