Assignment MATH4301/MATH7301; Coxeter groups

Due date: October 22

1. Let V be a Euclidean space with symmetric bilinear form $(\cdot,\cdot)$. In class we skipped the proof that each root system $\Phi \subset V$ admits a simple system Δ. The aim of this question is to find an explicit construction of such a system (as opposed to an existence proof). Throughout you may assume that Φ spans V. To keep the algebra/geometry as simple as possible we also make the (weak) assumption that Φ is crystallographic, i.e., that $(\alpha,\beta^\vee) \in \mathbb{Z}$ for all $\alpha,\beta \in \Phi$.

(a). Let $\alpha,\beta \in \Phi$ be distinct. Show that if Φ is crystallographic, then $(\alpha,\beta) > 0$ implies that $\alpha - \beta \in \Phi$. (**Hint:** Consider $(\alpha,\beta^\vee)(\alpha^\vee,\beta)$.)

(b). Give a geometric description of the set

$$R = \big\{\lambda \in V :\ (\alpha,\lambda) \neq 0 \text{ for all } \alpha \in \Phi\big\}.$$

and show that if $\lambda \in R$ then

$$\Phi = \{\alpha \in \Phi :\ (\alpha,\lambda) > 0\} \cup \{\alpha \in \Phi :\ (\alpha,\lambda) < 0\} =: \Psi^+(\lambda) \cup \Psi^-(\lambda).$$

(c). Given $\lambda \in R$, define $\Delta(\lambda) \subseteq \Psi^+(\lambda)$ as the set of those roots which cannot be written as the sum of two other roots in $\Psi^+(\lambda)$. Let $\beta \in \Psi^+(\lambda)$. Show that β admits the expansion

$$\beta = \sum_{\alpha\in\Delta(\lambda)} c_\alpha \alpha, \quad c_\alpha \in \mathbb{Z}_{\geq 0}.$$

(**Hint:** Proceed by contradiction.)

(d). For distinct $\alpha,\beta \in \Delta(\lambda)$, show that $(\alpha,\beta) \leq 0$. (**Hint:** Use part (a) and the definition of $\Delta(\lambda)$.

(e). For $\lambda \in V$ let $S(\lambda) \subset V$ be a set such that
(i) all $\mu \in S(\lambda)$ satisfy $(\lambda,\mu) > 0$,
(ii) $(\mu,\nu) \leq 0$ for all pairs $\mu,\nu \in S(\lambda)$.
Show that $S(\lambda)$ is linearly independent over $\mathbb{R}$.

(f). Show that $\Delta(\lambda)$ is a simple system for Φ.

2. For $k \in \mathbb{R}$ and $v \in \mathbb{R}^3$ let $H_{v,k}$ denote the (affine) hyperplane

$$H_{v,k} = \big\{\lambda \in \mathbb{R}^3 :\ (\lambda, v) = k\big\}.$$

For $\alpha = (1,-1,0)$, $\beta = (0,1,-1)$ and $\gamma = \alpha+\beta = (1,0,-1)$, consider the group $\widetilde{W}$ generated by the reflections $r_{\alpha,0}, r_{\beta,0}, r_{\gamma,2}$ in the hyperplanes $H_{\alpha,0}$, $H_{\beta,0}$ and $H_{\gamma,2}$.

(a). Describe all relations of the form $(s_i s_j)^{m_{i,j}} = 1$ among the three generators.

(b). Show that $\widetilde{W} = W \ltimes T$ where W is the group generated by the reflections $r_{\alpha,0}$, $r_{\beta,0}$ and $T \subset \mathbb{R}^3$ is a lattice. Give a full description of this lattice.

(c). (Challenge) Repeat the exercise in $\mathbb{R}^n$, i.e., $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$ with n generators $r_{\alpha_i,0}$ and $r_{\alpha_1+\cdots+\alpha_{n-1},2}$.

Solution to question 1.

(a). Since $(\alpha, \beta^\vee) \in \mathbb{Z}$ we have

$$I = (\alpha, \beta^\vee)(\alpha^\vee, \beta) = \frac{4(\alpha, \beta)^2}{\|\alpha\|^2\|\beta\|^2}.$$

We can now either proceed using Cauchy–Schwarz and the fact that α, β are distinct to conclude that $1 \leq I \leq 3$ or we may explicitly compute the admissible angles:

$$I = \frac{4(\alpha, \beta)^2}{\|\alpha\|^2\|\beta\|^2} = (2\cos\theta)^2 \in \mathbb{Z}.$$

Since α, β are distinct and $(\alpha, \beta) > 0$ this implies that $\theta \in \{\pi/6, \pi/4, \pi/3\}$. Hence

$$I = \begin{cases} 1 = 1 \times 1 & \text{if } \theta = \pi/3, \\ 2 = 1 \times 2 & \text{if } \theta = \pi/4, \\ 3 = 1 \times 3 & \text{if } \theta = \pi/6. \end{cases}$$

Both approaches lead us to conclude that one of $(\alpha, \beta^\vee)$ and $(\alpha^\vee, \beta)$ must be equal to 1.

Again there is a choice in how to exactly proceed. You may either appeal to symmetry (the question is symmetric in α and β) and assume without loss of generality that $(\alpha, \beta^\vee) = 1$. Or you can ignore symmetry and treat both cases separately. In the first case,

$$r_\beta(\alpha) = \alpha - (\alpha, \beta^\vee)\beta = \alpha - \beta \in \Phi$$

and in the second case

$$-r_\alpha(\beta) = -(\beta - (\beta, \alpha^\vee)\alpha) = -(\beta - \alpha) = \alpha - \beta \in \Phi.$$

(b). The set R consists of those vectors in V that do not lie on any of the hyperplanes H_α, $\alpha \in \Phi$:

$$R = V \setminus \cup_{\alpha \in \Phi} H_\alpha.$$

Elements of R are often called *regular*.

If $\lambda \in R$ then the hyperplane H_λ does not contain any roots (for such roots would satisfy $(\alpha, \lambda) = 0$). Hence H_λ divides Φ into two sets, $\Psi^+(\lambda)$ and $\Psi^-(\lambda)$ as defined above. Moreover, since roots come in pairs, $\Psi^-(\lambda) = -\Psi^+(\lambda)$ and Φ is the disjoint union of these two sets.

(c). Proceeding by contradiction, assume there are $\beta \in \Psi^+(\lambda)$ that do not admit the claimed expansion. Pick that particular β for which (β, λ) is minimal. Since each element of $\Delta(\lambda)$ *does* trivially admit the claimed expansion, our chosen β is *not* in $\Delta(\lambda)$. Hence it can be written as $\beta = \gamma + \delta$ with $\gamma, \delta \in \Psi^+(\lambda)$. But then

$$(\beta, \lambda) = (\gamma, \lambda) + (\delta, \lambda)$$

which means that the positive number on the left is expressed as a sum of two (necessarily smaller) positive numbers on the right. Since β was minimal, both γ and δ admit the claimed expansion. But then β also does, a contradiction.

(d). By contradiction, assume that $(\alpha, \beta) > 0$. By (a) we know that this implies that $\alpha - \beta$ and hence also $\beta - \alpha$ are in Φ. One of them must be in $\Psi^+(\lambda)$. If $\alpha - \beta \in \Psi^+(\lambda)$ then $\alpha = (\alpha - \beta) + \beta$ a contradiction. If $\beta - \alpha \in \Psi^+(\lambda)$ then $\beta = (\beta - \alpha) + \alpha$, again a contradiction. Hence $(\alpha, \beta) < 0$.

(e). Let

$$\sum_{\alpha \in S(\lambda)} c_\alpha \alpha = 0, \qquad c_\alpha \in \mathbb{R}$$

We want to show that all c_α are equal to zero.
Dismissing all the c_α which are 0, we rewrite the relation as

$$\sum_{\alpha \in A} a_\alpha \alpha = \sum_{\beta \in B} b_\beta \beta, \qquad a_\alpha, b_\alpha > 0$$

where A, B disjoint subsets of $S(\lambda)$. Clearly, we need to show that $A = B = \emptyset$. Now

$$0 \leq \Big\| \sum_{\alpha \in A} a_\alpha \alpha \Big\|^2 = \sum_{\substack{\alpha \in A \\ \beta \in B}} a_\alpha b_\beta (\alpha, \beta) \leq 0.$$

This implies that $A = \emptyset$. By symmetry this of course also implies that all $B = \emptyset$. Hence all c_α are equal to 0.

(f). By (e), $\Delta(\lambda)$ is a linearly independent set. By (c) it forms a simple system with positive roots Φ^+ given by $\Psi^+(\lambda)$ and negative roots Φ^- given by $\Psi^-(\lambda)$.

Solution to question 2.

Disclaimer: The solution below is by now means the shortest possible. For example, (1.3) *below is more general than necessary.*

(a). The three generators of $\widetilde{W}$ are $r_{\alpha,0}$, $r_{\beta,0}$ and $r_{\gamma,2}$. Since all are reflections, we trivially have

$$r_{\alpha,0}^2 = r_{\beta,0}^2 = r_{\gamma,2}^2 = 1.$$

Using that

$$r_{v,k}(\lambda) = \lambda + \big[k - (\lambda, v)\big] v^\vee, \tag{1.1}$$

it is readily checked that

$$r_{\alpha,k} r_{\beta,l} r_{\alpha,k} = r_{\gamma,k+l} \tag{1.2a}$$

$$r_{\alpha,k} r_{\gamma,l} r_{\alpha,k} = r_{\beta,l-k} \tag{1.2b}$$

$$r_{\gamma,l} r_{\alpha,k} r_{\gamma,l} = r_{\beta,l-k}. \tag{1.2c}$$

The right-hand side of (1.2a) is symmetric in α and β so we must have

$$r_{\alpha,k} r_{\beta,l} r_{\alpha,k} = r_{\beta,k} r_{\alpha,l} r_{\beta,k}$$

which is

$$(r_{\alpha,k} r_{\beta,l})^3 = 1. \tag{1.3a}$$

You were of course only asked to find this relation for $k = l = 0$. Equating (1.2b) and (1.2c) gives

$$(r_{\alpha,k} r_{\gamma,l})^3 = 1 \tag{1.3b}$$

and hence, by symmetry,

$$(r_{\beta,k} r_{\gamma,l})^3 = 1. \tag{1.3c}$$

You were only asked to find these relations for $k = 0$ and $l = 2$.

(b). Let t_μ denote the translation by $\mu \in \mathbb{R}^3$. Using (1.1) to compute the four two-letter words containing an $s_{\gamma,2}$ and one of $s_{\alpha,0}, s_{\beta,0}$, we find

$$r_{\alpha,0}r_{\gamma,2} = t_{(0,2,-2)}r_{\beta,0}r_{\alpha,0} = t_{2\beta}r_{\beta,0}r_{\alpha,0} \tag{1.4a}$$

$$r_{\beta,0}r_{\gamma,2} = t_{(2,-2,0)}r_{\alpha,0}r_{\beta,0} = t_{2\alpha}r_{\alpha,0}r_{\beta,0} \tag{1.4b}$$

$$r_{\gamma,2}r_{\beta,0} = t_{(2,0,-2)}r_{\beta,0}r_{\alpha,0} = t_{2\gamma}r_{\beta,0}r_{\alpha,0} \tag{1.4c}$$

$$r_{\gamma,2}r_{\alpha,0} = t_{(2,0,-2)}r_{\alpha,0}r_{\beta,0} = t_{2\gamma}r_{\alpha,0}r_{\beta,0}. \tag{1.4d}$$

This strongly suggests that any $w \in \widetilde{W}$ can be represented as

$$\tilde{w} = t_\lambda v$$

where $v \in W$ and $t_\lambda \in T$ with T the lattice

$$T = \{\lambda \in 2\mathbb{Z}^3 : \ \lambda_1 + \lambda_2 + \lambda_3 = 0\} = 2\mathbb{Z}\alpha + 2\mathbb{Z}\beta.$$

Note: There is some abuse of notation here. More precise would be to say that t_λ is a translation with $\lambda \in T$, but viewing the lattice T itself as a(n abelian) group acting on a vector space is standard mathematical sloppiness.

It is certainly true that any translation in T belongs to $\widetilde{W}$. From (1.4b) it follows that

$$t_{2\alpha} = r_{\beta,0}r_{\gamma,2}r_{\beta,0}r_{\alpha,0}$$

and from (1.4a) that

$$t_{2\beta} = r_{\alpha,0}r_{\gamma,2}r_{\alpha,0}r_{\beta,0}.$$

Since these two translations generate T[1] this proves that $T < \widetilde{W}$. But we saw in class (recall $r_{\alpha,\kappa}t_\lambda r_{\alpha,\kappa} = t_{r_\alpha(\lambda)}$) that translations are normalised by reflections, hence $T \lhd \widetilde{W}$. Since W follows from $\widetilde{W}$ by dropping one of the generators it is also obviously true that $W < \widetilde{W}$. There also is no issue with $W \cap T = \{1\}$, so that it remains to be shown that $WT = TW = \widetilde{W}$, i.e., that

$$\widetilde{W} = \{tv : \ t \in T \text{ and } v \in W\}.$$

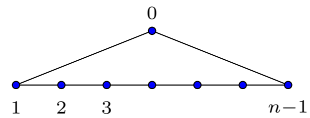
First we make the simple observation that $TW < \widetilde{W}$. Indeed, since T is normal there exists a v'' such that, $(tv)(t'v') = tt'v''v' \in TW$. But since $s_{\alpha,0}, s_{\beta,0}, s_{\gamma,2}$ are all in TW we must have $TW = \widetilde{W}$. (That $s_{\alpha,0}, s_{\beta,0}$ are in TW is trivial. That $s_{\gamma,2}$ is follows from (1.4c) and/or (1.4d): $r_{\gamma,2} = t_{2\gamma}r_{\beta,0}r_{\alpha,0}r_{\beta,0}$).

(c). Let $\alpha_0 = \alpha_1 + \cdots + \alpha_{n-1}$ and write $r_{i,k}$ for $r_{\alpha_i,k}$. Also identify $r_{n,k}$ with $r_{0,k}$. A direct computation using (1.1) shows the following list of relations

$$\begin{aligned} r_{i,k}r_{i+1,l}r_{i,k} &= r_{\alpha_i+\alpha_{i+1},k+l}, \qquad 1 \le i \le n-1 \\ r_{0,k}r_{1,l}r_{0,k} &= r_{1,l}r_{0,k}r_{1,l} = r_{\alpha_0-\alpha_1,k-l} \\ r_{0,k}r_{n-1,l}r_{0,k} &= r_{n-1,l}r_{0,k}r_{n-1,l} = r_{\alpha_0-\alpha_{n-1},k-l} \\ r_{i,k}r_{j,l} &= r_{j,l}r_{i,k}, \qquad 0 \le i,j \le n \quad |i-j| > 1. \end{aligned}$$

This implies relations consistent with the Coxeter graph

[1] An example of where we view T as a group acting on $\mathbb{R}^3$ rather than T being a lattice spanned by 2α and 2β.



and that $\tilde{W}$ contains all of the reflections $r_{i,k}$ with k even.
There are many ways to show that $T = 2\mathbb{Z}\alpha_1 + \cdots + 2\mathbb{Z}\alpha_{n-1}$ arises as a (normal) subgroup of $\tilde{W}$. For example,

$$\begin{aligned} r_{i,0}r_{i,2}(\lambda) &= r_{i,0}\big(r_{i,0}(\lambda) + 2k\alpha_i\big) \\ &= \lambda + 2r_{i,0}(\alpha_i) \\ &= \lambda - 2\alpha_i \\ &= t_{-2\alpha_i}(\lambda). \end{aligned}$$

If you prefer translations to be expressed in terms of the generators only, there is (prove!)

$$t_{2\alpha_i} = w(i)r_{0,2}w(i)^{-1}r_i,$$

where $w(i) = r_{i-1}\cdots r_2 r_1 \cdot r_{i+1}r_{i+2}\cdots r_{n-1}$.

The rest of (c) follows the $n = 3$ case mutatis mutandis.