PHYS2100: Dynamics, Chaos and Special Relativity

July 27, 2006

1 Course Outline

Dynamics of a single particle

- Vector calculus
- Newton's 2nd law
- Work and line integrals, arclength
- Conservative systems and conservation of energy
- Central forces and conservation of angular momentum
- Planetary motion and kepler's laws

Dynamics of many particle systems

- Systems with constraints and general coordinates
- Conservative systems, stable equilibria
- Lagrangian Mechanics and calculus of variations
- Hamiltonian mechanics
- Poissson brackets and canonical transformations

2 Vector Calculus

2.1 Curves in space

If components of a point (x, y, z) are functions of a variable t (time) then the point $(x(t), y(t), z(t))$ traces out a curve in 3-space. The coordinate equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \tag{1}$$

are called the **parametric equations** of the curve.

We put a natural orientation on a curve

$$\gamma : (x, y, z) = (x(t), y(t), z(t)),$$

and say that point $(x(t_1), y(t_1), z(t_1))$ *precedes* point $(x(t_2), y(t_2), z(t_2))$ if $t_1 < t_2$. We put arrow heads on a curve γ to mark the positive direction.

Examples 1,2.

2.2 Vector Functions

A vector

$$\mathbf{f}(t) = f_1(t)\hat{\mathbf{i}} + f_2(t)\hat{\mathbf{j}} + f_3(t)\hat{\mathbf{k}}$$

whose components are functions of one or more variables (in this case time t) is called a **vector function**. The basic concepts of the calculus of such functions eg. limits, differentiation etc. can be introduced in a natural way. For example

$$\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{a}$$

means that

$$\lim_{t \to t_0} f_i(t) = a_i, \quad i = 1, 2, 3.$$

Similarly, we differentiate vector functions component-wise:

$$\frac{d\mathbf{f}}{dt} = \frac{df_1}{dt}\hat{\mathbf{i}} + \frac{df_2}{dt}\hat{\mathbf{j}} + \frac{df_3}{dt}\hat{\mathbf{k}} = \dot{f}_1\hat{\mathbf{i}} + \dot{f}_2\hat{\mathbf{j}} + \dot{f}_3\hat{\mathbf{k}}.$$

Example: If $\mathbf{f}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$, then

$$\lim_{t \to 0} \mathbf{f}(t) = \hat{\mathbf{i}} + \hat{\mathbf{k}},$$

and

$$\dot{\mathbf{f}}(t) = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}.$$

Also note that the usual product rules of differentiation hold for vector functions.

i.e.

$$\frac{d}{dt}\{\mathbf{g}(t) \cdot \mathbf{f}(t)\} = \dot{\mathbf{g}}(t) \cdot \mathbf{f}(t) + \mathbf{g}(t) \cdot \dot{\mathbf{f}}(t)$$

$$\frac{d}{dt}\{\mathbf{g}(t) \times \mathbf{f}(t)\} = \dot{\mathbf{g}}(t) \times \mathbf{f}(t) + \mathbf{g}(t) \times \dot{\mathbf{f}}(t).$$

Note: Unless otherwise stated, we assume throughout the course that all functions are continuous and differentiable.

2.3 Position, Velocity, Acceleration

A particle moving in 3-space traces out a curve $(x(t), y(t), z(t))$, called the **path** of the particle, as t varies. The corresponding vector function

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

is called the **position vector** of the particle. The distance of the particle from the origin is therefore given by

$$r(t) = |\mathbf{r}(t)| = \sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} = \sqrt{x^2 + y^2 + z^2}.$$

Thus

$$\hat{\mathbf{r}}(t) = \frac{\mathbf{r}(t)}{r(t)}$$

determines the **unit vector** in the direction of the particle.

The vector

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\hat{\mathbf{i}} + \dot{y}(t)\hat{\mathbf{j}} + \dot{z}(t)\hat{\mathbf{k}}$$

is called the **velocity vector** of the particle; that is,

$$\mathbf{v}(t) = \lim_{dt \to 0} \frac{\mathbf{r}(t + dt) - \mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t).$$

Thus at any instant the velocity vector is tangent to the path of the particle and points in the direction of motion. It follows that

$$\hat{\mathbf{v}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

determines a *unit tangent vector* to the curve at the point (x(t), y(t),z(t)). We call

$$v(t) = |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2},$$

the **speed** of the particle and

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v})$$

the **kinetic energy** (K.E.) of the particle, where m is the particle's mass.

The vector

$$\mathbf{p} = m\mathbf{v}$$

is called the (linear) **momentum** of the particle. The **acceleration** vector of the particle is given by

$$\mathbf{a} = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}.$$

According to Newton's 2nd law, if $\mathbf{F}(t)$ is the force exerted on the particle then

$$m\ddot{\mathbf{r}} = \mathbf{F}(t)$$

which is called the **equation of motion**.

Note: In terms of momentum, we more accurately have

$$\dot{\mathbf{p}} = \mathbf{F}(t).$$

This is important only when mass m of the particle depends on time. In particular, if there is no force exerted on the particle, that is $\mathbf{F} = \mathbf{0}$, then

$$\dot{\mathbf{p}} = \mathbf{0}.$$

This is known as **conservation of linear momentum**.

Example 3.

2.4 Arclength

Let $\gamma : (x, y, z) = (x(t), y(t), z(t))$ be a curve in 3-space and let $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ be the corresponding position vector. Then recall that the **arclength** of the curve γ

between $t = a$ and $t = b$ is given by

$$L = \int_a^b |\dot{\mathbf{r}}| dt = \int_a^b v(t) dt.$$

This determines the distance travelled by a particle along path γ between times $t = a$ and $t = b$.

Example 4.

2.5 Work and Line Integrals

Recall from classical physics that the **work** W done by a constant force $\mathbf{F}$ in moving a particle along a *straight line* from point A to point B is

$$\begin{aligned} W &= \text{(component of force in direction of motion) x (distance)} \\ &= \mathbf{F} \cdot (\mathbf{r}_B - \mathbf{r}_A). \end{aligned}$$

In general $\mathbf{F}$ is a function of t and the path of the particle is no longer a straight line but is given by a *curve*

$$\gamma : (x, y, z) = (x(t), y(t), z(t)), \quad t \in [a, b].$$

Then the work done by F over path γ is equal to the **line integral** of $\mathbf{F}(t)$ over γ, which is given by

$$\begin{aligned} W &= \int_\gamma \mathbf{F}(t) \cdot d\mathbf{r} \\ &= \int_a^b \mathbf{F}(t) \cdot \dot{\mathbf{r}}(t) dt. \end{aligned}$$

Example 5.

Suppose a particle of mass m experiences a force $\mathbf{F}$, so equation of motion is

$$m\ddot{\mathbf{r}} = \mathbf{F}.$$

Then the work done by the force $\mathbf{F}$ in moving particle along its path γ from time 0 until time t is given by the line integral

$$W = \int_\gamma \mathbf{F} \cdot \dot{\mathbf{r}}\, dt = \int_0^t \mathbf{F}(t') \cdot \mathbf{v}(t')\, dt',$$

or

$$\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2} m \frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}).$$

Therefore,

$$\frac{dW}{dt} = \dot{T} \quad \text{where} \quad T = \frac{1}{2}mv^2 = \text{K.E.}$$

Hence

$$W = \int_a^b \dot{T}\, dt = T(b) - T(a).$$

i.e. The work done = increase in K.E.

2.6 Conservative systems and conservation of energy

We say that motion is **conservative** if there exists a function $V = V(x, y, z)$, called the **potential energy** (P.E.) such that

$$\mathbf{F} \cdot \mathbf{v} = -\frac{dV}{dt}.$$

This potential energy V is uniquely determined up to a constant. Now we have from above

$$\mathbf{F} \cdot \mathbf{v} = \frac{dT}{dt},$$

so

$$\frac{d}{dt}(T + V) = 0$$

or

$$T + V = E \text{ (const)}.$$

We call E the **energy** of the particle.

Examples 6,7.

2.7 Launching of artificial satellites

In certain cases of physical interest the approximation for gravitational force $F = -mg$ breaks down and we need to use the full Newtonian expression. Consider then launching a rocket ship from the earth's surface. In this case we have from

$$F = -\frac{mga^2}{(a+x)^2}$$

where a is the Earth's radius and x is the height of the rocket above the earth.

Hence

$$Fv = -\frac{mga^2}{(a+x)^2}\dot{x} = -\frac{dV}{dt},$$

where

$$V = -\frac{mga^2}{(a+x)}.$$

Therefore the system is still conservative but now with P.E. V, and the energy is

$$E = \frac{1}{2}mv^2 - \frac{mga^2}{(a+x)},$$

which is constant in time.

Note: Using Taylor series

$$V(x) = -\frac{mga}{(1+\frac{x}{a})} \approx -mga(1-\frac{x}{a}) = mgx - mga$$

valid for $\frac{x}{a} << 1$, which reduces to the Galilean expression (up to a constant).

If $E < 0$, the rocket is trapped in earth's gravitational field. Indeed $v = 0$ when

$$E = -\frac{mga^2}{(x+a)}.$$

Hence when

$$x = -\frac{a}{E}(mga + E)$$

the rocket ship falls back to earth.

Exercise: Problem Sheet 2 Question 4

The rocket ship will escape earth's gravitational field when $E = 0$, corresponding to

$$\frac{1}{2}mv^2 = \frac{mga^2}{(x+a)}$$

or

$$v = \sqrt{\frac{2ga^2}{(a+x)}}. \qquad (2)$$

In this case v never vanishes and rocket ship continues to rise indefinitely. Eqn.(2) gives the velocity needed to escape the earth's gravitational field at height x above earth's surface.

At the earth's surface ($x = 0$) this escape velocity is

$$v_e = \sqrt{2ga} = \sqrt{\frac{2GM}{a}} \qquad (3)$$

which is called the **escape velocity**. This last expression in fact gives the escape velocity from any spherical body of radius a and mass M.

2.8 Black Holes

When $v_e = c$ (speed of light) the body is called a **black hole**. Squaring equation (3) yields the black hole equation

$$c^2 = \frac{2GM}{R}$$

or

$$R = \frac{2GM}{c^2},$$

which is known as the Schwarzchild radius. Here R gives the radius to which a spherical body of mass M must be shrunk in order to become a black hole. This agrees with the expression from general relativity.

Note: Laplace predicted the existence of such objects as far back as the 18th century.

2.9 Gradient Function

If we have a function $g : \mathbf{R}^3 \mapsto \mathbf{R}^3$ (g(x,y,z)), we call the vector function

$$\nabla g = \frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}$$

the **gradient** of g. We also call

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$$

the **del operator**.

Example For $g(x,y,z) = x^2 + yz^2$ we have

$$\nabla g = 2x\,\hat{\mathbf{i}} + z^2\,\hat{\mathbf{j}} + 2yz\,\hat{\mathbf{k}}.$$

Note: ∇g is normal to the surface $g(x,y,z) = c$ at every point. Indeed let

$$\gamma : (x,y,z) = (x(t), y(t), z(t))$$

be any curve in the surface. Then

$$\dot{\mathbf{r}} \cdot \nabla g = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} = \frac{dg}{dt} = 0.$$

Therefore since $\dot{\mathbf{r}}$ is tangent to the surface, ∇g is orthogonal to the surface at the given point. The plane orthogonal to ∇g at a point $P(x,y,z)$ on the surface $g(x,y,z) = c$ is called the **tangent plane** to the surface at P.

2.10 Conservative Forces

A force $\mathbf{F}$ is called **conservative** if there exists a function $g(x,y,z)$ such that

$$\mathbf{F} = \nabla g.$$

We usually say more specifically, $\mathbf{F} = -\nabla V$, where $V(x,y,z)$ is called the **potential function**, as before.

Note: Suppose a particle of mass m moves under the influence of a conservative force $\mathbf{F} = -\nabla V$. Then the motion is conservative with P.E. V.

Proof:

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{v} &= -\nabla V \cdot \mathbf{v} \\
&= -\nabla V \cdot \dot{\mathbf{r}} \\
&= -\frac{\partial V}{\partial x}\dot{x} - \frac{\partial V}{\partial y}\dot{y} - \frac{\partial V}{\partial z}\dot{z} \\
&= -\frac{\partial V}{\partial x}\frac{dx}{dt} - \frac{\partial V}{\partial y}\frac{dy}{dt} - \frac{\partial V}{\partial z}\frac{dz}{dt} \\
&= -\frac{dV}{dt}
\end{aligned}$$

so motion is conservative with P.E. V.

We can generalise this by using the Jacobi Matrix. The **Jacobi Matrix** of a vector function $\mathbf{f} = f_1(x_1, x_2, \ldots x_n)\,\hat{\mathbf{i}} + f_2(x_1, x_2, \ldots x_n)\,\hat{\mathbf{j}} + f_3(x_1, x_2, \ldots x_n)\,\hat{\mathbf{k}}$ is the $n \times n$ matrix with elements:

$$[J_{\mathbf{f}}]_{ij} = \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

In this course we are usually working in 3 dimensions, in which case $n = 3$ and x_1, x_2 and x_3 correspond to x, y and z.

Lemma: A force $\mathbf{F}$ is conservative iff $J_{\mathbf{F}}$ is symmetric.
Proof: Suppose $\mathbf{F} = -\nabla V$ is conservative.

$$\begin{aligned} [J_{\mathbf{F}}]_{ij} = \frac{\partial F_i}{\partial x_j} &= -\frac{\partial^2 V}{\partial x_j \partial x_i} \\ &= -\frac{\partial^2 V}{\partial x_i \partial x_j} \\ &= [J_{\mathbf{F}}]_{ji}. \end{aligned}$$

So $J_{\mathbf{F}}$ is symmetric and conversely if $J_{\mathbf{F}}$ is symmetric the $\mathbf{F}$ is conservative.

Example 8.