1. Consider a paramagnetic system of $N$ non-interacting spin-1/2 atoms at temperature $T$ in a magnetic field $B$.

(a) Show that the entropy of the system is

$$S(B, T) = Nk_B \left[ \ln(2 \cosh x) - x \tanh x \right]$$

where $x \equiv \mu B / (k_B T)$.

(b) Find the entropy at high and low temperatures as a function of $x$ (i.e. expand the result to the first non-zero order in $x$). Are the results what you would expect? Why?

(c) For a fixed value of the magnetic field sketch the entropy as a function of the temperature.

2. A “classical” spin can be defined by a three-dimensional vector $\vec{S}$ which has fixed magnitude $S = |\vec{S}|$ and can point in any direction. This is in contrast to a quantum spin which has the property that the component of its spin in the $z$-direction can have only certain discrete values. The state of each classical spin in a non-interacting paramagnet is defined by the polar co-ordinates $\theta$ and $\phi$. Hence, in the presence of a magnetic field in the $z$-direction the energy of a spin with magnetic moment $\vec{m} = \gamma \vec{S}$ is

$$E(\theta, \phi) = -\vec{m} \cdot \vec{B} = -\gamma BS \cos \theta$$

(a) Evaluate the partition function for a set of $N$ non-interacting spins in thermal equilibrium at temperature $T$.

(b) Evaluate the internal energy $U$ as function of $B$ and $T$.

(c) Show that the magnetisation is $M(\vec{B}, T) = M(B, T)\hat{z}$ where (2 marks)

$$M(B, T) = \frac{N}{S} \coth \left( \frac{\gamma BS}{k_B T} \right) - \frac{k_B T}{B}$$

(d) Find expressions for $M(B, T)$ at high and low temperatures. Explain why your expressions make physical sense.

(e) Sketch the magnetisation as a function of temperature, for two different values of the magnetic field, $B_1$ and $B_2$, where $B_2 = 2B_1$.

**Group problems: to be presented Thursday 13th April 2006.**

Schroeder: Problems 2.42, 3.7, 7.53.
Problem 2.42. A black hole is a region of space where gravity is so strong that nothing, not even light, can escape. Throwing something into a black hole is therefore an irreversible process, at least in the everyday sense of the word. In fact, it is irreversible in the thermodynamic sense as well: Adding mass to a black hole increases the black hole’s entropy. It turns out that there’s no way to tell (at least from outside) what kind of matter has gone into making a black hole. Therefore, the entropy of a black hole must be greater than the entropy of any conceivable type of matter that could have been used to create it. Knowing this, it’s not hard to estimate the entropy of a black hole.

(a) Use dimensional analysis to show that a black hole of mass $M$ should have a radius of order $GM/c^2$, where $G$ is Newton’s gravitational constant and $c$ is the speed of light. Calculate the approximate radius of a one-solar-mass black hole ($M = 2 \times 10^{30}$ kg).

(b) In the spirit of Problem 2.36, explain why the entropy of a black hole, in fundamental units, should be of the order of the maximum number of particles that could have been used to make it.

(c) To make a black hole out of the maximum possible number of particles, you should use particles with the lowest possible energy: long-wavelength photons (or other massless particles). But the wavelength can’t be any longer than the size of the black hole. By setting the total energy of the photons equal to $Mc^2$, estimate the maximum number of photons that could be used to make a black hole of mass $M$. Aside from a factor of $8\pi^2$, your result should agree with the exact formula for the entropy of a black hole, obtained through a much more difficult calculation:

$$S_{b.h.} = \frac{8\pi^2 GM^2}{hc} - k.$$

(d) Calculate the entropy of a one-solar-mass black hole, and comment on the result.

Problem 3.7. Use the result of Problem 2.42 to calculate the temperature of a black hole, in terms of its mass $M$. (The energy is $Mc^2$.) Evaluate the resulting expression for a one-solar-mass black hole. Also sketch the entropy as a function of energy, and discuss the implications of the shape of the graph.

Problem 7.53. A black hole is a blackbody if ever there was one, so it should emit blackbody radiation, called Hawking radiation. A black hole of mass $M$ has a total energy of $Mc^2$, a surface area of $16\pi G^2 M^2/c^4$, and a temperature of $hc^3/16\pi^2 kGM$ (as shown in Problem 3.7).

(a) Estimate the typical wavelength of the Hawking radiation emitted by a one-solar-mass ($2 \times 10^{30}$ kg) black hole. Compare your answer to the size of the black hole.

(b) Calculate the total power radiated by a one-solar-mass black hole.

(c) Imagine a black hole in empty space, where it emits radiation but absorbs nothing. As it loses energy, its mass must decrease; one could say it “evaporates.” Derive a differential equation for the mass as a function of time, and solve this equation to obtain an expression for the lifetime of a black hole in terms of its initial mass.

(d) Calculate the lifetime of a one-solar-mass black hole, and compare to the estimated age of the known universe ($10^{10}$ years).

(e) Suppose that a black hole that was created early in the history of the universe finishes evaporating today. What was its initial mass? In what part of the electromagnetic spectrum would most of its radiation have been emitted?