1. Let $V$ be a Euclidean space with symmetric bilinear form $(\cdot, \cdot)$. In class we skipped the proof that each root system $\Phi \subset V$ admits a simple system $\Delta$. The aim of this question is to find an explicit construction of such a system (as opposed to an existence proof). Throughout you may assume that $\Phi$ spans $V$. To keep the algebra/geometry as simple as possible we also make the (weak) assumption that $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

(a). Let $\alpha, \beta \in \Phi$ be distinct. Show that if $\Phi$ is crystallographic, then $(\alpha, \beta) > 0$ implies that $\alpha - \beta \in \Phi$. (Hint: Consider $(\alpha, \beta^\vee)(\alpha^\vee, \beta)$.)

(b). Give a geometric description of the set $R = \{ \lambda \in V : (\alpha, \lambda) \neq 0 \text{ for all } \alpha \in \Phi \}$ and show that if $\lambda \in R$ then

$\Phi = \{ \alpha \in \Phi : (\alpha, \lambda) > 0 \} \cup \{ \alpha \in \Phi : (\alpha, \lambda) < 0 \} =: \Psi^+(\lambda) \cup \Psi^-(\lambda)$.

(c). Given $\lambda \in R$, define $\Delta(\lambda) \subseteq \Psi^+(\lambda)$ as the set of those roots which cannot be written as the sum of two other roots in $\Psi^+(\lambda)$. Let $\beta \in \Psi^+(\lambda)$. Show that $\beta$ admits the expansion

$$\beta = \sum_{\alpha \in \Delta(\lambda)} c_\alpha \alpha, \quad c_\alpha \in \mathbb{Z}_{\geq 0}.$$ (Hint: Proceed by contradiction.)

(d). For distinct $\alpha, \beta \in \Delta(\lambda)$, show that $(\alpha, \beta) \leq 0$. (Hint: Use part (a) and the definition of $\Delta(\lambda)$.)

(e). For $\lambda \in V$ let $S(\lambda) \subset V$ be a set such that

(i) all $\mu \in S(\lambda)$ satisfy $(\lambda, \mu) > 0$,

(ii) $(\mu, \nu) \leq 0$ for all pairs $\mu, \nu \in S(\lambda)$.

Show that $S(\lambda)$ is linearly independent over $\mathbb{R}$.

(f). Show that $\Delta(\lambda)$ is a simple system for $\Phi$.

2. For $k \in \mathbb{R}$ and $v \in \mathbb{R}^3$ let $H_{v,k}$ denote the (affine) hyperplane

$$H_{v,k} = \{ \lambda \in \mathbb{R}^3 : (\lambda, v) = k \}.$$ For $\alpha = (1, -1, 0)$, $\beta = (0, 1, -1)$ and $\gamma = \alpha + \beta = (1, 0, -1)$, consider the group $\widetilde{W}$ generated by the reflections $r_{\alpha,0}, r_{\beta,0}, r_{\gamma,2}$ in the hyperplanes $H_{\alpha,0}, H_{\beta,0}$ and $H_{\gamma,2}$.

(a). Describe all relations of the form $(s_i s_j)^{m_{i,j}} = 1$ among the three generators.

(b). Show that $\widetilde{W} = W \ltimes T$ where $W$ is the group generated by the reflections $r_{\alpha,0}, r_{\beta,0}$ and $T \subset \mathbb{R}^3$ is a lattice. Give a full description of this lattice.

(c). (Challenge) Repeat the exercise in $\mathbb{R}^n$, i.e., $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n - 1$ with $n$ generators $r_{\alpha_i,0}$ and $r_{\alpha_1 + \cdots + \alpha_{n-1},2}$. 

Solution to question 1.

(a). Since \((\alpha, \beta^\vee) \in \mathbb{Z}\) we have
\[
I = (\alpha, \beta^\vee)(\alpha^\vee, \beta) = \frac{4(\alpha, \beta)^2}{\|\alpha\|^2\|\beta\|^2}.
\]
We can now either proceed using Cauchy–Schwarz and the fact that \(\alpha, \beta\) are distinct to conclude that \(1 \leq I \leq 2\) or we may explicitly compute the admissible angles:
\[
I = \frac{4(\alpha, \beta)^2}{\|\alpha\|^2\|\beta\|^2} = (2 \cos \theta)^2 \in \mathbb{Z}.
\]
Since \(\alpha, \beta\) are distinct and \((\alpha, \beta) > 0\) this implies that \(\theta \in \{\pi/6, \pi/4, \pi/3\}\). Hence
\[
I = \begin{cases} 
1 = 1 \times 1 & \text{if } \theta = \pi/3, \\
2 = 1 \times 2 & \text{if } \theta = \pi/4, \\
3 = 1 \times 3 & \text{if } \theta = \pi/6.
\end{cases}
\]
Both approaches lead us to conclude that one of \((\alpha, \beta^\vee)\) and \((\alpha^\vee, \beta)\) must be equal to 1.

Again there is a choice in how to exactly proceed. You may either appeal to symmetry (the question is symmetric in \(\alpha\) and \(\beta\)) and assume without loss of generality that \((\alpha, \beta^\vee) = 1\). Or you can ignore symmetry and treat both cases separately. In the first case,
\[
r_\beta(\alpha) = \alpha - (\alpha, \beta^\vee)\beta = \alpha - \beta \in \Phi
\]
and in the second case
\[
-r_\alpha(\beta) = -((\beta, \alpha^\vee)\alpha) = -(\beta - \alpha) = \alpha - \beta \in \Phi.
\]

(b). The set \(R\) consists of those vectors in \(V\) that do not lie on any of the hyperplanes \(H_\alpha, \alpha \in \Phi\):
\[
R = V \setminus \cup_{\alpha \in \Phi} H_\alpha.
\]
Elements of \(R\) are often called regular.

If \(\lambda \in R\) then the hyperplane \(H_\lambda\) does not contain any roots (for such roots would satisfy \((\alpha, \lambda) = 0\)). Hence \(H_\lambda\) divides \(\Phi\) into two sets, \(\Psi^+(\lambda)\) and \(\Psi^-(-\lambda)\) as defined above. Moreover, since roots come in pairs, \(\Psi^-(-\lambda) = -\Psi^+(\lambda)\) and \(\Phi\) is the disjoint union of these two sets.

(c). Proceeding by contradiction, assume there are \(\beta \in \Psi^+(\lambda)\) that do not admit the claimed expansion. Pick that particular \(\beta\) for which \((\beta, \lambda)\) is minimal. Since each element of \(\Delta(\lambda)\) does trivially admit the claimed expansion, our chosen \(\beta\) is not in \(\Delta(\lambda)\). Hence it can be written as \(\beta = \gamma + \delta\) with \(\gamma, \delta \in \Psi^+(\lambda)\). But then
\[
(\beta, \lambda) = (\gamma, \lambda) + (\delta, \lambda)
\]
which means that the positive number on the left is expressed as a sum of two (necessarily smaller) positive numbers on the right. Since \(\beta\) was minimal, both \(\gamma\) and \(\delta\) admit the claimed expansion. But then \(\beta\) also does, a contradiction.

(d). By contradiction, assume that \((\alpha, \beta) > 0\). By (a) we know that this implies that \(\alpha - \beta\) and hence also \(\beta - \alpha\) are in \(\Phi\). One of them must be in \(\Psi^+(\lambda)\). If \(\alpha - \beta \in \Psi^+(\lambda)\) then \(\alpha = (\alpha - \beta) + \beta\) a contradiction. If \(\beta - \alpha \in \Psi^+(\lambda)\) then \(\beta = (\beta - \alpha) + \alpha\), again a contradiction. Hence \((\alpha, \beta) < 0\).
(e). Let

\[ \sum_{\alpha \in S(\lambda)} c_\alpha \alpha = 0, \quad c_\alpha \in \mathbb{R} \]

We want to show that all \( c_\alpha \) are equal to zero.

Dismissing all the \( c_\alpha \) which are 0, we rewrite the relation as

\[ \sum_{\alpha \in A} a_\alpha \alpha = \sum_{\beta \in B} b_\beta \beta, \quad a_\alpha, b_\beta > 0 \]

where \( A, B \) disjoint subsets of \( S(\lambda) \). Clearly, we need to show that \( A = B = \emptyset \).

Now

\[ 0 \leq \left\| \sum_{\alpha \in A} a_\alpha \alpha \right\|^2 = \sum_{\alpha \in A} a_\alpha b_\beta (\alpha, \beta) \leq 0. \]

This implies that \( A = \emptyset \). By symmetry this of course also implies that all \( B = \emptyset \).

Hence all \( c_\alpha \) are equal to 0.

(f). By (e), \( \Delta(\lambda) \) is a linearly independent set. By (c) it forms a simple system with positive roots \( \Phi^+ \) given by \( \Psi^+(\lambda) \) and negative roots \( \Phi^- \) given by \( \Psi^-(\lambda) \).

**Solution to question 2.**

Disclaimer: The solution below is by now means the shortest possible. For example, (1.3) below is more general than necessary.

(a). The three generators of \( \tilde{W} \) are \( r_{\alpha,0}, r_{\beta,0} \) and \( r_{\gamma,2} \). Since all are reflections, we trivially have

\[ r_{\alpha,0}^2 = r_{\beta,0}^2 = r_{\gamma,2}^2 = 1. \]

Using that

(1.1) \) \( r_{\nu,k}(\lambda) = \lambda + [k - (\lambda, \nu)] v^\nu, \)

it is readily checked that

(1.2a) \) \( r_{\alpha,k} r_{\beta,l} r_{\alpha,k} = r_{\gamma,k+l} \)
(1.2b) \) \( r_{\alpha,k} r_{\gamma,l} r_{\alpha,k} = r_{\beta,l-k} \)
(1.2c) \) \( r_{\gamma,l} r_{\alpha,k} r_{\gamma,l} = r_{\beta,l-k} \).

The right-hand side of (1.2a) is symmetric in \( \alpha \) and \( \beta \) so we must have

\[ r_{\alpha,k} r_{\beta,l} r_{\alpha,k} = r_{\beta,k} r_{\alpha,l} r_{\beta,k} \]

which is

(1.3a) \) \( (r_{\alpha,k} r_{\beta,l})^3 = 1. \)

You were of course only asked to find this relation for \( k = l = 0 \). Equating (1.2b) and (1.2c) gives

(1.3b) \) \( (r_{\alpha,k} r_{\gamma,l})^3 = 1 \)

and hence, by symmetry,

(1.3c) \) \( (r_{\beta,k} r_{\gamma,l})^3 = 1. \)

You were only asked to find these relations for \( k = 0 \) and \( l = 2. \).
(b). Let $t_\mu$ denote the translation by $\mu \in \mathbb{R}^3$. Using (1.1) to compute the four two-letter words containing an $s_{\gamma,2}$ and one of $s_{\alpha,0}, s_{\beta,0}$, we find

$$(1.4a) \quad r_{\alpha,0}r_{\gamma,2} = t_{(0,2,-2)}r_{\beta,0}r_{\alpha,0} = t_{2\beta}r_{\beta,0}r_{\alpha,0}$$

$$(1.4b) \quad r_{\beta,0}r_{\gamma,2} = t_{(2,2,0)}r_{\alpha,0}r_{\beta,0} = t_{2\alpha}r_{\alpha,0}r_{\beta,0}$$

$$(1.4c) \quad r_{\gamma,2}r_{\beta,0} = t_{(2,0,-2)}r_{\beta,0}r_{\alpha,0} = t_{2\gamma}r_{\beta,0}r_{\alpha,0}$$

$$(1.4d) \quad r_{\gamma,2}r_{\alpha,0} = t_{(2,2,-2)}r_{\alpha,0}r_{\beta,0} = t_{2\gamma}r_{\alpha,0}r_{\beta,0}.$$ 

This strongly suggests that any $w \in \tilde{W}$ can be represented as

$$\tilde{w} = t_\lambda v$$

where $v \in W$ and $t_\lambda \in T$ with $T$ the lattice

$$T = \{ \lambda \in 2\mathbb{Z}^3 : \lambda_1 + \lambda_2 + \lambda_3 = 0 \} = 2\mathbb{Z} \alpha + 2\mathbb{Z} \beta.$$ 

Note: There is some abuse of notation here. More precise would be to say that $t_\lambda$ is a translation with $\lambda \in T$, but viewing the lattice $T$ itself as a(n abelian) group acting on a vector space is standard mathematical sloppiness.

It is certainly true that any translation in $T$ belongs to $\tilde{W}$. From (1.4b) it follows that

$$t_{2\alpha} = r_{\beta,0}r_{\gamma,2}r_{\alpha,0},$$

and from (1.4a) that

$$t_{2\beta} = r_{\alpha,0}r_{\gamma,2}r_{\alpha,0}.$$ 

Since these two translations generate $T$, this proves that $T < \tilde{W}$. But we saw in class (recall $r_{\alpha,\kappa}t_{\lambda}r_{\alpha,\kappa} = t_{r_{\alpha}(\lambda)}$) that translations are normalised by reflections, hence $T < \tilde{W}$. Since $W$ follows from $\tilde{W}$ by dropping one of the generators it is also obviously true that $W < \tilde{W}$. There also is no issue with $W \cap T = \{1\}$, so that it remains to be shown that $WT = TW = \tilde{W}$, i.e., that

$$\tilde{W} = \{ tv : t \in T \text{ and } v \in W \}.$$ 

First we make the simple observation that $TW < \tilde{W}$. Indeed, since $T$ is normal there exists a $v''$ such that, $(tv)(tv') = t(vv')' \in TW$. But since $s_{\alpha,0}, s_{\beta,0}, s_{\gamma,2}$ are all in $TW$ we must have $TW = \tilde{W}$. (That $s_{\alpha,0}, s_{\beta,0}$ are in $TW$ is trivial. That $s_{\gamma,2}$ is follows from (1.4c) and/or (1.4d): $r_{\gamma,2} = t_{2\gamma}r_{\beta,0}r_{\alpha,0}r_{\beta,0}$).

(c). Let $\alpha_0 = \alpha_1 + \cdots + \alpha_{n-1}$ and write $r_{i,k}$ for $r_{\alpha_i,k}$. Also identify $r_{n,k}$ with $r_{0,k}$.

A direct computation using (1.1) shows the following list of relations

$$r_{i,k}r_{i+1,k}r_{i,k} = r_{\alpha_1+\alpha_i+1,k+i}, \quad 1 \leq i \leq n - 1$$

$$r_{0,k}r_{1,k}r_{0,k} = r_{1,0}r_{0,0}r_{1,l} = r_{\alpha_0-\alpha_1,k-l}$$

$$r_{0,k}r_{n-1,k}r_{0,k} = r_{n-1,0}r_{0,0}r_{n-1,l} = r_{\alpha_0-\alpha_{n-1},k-l}$$

$$r_{i,k}r_{j,l} = r_{j,l}r_{i,k}, \quad 0 \leq i,j \leq n \quad |i-j| > 1.$$ 

This implies relations consistent with the Coxeter graph

\[1\] An example of where we view $T$ as a group acting on $\mathbb{R}^3$ rather than $T$ being a lattice spanned by $2\alpha$ and $2\beta$. 


and that \( \tilde{W} \) contains all of the reflections \( r_{i,k} \) with \( k \) even. There are many ways to show that \( T = 2\mathbb{Z}\alpha_1 + \cdots + 2\mathbb{Z}\alpha_{n-1} \) arises as a (normal) subgroup of \( \tilde{W} \). For example,

\[
\begin{align*}
    r_{i,0}r_{i,2}(\lambda) &= r_{i,0}(r_{i,0}(\lambda) + 2k\alpha_i) \\
    &= \lambda + 2r_{i,0}(\alpha_i) \\
    &= \lambda - 2\alpha_i \\
    &= t_{-2\alpha_i}(\lambda).
\end{align*}
\]

If you prefer translations to be expressed in terms of the generators only, there is (prove!)

\[
t_{2\alpha_i} = w(i)r_{0,2}w(i)^{-1}r_{i},
\]

where \( w(i) = r_{i-1} \cdots r_2r_1 \cdot r_{i+1}r_{i+2} \cdots r_{n-1} \).

The rest of (c) follows the \( n = 3 \) case mutatis mutandis.