8 Stability

Often a system will have a steady-state solution where one or more variables remain constant. Assume a system has a steady-state solution \( q_i = x \). Then we can learn about the behaviour of the system by solving the equations of motion for \( q_i = x + \delta x \) where \( \delta x \) is a small perturbation.

In order to understand the nature of the solutions we usually make a first-order Taylor approximation. i.e.

\[
 f(x + \delta x) \approx f(x) + f'(x)\delta x.
\]

This is valid provided \( \delta x \) is sufficiently small. Note that this is equivalent to stating

\[
 f'(x) \approx \frac{f(x + \delta x) - f(x)}{\delta x}
\]

for small \( \delta x \).

**Example 21.** Spherical Pendulum.

\[
 V = mga(1 - \cos \theta), \quad T = \frac{1}{2} m[(\dot{a} \theta)^2 + (a \sin \theta \dot{\phi})^2]
\]

and thus

\[
 L = T - V = \frac{1}{2} m(a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2) - mga(1 - \cos \theta).
\]

In this case \( L = L(\theta, \dot{\theta}, \dot{\phi}) \) is independent of \( \phi \) which is thus a cyclic coordinate. The corresponding generalised momentum is

\[
 p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \sin^2 \theta \dot{\phi} = \text{constant}.
\]

Also,

\[
 p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = ma^2 \dot{\theta}
\]

and satisfies

\[
 \frac{dp_{\theta}}{dt} = \frac{\partial L}{\partial \dot{\theta}} = ma^2 \sin \theta \cos \theta \dot{\phi}^2 - mga \sin \theta,
\]

or

\[
 \dot{\theta} = \sin \theta \cos \theta \dot{\phi}^2 - \frac{g}{a} \sin \theta
\]

which is the equation of motion.

A solution to the equations of motion is given by circular motion at a fixed angle of inclination \( \theta = \alpha \) (constant) provided that \( \phi \) is constant and

\[
 0 = ma^2 \sin \alpha \cos \alpha \dot{\phi}^2 - mga \sin \alpha \quad \Rightarrow \dot{\phi}^2 = \frac{g}{a \cos \alpha}.
\]
Thus the mass moves uniformly in a circle with angular velocity \( \dot{\phi} = \omega = \sqrt{\frac{g}{a \cos \alpha}} \) where \(-\pi/2 < \alpha < \pi/2\).

**Stability:** Consider a small perturbation away from circular motion so that \( \theta = \alpha + \delta \theta \) with \( \delta \theta \) small. Then the above equation for \( p_\phi \) becomes

\[
ma^2 \sin^2(\alpha + \delta \theta) \dot{\phi} = \text{const.} = ma^2 \omega \sin^2 \alpha,
\]

or

\[
\dot{\phi} = \frac{\omega \sin^2 \alpha}{\sin^2(\alpha + \delta \theta)}.
\]

Substituting this into the second equation of motion, we find

\[
\delta \ddot{\theta} = \frac{\omega^2 \sin^4 \alpha}{\sin^3(\alpha + \delta \theta)} \cos(\alpha + \delta \theta) - \frac{g}{a} \sin(\alpha + \delta \theta)
\]

But for \( \delta \theta \) small we have the first order approximations

\[
\sin(\alpha + \delta \theta) \approx \sin \alpha + \cos \alpha \delta \theta, \quad \cos(\alpha + \delta \theta) \approx \cos \alpha - \sin \alpha \delta \theta
\]

and

\[
\frac{1}{\sin^3(\alpha + \delta \theta)} \approx \frac{\sin \alpha - 3 \cos \alpha \delta \theta}{\sin^4 \alpha}.
\]

Hence

\[
\delta \ddot{\theta} \approx \omega^2 (\sin \alpha - 3 \cos \alpha \delta \theta)(\cos \alpha - \sin \alpha \delta \theta) - \omega^2 \cos \alpha (\sin \alpha + \cos \alpha \delta \theta)
\]

\[
\approx -\omega^2 (\sin^2 \alpha + 4 \cos^2 \alpha) \delta \theta
\]

\[
= -\omega^2 (1 + 3 \cos^2 \alpha) \delta \theta.
\]

Thus \( \theta \) will oscillate around \( \alpha \) with frequency \( \omega \sqrt{1 + 3 \cos^2 \alpha} \) (simple harmonic motion), and the circular orbits are stable.
Example 22. Two masses $m_1$ and $m_2$ are connected by a (massless) spring of stiffness $k$. The system is set rotating about the centre of mass with an angular velocity $\omega$ and then released. If the masses are slightly disturbed along the line joining them, show that the angular frequency of oscillation is

$$\sqrt{\frac{3\omega^2 m_1 m_2 + k(m_1 + m_2)}{m_1 m_2}}.$$

Solution: Let $r_1, r_2$ be the distances of masses $m_1, m_2$ respectively from the centre of mass.

As the system is rotating around the centre of mass, we know $m_1 r_1 = m_2 r_2$, i.e. $r_2 = \frac{m_1}{m_2} r_1$.

Then

$$T = \frac{1}{2} m_1 (r_1^2 + r_1^2 \dot{\theta}^2) + \frac{1}{2} m_2 (r_2^2 + r_2^2 \dot{\theta}^2)$$

$$= \frac{1}{2} (1 + \frac{m_1}{m_2})(r_1^2 + r_1^2 \dot{\theta}^2)$$

and

$$V = \frac{1}{2} k(r_1 + r_2 - l)^2 = \frac{1}{2} \left[ r_1 (1 + \frac{m_1}{m_2}) - l \right]^2,$$

so

$$L = T - V = \frac{1}{2} (1 + \frac{m_1}{m_2})(r_1^2 + r_1^2 \dot{\theta}^2) - \frac{1}{2} \left[ r_1 (1 + \frac{m_1}{m_2}) - l \right]^2.$$

$L$ has no explicit dependence on $\theta$, so $\theta$ is cyclic and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_1 (1 + \frac{m_1}{m_2}) r_1 \dot{\theta}$ is constant.

i.e. $r_1^2 \dot{\theta} = c$ for some constant $c$. Let the distance of $m_1$ from the centre of mass before the disturbance be $d$. Then $c = d^2 \omega$, and $\dot{\theta} = \frac{d^2}{r_1^2} \omega$.

The other equation of motion is

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r}$$

$$= m_1 (1 + \frac{m_1}{m_2}) \dot{r} - m_1 (1 + \frac{m_1}{m_2}) r_1 \dot{\theta}^2 + k(1 + \frac{m_1}{m_2})[r_1 (1 + \frac{m_1}{m_2}) - l]$$

or

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\[ \ddot{r}_1 = \frac{\delta^2 r_1}{m_1} \left[ r_1 (1 + \frac{m_1}{m_2}) - l \right] \\
= \frac{d^4}{r_1^3} \omega^2 - \frac{k}{m_1} \left[ r_1 (1 + \frac{m_1}{m_2}) - l \right]. \]

As the masses are only slightly disturbed, we can set \( r_1 = d + \delta r \) where \( \delta r \) is small. Then

\[ \frac{1}{r_1^3} \approx \frac{1}{(d + \delta r)^3} \approx 1 - 3\delta r \frac{d}{d^4} = \frac{d - 3\delta r}{d^4}, \]

using a first order Taylor expansion. Hence the second equation of motion becomes

\[ \delta \ddot{r} \approx (d - 3\delta r)\omega^2 - \frac{k}{m_1} [(d + \delta r)(1 + \frac{m_1}{m_2}) - l] \]

\[ = - \left[ 3\omega^2 + \frac{k}{m_1} \left(1 + \frac{m_1}{m_2}\right) \right] \delta r + c' \quad c' \text{ a constant} \]

\[ = - \left[ 3\omega^2 m_1 m_2 + k(m_1 + m_2) \right] \frac{\delta r}{m_1 m_2} + c'. \]

Hence the mass will oscillate with angular frequency

\[ \sqrt{\frac{3\omega^2 m_1 m_2 + k(m_1 + m_2)}{m_1 m_2}}. \]

**Exercise:** A particle moves on the inside surface of a cone of half angle \( \alpha \). The axis of the cone is vertical with the vertex downwards. Determine the condition on the angular velocity \( \omega \) such that the particle can describe a horizontal circle \( h \) above the vertex. Show that the period of small oscillations about this circular path is

\[ \frac{2\pi}{\cos \alpha} \sqrt{\frac{h}{3g}}. \]