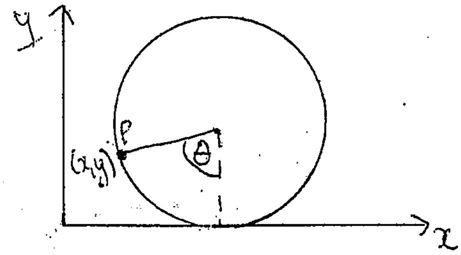
Questions 1, 5 and 7 due in Friday 11th August

PHYS2100 Problem Sheet 2

Semester 2, 2006

1. Let P be a fixed point on the rim of a wheel of radius a. Suppose the wheel rolls along the positive x-axis and that P is initially at the origin. When the wheel rolls a given distance to point P rotates through a certain angle θ as shown.



   (i) Show that the curve traced out by the point P is described by the parametric equations

$$x = a\theta - a\sin\theta, \qquad y = a - a\cos\theta.$$

   (ii) Hence find the length of arc of the curve (called a *cycloid*) between $\theta = 0$ and $\theta = \frac{\pi}{2}$. [**Hint:** Use $|\dot{\mathbf{r}}|\ dt = |\frac{d\mathbf{r}}{d\theta}|\frac{d\theta}{dt}\ dt = |\frac{d\mathbf{r}}{d\theta}|\ d\theta$.]

   (iii) Given that the wheel rotates with a constant angular velocity $\dot{\theta} = \omega$, find the velocity and acceleration vectors of the point P.

2. A particle of charge q moving through a magnetic field $\mathbf{B}$ experiences the force

$$\mathbf{F} = q(\mathbf{B} \times \mathbf{v})$$

   where $\mathbf{v}$ is the velocity of the particle. Show that:

   (i) This force does no work in moving the particle along its path from $t = 0$ to $t = T$.

   (ii) The speed $v(t) = |\mathbf{v}(t)|$ of the particle is constant in time.

3. Find the work done by the force

$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\hat{\mathbf{i}} + \frac{4}{x^2 + y^2}\hat{\mathbf{j}}$$

   acting on a particle that moves along each of the curves γ given below:

   (i) $(x, y) = (2\cos t, 2\sin t), \quad 0 \le t \le \frac{\pi}{2}$.

   (ii) $(x, y) = (t, 2t), \quad 1 \le t \le 2$.

   (iii) $(x, y) = \begin{cases} (t, t), & 1 \le t \le 2, \\ 2\sqrt{2}(\cos(\frac{\pi}{8}t), \sin(\frac{\pi}{8}t)), & 2 \le t \le 4. \end{cases}$

4. A rocket ship of mass m is launched vertically from the Earth's surface with velocity v_0. Show that:

(i) The energy E of the rocket must satisfy $E \geq -mga$ where a is the radius of the earth and g the acceleration due to gravity.

(ii) The velocity $\mathbf{v}(t)$ is a monotonically decreasing function of time.

(iii) The maximum height h reached by the rocket is given by

$$h = \frac{av_0^2}{2ga - v_0^2}$$

provided its energy E is strictly negative.

5. A non-linear oscillator consisting of a mass on a spring has a potential energy of the form $\frac{1}{2}kx^2 - \frac{1}{3}\alpha x^3$, where k and α are positive constants. Sketch the graph of the potential as a function of the displacement x. Using conservation of energy, show that the motion is oscillatory if the initial position x_0 satisfies $0 < x_0 < \frac{k}{\alpha}$ and the initial velocity is small enough. Show that the initial velocity v_0 must satisfy $v_0 < \frac{k}{\alpha}\sqrt{\frac{k}{m}}$.

6. A simple pendulum of length l is made to oscillate by giving it a velocity v_0 when it is at the bottom (equilibrium) point. How high will it swing? Under what conditions will it never come to rest?

7. (i) In polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2},$$

show that the velocity and acceleration vectors of a particle moving in the $x-y$ plane are given respectively by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}, \quad \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$$

where $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$ is the unit vector in the direction of the particle and $\hat{\boldsymbol{\theta}} = \frac{d\hat{\mathbf{r}}}{d\theta}$.

(ii) Deduce that

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0, \quad \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}}, \quad \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$$

where $\hat{\mathbf{k}}$ is the unit vector in the z-direction.

(iii) Determine the kinetic energy T and angular momentum vector L of the particle in polar coordinates.

8. A projectile is fired from the earth's surface with a velocity v_0 at an angle α to the horizontal. Solve the equation of motion in the Galilean approximation and show that the path of the projectile is a parabola. Determine the horizontal distance travelled by the projectile. At what angle should the projectile be fired in order to maximise this horizontal range?