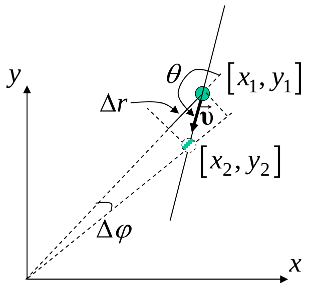
Lectures 5-6

How does an object appear for an observer?



Determine the angular velocity observed from the origin of the reference frame for object shown in the figure. The observed angular velocity is defined as the rate with which the observed direction on the object (measured in radians) changes in time.

The object is located at [x_1,y_1] at time t_1 and is located at [x_2,y_2] at time t_2 . However, the observation times for these two events are different from t_1 and t_2 because light wave takes a certain time to propagate from the object to the origin. The observation times can be calculated as follows

$$t_{O1} = t_1 + r_1/c \quad \text{and} \quad t_{O2} = t_2 + r_2/c$$

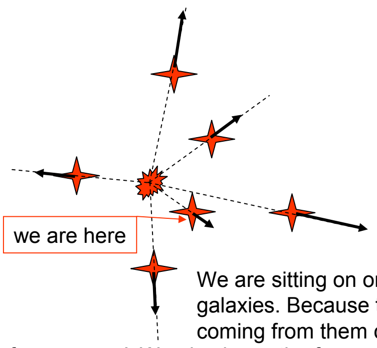
The change of the distance from the object to the origin is $\Delta r = \upsilon\left(t_2 - t_1\right)\cos\theta = \upsilon\Delta t\cos\theta$

The change of the direction is $\Delta\varphi = \upsilon\Delta t\sin\theta/r$ (the angle increases counter clockwise).

Therefore $$\omega_O = \frac{\Delta\varphi_O}{\Delta t_O} = \frac{\Delta\varphi}{\Delta t + \Delta r/c} = \frac{\upsilon\Delta t\sin\theta}{r\left(1+\frac{\upsilon}{c}\cos\theta\right)\Delta t} = \frac{\upsilon\sin\theta}{r\left(1+\frac{\upsilon}{c}\cos\theta\right)}$$

Note: If we define the observed speed as $\upsilon_O=\omega_O r$, then this speed can be larger than *c.*

Big Bang and the red shift



After a Big Bang, many fragments were formed. These fragments move with different but constant velocities (we neglect gravitational forces between them) and in different directions. The time is set to zero at the moment of the Big Bang. The origin of the reference frame is placed at the Big Bang location. At a later time *t* the distance of each fragment from the origin is proportional to the velocity of the fragment. That is $\vec{\mathbf{r}} = \vec{\mathbf{u}}t$

We are sitting on one of these fragments (or own galaxy) and observe other galaxies. Because the other galaxies are moving relative to us, the light coming from them contains atomic spectra but these spectra are shifted frequency ω′. We also know the frequencies of unsifted spectral lines. Therefore the ratio β = ω′/ω can be calculated for every object in the visible universe.

Given β and the time passed after the Big Bang, calculate the distance from Earth to that object.

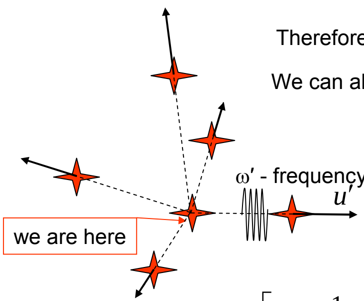
In our reference frame (primed), the distance to the object is proportional to its primed velocity and the primed time. $\vec{\mathbf{r}}' = \vec{\mathbf{u}}'t'$

This is because in our reference frame the object is moving with constant velocity and its trajectory starts at the origin of our reference frame (at time zero we were at the Big Bang location). First we take into account that the observed location of the object does not coincide with its actual location

$$r' = r'_O + u'\Delta t' = r' = r'_O + u' \cdot r'_O / c = r'_O\left(1 + u'/c\right)$$

where Δt′ is the time interval which light needs to cover the distance r'_O

Big Bang and the red shift (cont)



Therefore the observed distance is $r'_O = \dfrac{u't'}{1+u'/c}$

We can also relate β to the velocity u′

$$\beta \equiv \frac{\omega'}{\omega} = \gamma\left(1 - \frac{u'}{c}\right) = \sqrt{\frac{1-u'/c}{1+u'/c}}$$

ω′ - frequency measured on Earth ω - frequency measured on the moving star or in a laboratory on Earth

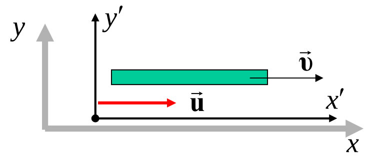
Note that 0 < β <1

Solve this equation for *u*′ $\left[\beta^2 = \dfrac{1-u'/c}{1+u'/c}\right] \Rightarrow \left[1-\beta^2 = u'/c + \beta^2 u'/c\right] \Rightarrow \left[\dfrac{1-\beta^2}{1+\beta^2}c = u'\right]$

Answer. The observed distance to the object is

$$r'_O = \frac{\dfrac{1-\beta^2}{\beta^2+1}}{1+\dfrac{1-\beta^2}{\beta^2+1}}ct' = \frac{1-\beta^2}{2}ct'$$

Length of a stick



A stick is moving with velocity υ in a not primed reference frame where the length of the stick is *L*. Find the length of the stick in the primed reference frame (standard configuration).

Consider a point moving with velocity υ in the not primed reference frame. The equation of motion for this point is

$$x = \upsilon t + x_0$$

We transform this equation of motion into the primed reference frame.

We substitute the Lorentz relations $x = \gamma\left[x' + ut'\right]; \quad t = \gamma\left[t' + \frac{u}{c^2}x'\right]$

into the not primed equation of motion and solve the obtained equation for x'

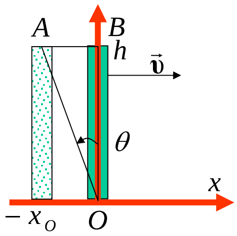
This gives the equation of motion in the primed reference frame $x' = \frac{\upsilon - u}{1 - u\upsilon/c^2}t' + \frac{x_0/\gamma}{1 - u\upsilon/c^2}$

For the two ends of the stick we have two equations of motion $x_1 = \upsilon t + x_0$ and $x_2 = \upsilon t + x_0 + L$

Using the previously derived transformation to the primed reference frame

$$L' \equiv x_2' - x_1' = \frac{L/\gamma}{1 - u\upsilon/c}$$

Observed shape of a stick



A vertical stick has a height *h* and moves with a horizontal velocity υ (see Figure). What will an observer sitting at the origin of the reference frame see when the bottom of the stick crosses on the y-axis?

When light emitted by the bottom point of the stick crosses the y-axis, only light emitted by the top of the stick Δt seconds earlier can reach the observer. This light does not come from the vertical direction.

Simple kinematics and geometrical considerations leads to the following relations

$$|AO| = c\Delta t$$
$$\Delta t = x_O/\upsilon$$
$$x_O = |AO|\sin\theta$$

Substituting the third equality in the second and the second in the first on gets

$$|AO| = |AO|\sin\theta\, c/\upsilon$$
$$\sin\theta = \upsilon/c$$

The stick will appear tilted (therefore the observed will be able actually to see it)

Constant proper acceleration

Problem. *At any instance, a particle has a constant acceleration a in a reference frame where the particle is instantaneously at rest. Calculate the velocity and the position of that particle in the laboratory reference frame*

In the reference frame where the particle is instantaneously at rest $\vec{\mathbf{A}}' = [a', 0, 0, 0]$

In the laboratory reference frame $A_x \equiv \gamma \frac{\upsilon^2/c^2}{\left(1-\upsilon^2/c^2\right)^{3/2}}\frac{d\upsilon}{dt} + \gamma^2 \frac{d\upsilon}{dt} = \gamma A'_x$

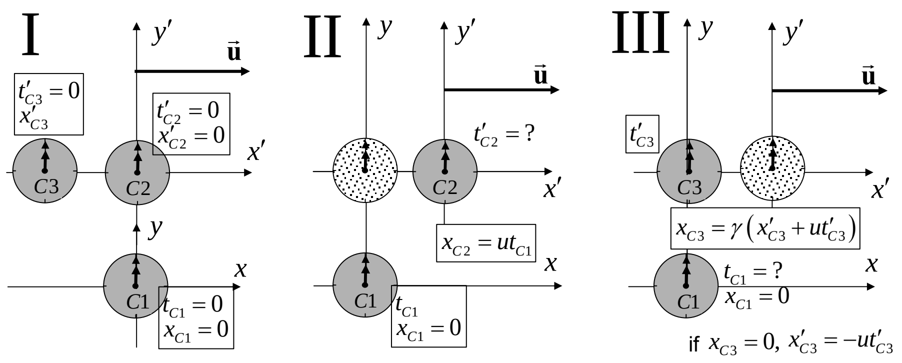
We solve this for dυ/dt to get $d\upsilon\left(1-\upsilon^2/c^2\right)^{-3/2} = a'dt.$ Integrating both sides, one gets $\left(1-\upsilon^2/c^2\right)^{-1/2}\upsilon = a't.$ This can be solved for the velocity $\upsilon = a't\left[1+\left(a't/c\right)^2\right]^{-1/2}$

Using $dx = \upsilon dt$ and integration over time we get the position $x = \frac{c^2}{a'}\left(1+t^2a'^2/c^2\right)^{1/2}$

This is how the integrals were calculated $\int_0^t \frac{a'\xi d\xi}{\sqrt{1+\xi^2 a'^2/c^2}} = \frac{1}{2a'/c^2}\int_0^{t^2a'^2/c^2}\frac{dy}{(1+y)^{1/2}} = \frac{\left(1+t^2a'^2/c^2\right)^{1/2}}{a'/c^2}$

$$\int_0^{\upsilon}\frac{d\xi}{\left(1-\xi^2/c^2\right)^{3/2}} \int_{\cos\theta=0}^{\cos\theta=\frac{\upsilon}{c}}\frac{\sin\theta d\theta}{\left(1-\cos^2\theta\right)^{3/2}} = \int_{\cos\theta=0}^{\cos\theta=\frac{\upsilon}{c}}\frac{d\theta}{\sin^2\theta} = \left.\frac{\cos\theta}{\sin\theta}\right|_{\cos\theta=0}^{\cos\theta=\frac{\upsilon}{c}} = \left.\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right|_{\cos\theta=0}^{\cos\theta=\upsilon/c} = \frac{\upsilon/c}{\sqrt{1-\upsilon^2/c^2}}$$

Comparison of two clocks in one frame to the same clock in other frame



Analyse Situation II. Use $t'_{C2} = \gamma\left(t_{C1} - \frac{u}{c^2}x_{C2}\right)$ and $x_{C2} = ut_{C1}$ to show that $t'_{C2} = t_{C1}/\gamma$

Analyse Situation III. Use $t_{C1} = \gamma\left(t'_{C3} + \frac{u}{c^2}x'_{C3}\right)$ and $x'_{C3} = -ut'_{C3}$ to show that $t_{C1} = t'_{C3}/\gamma$

Because the clocks are synchronized $t'_{C3} = t'_{C2}$

one gets in Situation III $t_{C1} = t'_{C2}/\gamma$

Resolve the contradiction