S. Ole Warnaar

An A_2 Bailey tree and $A_2^{(1)}$ Rogers–Ramanujan-type identities

Abstract. The A_2 Bailey chain of Andrews, Schilling and the author is extended to a four-parameter A_2 Bailey tree. As main application of this tree, we prove the Kanade–Russell conjecture for a three-parameter family of Rogers–Ramanujan-type identities related to the principal characters of the affine Lie algebra $A_2^{(1)}$. Combined with known q-series results, this further implies an $A_2^{(1)}$-analogue of the celebrated Andrews–Gordon q-series identities. We also use the A_2 Bailey tree to prove a Rogers–Selberg-type identity for the characters of the principal subspaces of $A_2^{(1)}$ indexed by arbitrary level-k dominant integral weights λ. This generalises a result of Feigin, Feigin, Jimbo, Miwa and Mukhin for $\lambda = k\Lambda_0$.

Keywords: $A_2^{(1)}$ and $\mathcal{W}_3$ character formulas, Bailey's lemma, Kanade–Russell conjecture, principal subspaces of $A_2^{(1)}$, Rogers–Ramanujan-type identities.

1. Introduction

Let $(a;q)_\infty := (1-a)(1-aq)\cdots$ and $(a;q)_n := (a;q)_\infty/(aq^n;q)_\infty$ for n an integer. In particular, $(a;q)_0 = 1$, $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n > 0$ and $1/(q;q)_n = 0$ for $n < 0$. Further let a, k, τ be integers such that $k \geqslant 1$, $0 \leqslant a \leqslant k$, $\tau \in \{0,1\}$, and fix $K := 2k+\tau+2$. Then the modulus-K Andrews–Gordon–Bressoud q-series identities are given by

$$\sum_{\lambda_1\geqslant\cdots\geqslant\lambda_k\geqslant 0} \frac{q^{\lambda_1^2+\cdots+\lambda_k^2+\lambda_{a+1}+\cdots+\lambda_k}}{(q;q)_{\lambda_1-\lambda_2}\cdots(q;q)_{\lambda_{k-1}-\lambda_k}(q^{2-\tau};q^{2-\tau})_{\lambda_k}} = \frac{(q^{a+1};q^K)_\infty(q^{K-a-1};q^K)_\infty(q^K;q^K)_\infty}{(q;q)_\infty}, \tag{1.1}$$

where $\tau = 1$ corresponds to the Andrews–Gordon or odd modulus case [3] and $\tau = 0$ to the Bressoud or even modulus case [19]. The Andrews–Gordon identities for $k = 1$ simplify

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to the famous Rogers–Ramanujan identities [72–74]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{1.2a}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{1.2b}$$

These identities and their generalisations due to Andrews, Gordon and Bressoud have a rich history. They are the analytic counterpart of well-known theorems for integer partitions [18, 19, 43, 64, 76], have numerous important interpretations in terms of the representation theory of affine Lie algebras and vertex operator algebras [23, 24, 29, 33, 45, 54, 57–61, 65, 66, 81], and have arisen in a variety of other contexts such as in algebraic geometry [21, 69], combinatorics [26, 36], commutative algebra [1, 11, 68], group theory [25], knot theory [10, 46], number theory [20, 67], statistical mechanics [14, 15, 86], the theory of orthogonal polynomials [39, 49], and symmetric function theory [13, 48, 71, 80]. For a comprehensive introduction to the Rogers–Ramanujan identities and their generalisations we refer the reader to *An invitation to the Rogers–Ramanujan identities*, by Sills [78].

The representation-theoretic interpretations of the Andrews–Gordon–Bressoud identities based on the affine Lie algebra $\mathrm{A}_1^{(1)}$ make it a natural problem to try to extend (1.1) to $\mathrm{A}_{r-1}^{(1)}$. Despite the long history of the subject, this is very much an open problem. In 1999 Andrews, Schilling and the author succeeded in finding (some) analogues of (1.1) for $\mathrm{A}_2^{(1)}$ for all moduli [8]. To succinctly describe these results, we require the modified theta functions $\theta(z;q) := (z;q)_\infty(q/z;q)_\infty$ and $\theta(z_1,\dots,z_n;q) := \theta(z_1;q)\cdots\theta(z_n;q)$, and the q-binomial coefficients

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}$$

for integers n, m such that $0 \leqslant m \leqslant n$ and zero otherwise. We also need the appropriate $\mathrm{A}_2^{(1)}$-analogue of $1/(q^{2-\tau};q^{2-\tau})_n$ (which occurs in (1.1) with $n = \lambda_k$), and for n, m nonnegative integers and $\tau \in \{-1, 0, 1\}$, we define

$$g_{n,m;\tau}(q) := \frac{q^{\tau(\tau-1)nm}}{(q;q)_{n+m}(q^2;q)_{n+m}} \begin{bmatrix} n+m \\ n \end{bmatrix}_p, \tag{1.3}$$

where $p = q$ if $\tau^2 = 1$ and $p = q^3$ if $\tau = 0$. Thus, in the simplest and perhaps most important case, $g_{n,m;1}(q) = 1/((q;q)_n(q;q)_m(q^2;q)_{n+m})$. Then, for a, k, τ integers such that $k \geqslant 1$, $0 \leqslant a \leqslant k$ and $\tau \in \{-1, 0, 1\}$, it was shown in [8] that

$$\begin{aligned} \sum_{\substack{\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0 \\ \mu_1 \geqslant \cdots \geqslant \mu_k \geqslant 0}} \frac{1-q^{\lambda_a+\mu_a+1}}{1-q} \, \frac{q^{\sum_{i=1}^k (\lambda_i^2 - \lambda_i\mu_i + \mu_i^2) + \sum_{i=a+1}^k (\lambda_i + \mu_i)}}{\prod_{i=1}^{k-1} (q;q)_{\lambda_i-\lambda_{i+1}} (q;q)_{\mu_i-\mu_{i+1}}} \, g_{\lambda_k,\mu_k;\tau}(q) \\ = \frac{(q^K;q^K)_\infty^2}{(q;q)_\infty^3} \, \theta\big(q^{a+1}, q^{a+1}, q^{2a+2}; q^K\big), \end{aligned} \tag{1.4}$$

where $K := 3k + \tau + 3$ and $q^{\lambda_0} = q^{\mu_0} := 0$. From a q-series as well as combinatorial point of view this is a perfectly good analogue of (1.1). For example, by the Borodin product formula [17], the right-hand side corresponds to the generating function of cylindric partitions [42] of three rows with 'profile' given by $(K-2a-3, a, a)$. If, however, one wishes to interpret (1.4) as an identity for the principal characters of $\mathrm{A}_2^{(1)}$ (characters of the principally graded subspaces of basic $\mathrm{A}_2^{(1)}$ modules in the sense of [37, 60]) or, for $3 \nmid K$, as branching functions of $\mathrm{A}_2^{(1)}$ and characters of the $\mathcal{W}_3(3, K)$ vertex operator algebra (see [90, Section 4]), then one should multiply both sides of (1.4) by $(q;q)_\infty$.[1] This would obscure the positivity of the left-hand side, and for this reason we will not view the above as the "proper" $\mathrm{A}_2^{(1)}$-analogues of the Andrews–Gordon–Bressoud identities. Instead we follow Kanade and Russell [51] and refer to (1.4) as the Andrews–Schilling–Warnaar identities, or ASW identities for short. From both a representation theoretic and cylindric partition point of view it is clear that the above set of ASW identities is not complete, and there should be an appropriate multisum expression for each dominant integral weight $(K-a-b-3)\Lambda_0 + a\Lambda_1 + b\Lambda_2$ of $\mathrm{A}_2^{(1)}$ or each cylindric-partition profile $(K-a-b-3, a, b)$, with corresponding product form as above but with theta function given by $\theta(q^{a+1}, q^{b+1}, q^{a+b+2}; q^K)$. Recently Kanade and Russell [51, Conjecture 5.1] (see also [52]) posed the following beautiful conjecture that covers all cases for which $0 \leqslant a, b \leqslant k$.

Conjecture 1.1 (Kanade–Russell). *Let a, b, k be integers such that $0 \leqslant a, b \leqslant k$, and let $K := 3k + \tau + 3$ for $\tau \in \{-1, 0, 1\}$. Then*

$$\sum_{\substack{\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0 \\ \mu_1 \geqslant \cdots \geqslant \mu_k \geqslant 0}} \frac{1-q^{\lambda_a+\mu_b+1}}{1-q} \, \frac{q^{\sum_{i=1}^k (\lambda_i^2 - \lambda_i\mu_i + \mu_i^2) + \sum_{i=a+1}^k \lambda_i + \sum_{i=b+1}^k \mu_i}}{\prod_{i=1}^{k-1} (q;q)_{\lambda_i - \lambda_{i+1}} (q;q)_{\mu_i - \mu_{i+1}}} \, g_{\lambda_k, \mu_k; \tau}(q) \tag{1.5}$$
$$= \frac{(q^K;q^K)_\infty^2}{(q;q)_\infty^3} \, \theta\big(q^{a+1}, q^{b+1}, q^{a+b+2}; q^K\big),$$

where $q^{\lambda_0} = q^{\mu_0} := 0$.

For $b = 0$ and $\tau^2 = 1$ this was previously conjectured in [90, Conjecture 7.4]. By symmetry in a and b, there are $\binom{k+2}{2}$ distinct identities for fixed k, where it is noted that the right-hand sides for $(a,b) = (k,k)$ and $(a,b) = (k, k-1)$ are the same if $\tau = -1$ due to the simple relation $\theta(z;q) = \theta(q/z;q)$. In the following we may thus without loss of generality assume that $a \geqslant b$. For $\tau = -1$ the sum over μ_k can be carried out by a limiting case of the

[1] The result (1.4) may be interpreted as an identity for the principally specialised characters of $\widehat{\mathfrak{gl}(3)}$ indexed by $(K-2a-3)\Lambda_0 + a(\Lambda_1 + \Lambda_2)$ for $0 \leqslant a \leqslant k$, see e.g., [37, 82]. This, however, does not match the interpretation of the Andrews–Gordon–Bressoud identities as character identities for the principal characters of $\widehat{\mathfrak{sl}(2)} = \mathrm{A}_1^{(1)}$.

q-Chu–Vandermonde summation (see e.g., (3.8) below), resulting in the slightly simpler

$$\sum_{\substack{\lambda_1\geqslant\dots\geqslant\lambda_k\geqslant 0\\ \mu_1\geqslant\dots\geqslant\mu_{k-1}\geqslant 0}} \frac{1-q^{\lambda_a+\mu_b+1}}{1-q} \frac{q^{\sum_{i=1}^k(\lambda_i^2-\lambda_i\mu_i+\mu_i^2)+\sum_{i=a+1}^k\lambda_i+\sum_{i=b+1}^{k-1}\mu_i}}{(q^2;q)_{\lambda_k+\mu_{k-1}}\prod_{i=1}^k(q;q)_{\lambda_i-\lambda_{i+1}}\prod_{i=1}^{k-1}(q;q)_{\mu_i-\mu_{i+1}}}$$
$$= \frac{(q^{2k+2};q^{2k+2})_\infty^2}{(q;q)_\infty^3}\,\theta\big(q^{a+1},q^{b+1},q^{a+b+2};q^{3k+2}\big),$$

where $0\leqslant b\leqslant a\leqslant k$ $(b\neq k)$, $\lambda_{k+1}=\mu_k:=0$, and, for $k=1$, $\mu_0:=\infty$.

Besides (1.4), also the $(a,b)=(k,0)$ and $(k-1,0)$ instances of (1.5) for $\tau^2=1$ were proved in [8]. For the moduli 5 and 7 this covers all identities in (1.5). The identity of smallest modulus missing from [8] corresponds to $(a,b,k,\tau)=(1,0,1,0)$ which has modulus 6. Kanade and Russell proved this by solving the Corteel–Welsh system of functional equations [28] for cylindric partitions of profile $(d-a-b,a,b)$ for $d=3$, see [51, Corollary 7.5]. For the moduli 8 and 10 they again solved the corresponding Corteel–Welsh systems (in these cases $d=5$ and $d=7$ respectively) confirming the conjecture. Alternatively, the modulus-8 case is implied by combining the recent results of Corteel–Dousse–Uncu [27] and the author [90] on modulus-8 Rogers–Ramanujan-type identities for $\mathrm{A}_2^{(1)}$. Finally, Uncu [85, Theorems 4.4 & 5.4] settled the moduli 11 and 13 by algorithmically confirming and complementing a conjectured partial solution to the Corteel–Welsh system due to Kanade and Russell.

The first main result of this paper is a case-free proof of the Kanade–Russell conjecture for arbitrary modulus.

Theorem 1.2. *The Kanade–Russell conjecture holds for all moduli.*

The three cases of smallest modulus not previously proved in the literature are $k=2$, $\tau=0$ and $(a,b)\in\{(1,0),(2,0),(2,1)\}$. For example, for $(a,b)=(2,0)$ the theorem confirms the modulus-9 identity

$$\sum_{\lambda_1,\lambda_2,\mu_1,\mu_2=0}^\infty \frac{q^{\lambda_1^2-\lambda_1\mu_1+\mu_1^2+\lambda_2^2-\lambda_2\mu_2+\mu_2^2+\mu_1+\mu_2}(q^3;q^3)_{\lambda_2+\mu_2}}{(q;q)_{\lambda_1-\lambda_2}(q;q)_{\mu_1-\mu_2}(q^3;q^3)_{\lambda_2}(q^3;q^3)_{\mu_2}(q;q)_{\lambda_2+\mu_2}(q;q)_{\lambda_2+\mu_2+1}}$$
$$=\prod_{n=1}^\infty\frac{(1-q^{9n})}{(1-q^n)^2(1-q^{9n-7})(1-q^{9n-2})},$$

where we recall that $1/(q;q)_n=0$ if n is a negative integer, so that the summand vanishes unless $\lambda_1\geqslant\lambda_2$ and $\mu_1\geqslant\mu_2$.

As mentioned above, from a representation theoretic point of view the ASW identities should be multiplied by a factor $(q;q)_\infty$. For $\tau^2=1$ this factor can be absorbed in the multisum using a transformation formula from [90]. This gives what we view as the Andrews–Gordon identities for $\mathrm{A}_2^{(1)}$. In full generality this result is too involved to be stated in the introduction and below we restrict ourselves to the special case $b=0$. For the full result the reader is referred to Theorems 7.2 and 7.3.

Theorem 1.3 ($A_2^{(1)}$ Andrews–Gordon identities; $b=0$ case). *Let a,k be integers such that $0 \leqslant a \leqslant k$. Then*

$$\sum_{\substack{\lambda_1,\dots,\lambda_k\geqslant 0\\ \mu_1,\dots,\mu_{k-1}\geqslant 0}} \frac{q^{\lambda_k^2+\sum_{i=a+1}^k \lambda_i}}{(q;q)_{\lambda_1}} \prod_{i=1}^{k-1} q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2+\mu_i} \begin{bmatrix} \lambda_i \\ \lambda_{i+1} \end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1} \\ \mu_i \end{bmatrix} \tag{1.6a}$$
$$= \frac{(q^K;q^K)_\infty^2}{(q;q)_\infty^2}\, \theta\big(q,q^{a+1},q^{a+2};q^K\big),$$

where $\mu_k := 2\lambda_k$ and $K := 3k+2$, and

$$\sum_{\substack{\lambda_1,\dots,\lambda_k\geqslant 0\\ \mu_1,\dots,\mu_k\geqslant 0}} \frac{q^{\sum_{i=a+1}^k \lambda_i}}{(q;q)_{\lambda_1}} \prod_{i=1}^{k} q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2+\mu_i} \begin{bmatrix} \lambda_i \\ \lambda_{i+1} \end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1} \\ \mu_i \end{bmatrix} \tag{1.6b}$$
$$= \frac{(q^K;q^K)_\infty^2}{(q;q)_\infty^2}\, \theta\big(q,q^{a+1},q^{a+2};q^K\big),$$

where $\lambda_{k+1} := 0$, $\mu_{k+1} := \lambda_k$ and $K := 3k+4$.

These results were conditionally proved in [90] assuming the truth of (1.5) for $b=0$ and $\tau^2 = 1$. The q-series in (1.6a) and (1.6b) correspond to the principal characters of the $A_2^{(1)}$-highest weight module $L((K-a)\Lambda_0 + a\Lambda_1)$ for $K = 3k+2$ and $K = 3k+4$, respectively. Alternatively, they may be recognised as the normalised characters of the $\mathcal{W}_3(3,K)$ vertex operator algebra of conformal weight $a(a+3)/K - a$.

One of the most streamlined proofs of the Andrews–Gordon–Bressoud identities (1.1) is based on what is known as the Bailey lattice [2], which is a generalisation of the well-known Bailey chain [5]. Our proof of Theorem 1.2 presented in Section 5 is based on an A_2-analogue of a special case of the Bailey lattice which, due to its tree-like structure, we refer to as the A_2 Bailey tree. A single branch of the A_2 Bailey tree corresponds to the A_2 Bailey chain developed in [8] to prove the ASW identities (1.5). Andrews' original proof of the Andrews–Gordon identities [3] predates the discoveries of the Bailey chain and Bailey lattice, and instead is based on recursion relations for the Rogers–Selberg function $Q_{k,i}(z;q)$ defined by [74, 77]

$$Q_{k,i}(z;q) := \frac{1}{(zq;q)_\infty} \sum_{n=0}^\infty \big(1-z^i q^{(2n+1)i}\big)(-1)^n z^{kn} q^{(2k+1)\binom{n+1}{2}-in} \frac{(zq;q)_n}{(q;q)_n}, \tag{1.7}$$

for integers i,k such that $1 \leqslant i \leqslant k$. These recursions were solved by Andrews to give the multisum representation [3, Equation (2.5)]

$$Q_{k,i}(z;q) = \sum_{\lambda_1\geqslant\cdots\geqslant\lambda_{k-1}\geqslant 0} \frac{z^{\lambda_1+\cdots+\lambda_{k-1}} q^{\lambda_1^2+\cdots+\lambda_{k-1}+\lambda_i+\cdots+\lambda_{k-1}}}{(q;q)_{\lambda_1-\lambda_2}\cdots(q;q)_{\lambda_{k-2}-\lambda_{k-1}}(q;q)_{\lambda_{k-1}}}. \tag{1.8}$$

Equating the two expressions for $Q_{k,i}$, specialising $z=1$ and using the Jacobi-triple product identity yields (1.1) with $(a,k) \mapsto (i-1,k-1)$ and $\tau = 1$. The equality of (1.7) and (1.8) may also be proved by the Bailey lattice, and by lifting this proof to the A_2-setting we obtain the following identity for the character of the level-k principal subspace W_λ of $\mathrm{A}_2^{(1)}$ indexed by $\lambda=(k-a-b)\Lambda_0+a\Lambda_1+b\Lambda_2$ (see Section 8 for details). Let $Q_+:=\{y=(y_1,y_2,y_3)\in\mathbb{Z}^3:\ y_1+y_2+y_3=0,\ y_1\geqslant 0,\ y_1+y_2\geqslant 0\}$.

Theorem 1.4. *For a,b,k integers such that $0\leqslant a,b\leqslant k$, let v be the strict partition $v:=(a+b+2,b+1,0)$. Then*

$$\sum_{\substack{\lambda_1\geqslant\cdots\geqslant\lambda_k\geqslant 0\\ \mu_1\geqslant\cdots\geqslant\mu_k\geqslant 0}}\Big(1-\frac{x_1}{x_3}\,q^{\lambda_a+\mu_b-1}\Big)\prod_{i=1}^{k}\frac{\big(\frac{x_1}{x_2}\big)^{\lambda_i}\big(\frac{x_2}{x_3}\big)^{\mu_i}q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2-\chi(i\leqslant a)\lambda_i-\chi(i\leqslant b)\mu_i}}{(q;q)_{\lambda_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}}$$
$$=\sum_{y\in Q_+}\frac{\det_{1\leqslant i,j\leqslant 3}\big((x_iq^{y_i})^{v_i-v_j}\big)}{\prod_{1\leqslant i<j\leqslant 3}(x_i/x_j;q)_\infty}\prod_{i=1}^{3}\frac{x_i^{(k+2)y_i}q^{(k+2)\binom{y_i}{2}-v_iy_i}(x_i/x_3;q)_{y_i}}{(qx_i/x_1;q)_{y_i}},$$

where $q^{\lambda_0}=q^{\mu_0}=\lambda_{k+1}=\mu_{k+1}:=0$.

Setting $(x_1,x_2,x_3)=(zw,w,1)$ and letting w tend to 0, the summand on the left vanishes unless $\mu_1=\cdots=\mu_k=0$, resulting in $Q_{k+1,a+1}(z/q;q)$ in its multisum representation (1.8). In this same limit the summand on the right vanishes unless $(y_1,y_2,y_3)\in Q_+$ is of the form $(n,-n,0)$ for $n\in\mathbb{N}_0$. After some simplifications this yields $Q_{k+1,a+1}(z/q;q)$ as defined in (1.7). In contrast to the $\mathrm{A}_1^{(1)}$ case, (1.5) does not follow from Theorem 1.4 by specialisation of the x_i. For $b=a$ the determinant on the right (which up to normalisation is a Schur function [63]) factorises, resulting in the simpler

$$\sum_{\substack{\lambda_1\geqslant\cdots\geqslant\lambda_k\geqslant 0\\ \mu_1\geqslant\cdots\geqslant\mu_k\geqslant 0}}\Big(1-\frac{x_1}{x_3}\,q^{\lambda_a+\mu_a-1}\Big)\prod_{i=1}^{k}\frac{\big(\frac{x_1}{x_2}\big)^{\lambda_i}\big(\frac{x_2}{x_3}\big)^{\mu_i}q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2-\chi(i\leqslant a)(\lambda_i+\mu_i)}}{(q;q)_{\lambda_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}}$$
$$=\sum_{y\in Q_+}\bigg(\prod_{1\leqslant i<j\leqslant 3}\frac{1-\big(q^{y_i-y_j}x_i/x_j\big)^{a+1}}{(x_i/x_j;q)_\infty}\prod_{i=1}^{3}\frac{x_i^{(k+2)y_i}q^{(k+2)\binom{y_i}{2}+(a+1)iy_i}(x_i/x_3;q)_{y_i}}{(qx_i/x_1;q)_{y_i}}\bigg).$$

For $a=0$ this is [31, Corollary 7.8] by Feigin et al. The large-k limit of Theorem 1.4 gives our next result, where $\mathscr{P}$ denotes the set of integer partitions.

Corollary 1.5. *For a,b nonnegative integers and $v:=(a+b+2,b+1,0)$,*

$$\sum_{\lambda,\mu\in\mathscr{P}}\Big(1-\frac{x_1}{x_3}\,q^{\lambda_a+\mu_b-1}\Big)\prod_{i\geqslant 1}\frac{\big(\frac{x_1}{x_2}\big)^{\lambda_i}\big(\frac{x_2}{x_3}\big)^{\mu_i}q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2-\chi(i\leqslant a)\lambda_i-\chi(i\leqslant b)\mu_i}}{(q;q)_{\lambda_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}}$$
$$=\frac{1}{\prod_{1\leqslant i<j\leqslant 3}(x_i/x_j;q)_\infty}\det_{1\leqslant i,j\leqslant 3}\big(x_i^{v_i-v_j}\big),$$

where $q^{\lambda_0}=q^{\mu_0}:=0$.

For $a=b$ the right-hand side simplifies to

$$\prod_{1\leqslant i<j\leqslant 3}\frac{1-(x_i/x_j)^{a+1}}{1-x_i/x_j}\,\frac{1}{(qx_i/x_j;q)_\infty}$$

so that the $a=b=0$ case of Corollary 1.5 gives the A_2 instance of Hua's combinatorial identity for quivers of arbitrary finite type, see [47, Theorem 4.9] and the minor correction pointed out in [38]. The determinant in Theorem 1.4 also simplifies for $(x_1,x_2,x_3)=(z^2,z,1)$, resulting in

$$\sum_{\lambda,\mu\in\mathscr{P}} \big(1-z^2\,q^{\lambda_a+\mu_b-1}\big) \prod_{i\geqslant 1} \frac{z^{\lambda_i+\mu_i}\,q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2-\chi(i\leqslant a)\lambda_i-\chi(i\leqslant b)\mu_i}}{(q;q)_{\lambda_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}} \\ = \frac{1}{(zq,zq,z^2q;q)_\infty}\,\frac{(1-z^{a+1})(1-z^{b+1})(1-z^{a+b+2})}{(1-z)(1-z)(1-z^2)},$$

where $(a_1,\dots,a_k;q)_\infty := (a_1;q)_\infty\cdots(a_k;q)_\infty$. For $z=q$ this proves another conjecture by Kanade and Russell, stated as Conjecture 3.1 in [51].

The rest of this paper is organised as follows. In Section 2 we recall some standard material from the theory of q-series, root systems and symmetric functions that is used throughout the paper. Then, in Section 3, we review the classical or A_1 Bailey chain and a special case of the Bailey lattice which in this paper will be referred to as the A_1 Bailey tree. Although all of the material in this section is essentially known, some results are formulated in a form that is new. In particular, the Bailey tree will be recast as a one-parameter deformation of the Bailey chain. In Section 4 the A_2 Bailey chain of [8] is generalised to an A_2 Bailey tree. The simplest part of this tree consists of a two-parameter deformation of the A_2 Bailey chain, analogous to the one-parameter deformation described in Section 3. As it turns out, this two-parameter Bailey tree can only prove the Kanade–Russell conjecture for $b=0$, and to obtain the full set of identities we develop an additional and more complicated four-parameter deformation of the A_2 Bailey chain. In Section 5 we apply the A_2 Bailey tree to a suitable root identity to prove Theorem 1.2. As mentioned just above Conjecture 1.1, there should be an ASW identity for each dominant integral weight $(K-a-b-3)\Lambda_0+a\Lambda_1+b\Lambda_2$ of $A_2^{(1)}$, and in Theorem 6.1 of Section 6 the missing cases for $\tau=0$ are obtained using a key observation due to Kanade and Russell. Then, in Section 7, we prove the $A_2^{(1)}$-analogues of the Andrews–Gordon identities, stated in Theorems 7.2 and 7.3. In Section 8 we give a short introduction to the principal subspaces of $A_{r-1}^{(1)}$ in the sense of Feigin and Stoyanovsky, and then apply the A_2 Bailey tree to prove Theorem 1.4. Finally, in Section 9 we discuss the prospects of an A_{r-1} Bailey tree and a generalisation of (1.5) to arbitrary rank r.

2. Preliminaries

A partition $\lambda=(\lambda_1,\lambda_2,\dots)$ is a sequence of weakly decreasing integers such that $|\lambda|:=\lambda_1+\lambda_2+\cdots$ is finite. We will follow the convention to omit the infinite string of zeros in a partition, writing $(4,3,2,2,1)$ instead of $(4,3,2,2,1,0,\dots)$. If λ is a partition such that $|\lambda|=n$, we say that λ is a partition of n and write $\lambda\vdash n$. The set of all partitions, including the unique partition of 0, is denoted by $\mathscr{P}$. The length $l(\lambda)$ of a partition λ is defined as the number of positive λ_i. A rectangular partition is a partition λ such that $\lambda_1 = \cdots = \lambda_r = m$ for some positive integer m and $\lambda_{r+1} = 0$. We will typically denote such a λ by (m^r). The partition μ is said to be contained in the partition λ, denoted $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$.

Many of the identities in this paper involve a sum over the root lattice Q of A_{r-1} or a subset thereof, mostly for $r = 3$. It will be convenient to employ the standard embedding of this lattice in $\mathbb{Z}^r$, and we set

$$Q := \{(y_1, y_2, \dots, y_r) \in \mathbb{Z}^r : y_1 + y_2 + \cdots + y_r = 0\}, \tag{2.1a}$$

$$Q_+ := \{(y_1, y_2, \dots, y_r) \in Q : y_1 + \cdots + y_i \geqslant 0 \text{ for all } 1 \leqslant i \leqslant r\}, \tag{2.1b}$$

$$Q_{++} := \{(y_1, y_2, \dots, y_r) \in Q : y_1 \geqslant y_2 \geqslant \cdots \geqslant y_r\}. \tag{2.1c}$$

For $y \in Q$ we also define $y_{ij} := y_i - y_j$ for $1 \leqslant i < j \leqslant r$, where the reader is warned that for the sake of brevity the two indices i and j will not be separated by a comma. Let ε_i denote the ith standard unit vector in $\mathbb{R}^r$ and $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^r$, so that $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}$, with $\delta_{i,j}$ the Kronecker delta. For $i \in I := \{1, \dots, r-1\}$, let

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{and} \quad \omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{r}(\varepsilon_1 + \cdots + \varepsilon_r)$$

be the ith simple root and ith fundamental weight of $\mathfrak{sl}_r$ respectively, so that $\langle \alpha_i, \omega_j \rangle = \delta_{i,j}$. Then Q_+ corresponds to $\sum_{i \in I} \mathbb{N}_0 \alpha_i$ and $Q_{++} = Q \cap P_+$, where $P_+ := \sum_{i \in I} \mathbb{N}_0 \omega_i$ is the set of dominant (integral) weights of $\mathfrak{sl}_r$.

In this paper, q-series are typically viewed as elements of the formal power series ring $R[[q]]$ with R an appropriate coefficient ring or field, such as $\mathbb{Z}$, $\mathbb{Q}(a)$ or $\mathbb{Q}(z, w)$. A notable exception will be the q-series featured in Gustafson's ${}_6\psi_6$ summation (4.4) for the affine root system $\mathrm{A}_{r-1}^{(1)}$. This require complex q such that $|q| < 1$. Many of our proofs rely on identities for basic hypergeometric functions [40]. Using the condensed notation

$$(a_1, \dots, a_k; q)_n = \prod_{i=1}^{k} (a_i; q)_n,$$

for $n \in \mathbb{Z} \cup \{\infty\}$, the ${}_r\phi_s$ basic hypergeometric function is defined as

$${}_r\phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \Big((-1)^k q^{\binom{k}{2}} \Big)^{s-r+1} z^k. \tag{2.2}$$

This will only ever be used for terminating series, i.e., for series such that one of the numerator variables a_i is of the form q^{-n} for n a nonnegative integer. This ensures the summand vanishes unless $k \in \{0, 1, \dots, n\}$. We also adopt the standard one-line notation

$${}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z)$$

for the series (2.2) and abbreviate the very-well-poised basic hypergeometric function

$${}_r\phi_{r-1} \left[\begin{matrix} a_1, a_1^{1/2} q, -a_1^{1/2} q, a_4, \dots, a_r \\ a_1^{1/2}, -a_1^{1/2}, a_1 q/a_4, \dots, a_1 q/a_r \end{matrix} ; q, z \right]$$

as ${}_rW_{r-1}(a_1; a_4, \dots, a_r; q, z)$.

3. The A_1 Bailey lemma

To motivate the A_2 Bailey tree presented in the next section, we first review the classical A_1 case. Since the aim is to prove $A_2^{(1)}$ generalisations of the Andrews–Gordon–Bressoud identities (1.1), we will focus on that part of the Bailey machinery needed for proving (1.1). This allows us to adopt simpler notation than is typically found in treatments of the Bailey lemma such as in [5, 6, 88]. This notation is also more suited to generalisation to A_2 and, ultimately, A_{r-1}, since for higher rank the use of actual Bailey pairs often is notationally very cumbersome. The reader familiar with the existing literature should have no difficulties translating most of the results presented below in terms of Bailey pairs and transformations of such pairs.

Recall that $1/(q;q)_n = 0$ for n a negative integer. The main ingredients in our treatment of the Bailey lemma are the following three rational functions:

$$\Phi_n(z;q) := \frac{1}{(q,zq;q)_n}, \qquad \Phi_n(u;z;q) := \frac{1-uz-(1-u)zq^n}{(q;q)_n(z;q)_{n+1}} \tag{3.1}$$

and

$$\mathcal{K}_{n;r}(z;q) := \frac{z^r q^{r^2}}{(q;q)_{n-r}},$$

where $n, r \in \mathbb{Z}$. The reason for separating u and z as well as n and r by semicolons is that n, r, u and z all become sequences in the higher-rank case. For later reference we note that

$$\Phi_n(z^{-1};q^{-1}) = (zq)^n q^{n^2}\Phi_n(z;q), \tag{3.2a}$$

$$\Phi_n(1;z;q) = \Phi_n(z;q), \qquad \Phi_n(z^{-1};z;q) = q^n\Phi_n(z;q) \tag{3.2b}$$

and

$$\Phi_n(u;z;q) = \Phi_n(z/q;q) - \frac{uz}{(z;q)_2}\Phi_{n-1}(zq;q). \tag{3.3}$$

From [40, Equation (2.3.4)] it follows that

$$\sum_{r=N}^{n} q^{n-r}\Phi_{n-r}(zq^{2r};q)\Phi_{r-N}(zq^{2r};q^{-1}) = \delta_{n,N}, \tag{3.4}$$

which is Andrews' A_1 matrix inversion [4, Lemma 3] in disguise.

A key role in the Bailey lemma [12] is played by the above-mentioned Bailey pairs. These are pairs of sequences $(\alpha(z;q), \beta(z;q))$ indexed by nonnegative integers and depending on parameters z and q such that[2]

$$\beta_n(z;q) = \sum_{r=0}^{n} \frac{\alpha_r(z;q)}{(q;q)_{n-r}(zq;q)_{n+r}}, \tag{3.5a}$$

[2]It the literature on the Bailey lemma it is customary to use a instead of z, and to refer to a pair satisfying (3.5a) as a Bailey pair relative to a.

or, equivalently, [4, Lemma 3]

$$\alpha_n(z;q)=\sum_{r=0}^{n}\frac{1-zq^{2n}}{1-z}\,(-1)^{n-r}q^{\binom{n-r}{2}}\,\frac{(z;q)_{n+r}}{(q;q)_{n-r}}\,\beta_r(z;q). \tag{3.5b}$$

In [5] Andrews discovered that, given a Bailey pair relative to z, there is a simple transformation (already implicit in the work of Bailey) that turns this pair into a new Bailey pair relative to z. This can be iterated to yield what Andrews termed the Bailey chain:

$$\big(\alpha(z;q),\beta(z;q)\big)\mapsto\big(\alpha'(z;q),\beta'(z;q)\big)\mapsto\big(\alpha''(z;q),\beta''(z;q)\big)\mapsto\cdots. \tag{3.6}$$

The essence of (a special case of) this transformation is captured in the following lemma, which by abuse of terminology we also refer to as the A_1 Bailey chain. In particular it should be clear that the result below lends itself to iteration thanks to its reproducing nature.

Lemma 3.1 (A_1 Bailey chain). *For n a nonnegative integer,*

$$\sum_{r=0}^{n}\mathcal{K}_{n;r}(z;q)\Phi_r(z;q)=\Phi_n(z;q). \tag{3.7}$$

Proof. In q-hypergeometric notation the identity (3.7) is

$${}_1\phi_1(q^{-n};zq;q;zq^{n+1})=\frac{1}{(zq;q)_n}, \tag{3.8}$$

which is the terminating form of [40, Equation (II.5)]. ∎

Corollary 3.2. *We have*

$$\Phi_n(z;q)=\sum_{\lambda\in\mathscr{P}}\prod_{i\geqslant 1}\frac{z^{\lambda_i}q^{\lambda_i^2}}{(q;q)_{\lambda_{i-1}-\lambda_i}},$$

where $\lambda_0:=n$.

Proof. By a k-fold application of (3.7),

$$\Phi_n(z;q)=\sum_{\substack{\lambda\in\mathscr{P}\\ l(\lambda)\leqslant k}}\Phi_{\lambda_k}(z;q)\prod_{i=1}^{k}\mathcal{K}_{\lambda_{i-1};\lambda_i}(z;q),$$

where $\lambda_0:=n$. Letting k tend to infinite yields the claim. ∎

The Bailey chains (3.6) or (3.7) alone are not enough to prove the full set of Andrews–Gordon–Bressoud identities (1.1), and in [2] Agarwal, Andrews and Bressoud found a further transformation for Bailey pairs, this time scaling the parameter z by a factor q:

$$\big(\alpha(z;q),\beta(z;q)\big)\mapsto\big(\alpha'(z/q;q),\beta'(z/q;q)\big).$$

Combining this with the original transformation allows for more complicated patterns of iteration which are not linear in nature. This led Agarwal, Andrews and Bressoud to refer to their discovery as the Bailey lattice. Equipped with the Bailey lattice it is a simple exercise to prove (1.1) in full. The part of the Bailey lattice needed for proving the Andrews–Gordon–Bressoud identities has the structure of a simple binary tree, and is captured in the following lemma.

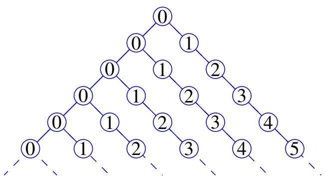
Lemma 3.3 (A_1 Bailey tree). *For n a nonnegative integer,*

$$\sum_{r=0}^{n} \mathcal{K}_{n;r}(z;q)\Phi_r(1;z;q) = \Phi_n(1;z;q), \tag{3.9a}$$

and

$$\sum_{r=0}^{n} \mathcal{K}_{n;r}(z/q;q)\Phi_r(u;z;q) = \Phi_n(uz;z;q). \tag{3.9b}$$

By $\Phi_n(1;z;q) = \Phi_n(z;q)$, the first claim is merely a restatement of the Bailey chain. The crucial part of Lemma 3.3 is that one can first repeatedly apply (3.9a) (or (3.7)) and then change the nature of the iteration by continuing with (3.9b), initially with $u = 1$, then $u = z$, $u = z^2$ and so on, changing the linear nature of Lemma 3.1, instead generating the binary tree



where the label $i \in \mathbb{N}_0$ represents the rational function $\Phi_n(z^i;z;q)$. Of course, (3.9b) in isolation allows for

$$\Phi_n(u;z;q) \mapsto \Phi_n(uz;z;q) \mapsto \Phi_n(uz^2;z;q) \mapsto \Phi_n(uz^3;z;q) \mapsto \cdots,$$

but if one wishes to combine (3.9a) and (3.9b) then this fixes $u = 1$.

Proof of Lemma 3.3. Since (3.9a) is a restatement of (3.7), only (3.9b) requires proof.

By (3.1) or (3.3) it is clear that both sides of (3.9b) are polynomials in u of degree one. Taking the constant term using (3.3) yields (3.7) with $z \mapsto z/q$. Similarly, extracting the coefficient of u in (3.9b) and dividing both sides by $-z/(z;q)_2$, gives

$$\sum_{r=1}^{n} \mathcal{K}_{n;r}(z/q;q)\Phi_{r-1}(zq;q) = z\Phi_{n-1}(zq;q).$$

Here we have also used that $\Phi_{-1} = 0$ to change the lower bound on the sum from 0 to 1. Shifting $r \mapsto r+1$ and noting that

$$\mathcal{K}_{n;r+1}(z/q;q) = z\mathcal{K}_{n-1;r}(zq;q),$$

results in (3.7) with $(z,n) \mapsto (zq, n-1)$. ∎

Before we are ready to demonstrate how the Andrews–Gordon–Bressoud identities (1.1) arise from the above results, a slight reformulation of the previous two lemmas is needed. For this purpose we define

$$\Phi_{n;y}(z;q) := \frac{\Phi_{n-y}(zq^{2y};q)}{(zq;q)_{2y}} = \frac{1}{(q;q)_{n-y}(zq;q)_{n+y}},$$
$$\Phi_{n;y}(u;z;q) := \frac{\Phi_{n-y}(u;zq^{2y};q)}{(zq;q)_{2y}} = \frac{1-uzq^{2y}-(1-u)zq^{n+y}}{(q;q)_{n-y}(zq;q)_{n+y}(1-zq^{2y})},$$

where $n,y\in\mathbb{Z}$. Note that once again $\Phi_{n;y}(1;z;q)=\Phi_{n;y}(z;q)$, and that $\Phi_{n;y}$ vanishes unless $n\geqslant y$. Further note that (3.5a) and (3.5b) can be recast in terms of $\Phi_{n;y}(z;q)$ and $\Phi_n(z;q)$ as

$$\beta_n(z;q) = \sum_{r=0}^{n} \Phi_{n;r}(z;q)\alpha_r(z;q), \tag{3.10a}$$

$$\alpha_n(z;q) = q^{-n}(zq;q)_{2n}\sum_{r=0}^{n} q^r\Phi_{n-r}\big(zq^{2n};q^{-1}\big)\beta_r(z;q). \tag{3.10b}$$

By replacing $(z,n)\mapsto(zq^{2y},n-y)$ in Lemmas 3.1 and 3.3 and then shifting the summation index $r\mapsto r-y$, the following two corollaries arise.[3]

Corollary 3.4. *For $n,y\in\mathbb{Z}$,*

$$\sum_{r=y}^{n}\mathcal{K}_{n;r}(z;q)\Phi_{r;y}(z;q) = z^y q^{y^2}\Phi_{n;y}(z;q). \tag{3.11}$$

Corollary 3.5. *For $n,y\in\mathbb{Z}$,*

$$\sum_{r=y}^{n}\mathcal{K}_{n;r}(z;q)\Phi_{r;y}(1;z;q) = z^y q^{y^2}\Phi_{n;y}(1;z;q), \tag{3.12a}$$

and

$$\sum_{r=y}^{n}\mathcal{K}_{n;r}(z/q;q)\Phi_{r;y}(u;z;q) = z^y q^{y^2-y}\Phi_{n;y}(uzq^{2y};z;q). \tag{3.12b}$$

We are now ready to give a short proof of (1.1).

[3] Corollary 3.4 for $z=1$ and $z=q$ is equivalent to [70, (R1) & (R2)] in that (3.11) for these two values of z corresponds to the coefficient of a_n and b_n in equations (R1) and (R2) of [70].

Proof. Slater's Bailey pairs B(3) and E(3) [79] are equivalent to the following pair of polynomial identities:[4]

$$\sum_{y=-n-1}^{n}(-1)^y q^{3\binom{y}{2}+2y}\begin{bmatrix}2n+1\\ n-y\end{bmatrix}=\frac{(q;q)_{2n+1}}{(q;q)_n} \tag{3.13}$$

and

$$\sum_{y=-n-1}^{n}(-1)^y q^{2\binom{y}{2}+y}\begin{bmatrix}2n+1\\ n-y\end{bmatrix}=\frac{(q;q)_{2n+1}}{(q^2;q^2)_n},$$

where n is a nonnegative integer. In terms of the rational function $\Phi_{n;y}(z;q)$, Slater's identities can be written as

$$\sum_{y=-n-1}^{n}(-1)^y q^{(2+\tau)\binom{y+1}{2}-y}\Phi_{n;y}(q;q)=\frac{1-q}{(q^{2-\tau};q^{2-\tau})_n}, \tag{3.14a}$$

where $\tau\in\{0,1\}$. Although this form of the identity is perfectly suitable for the application of the Bailey tree, we will rewrite it further to more closely mimic its A$_2$-analogue, given by (5.2) on page 25. To this end, let t_y be the summand of (3.14a) and rewrite the sum as $\sum_y t_y=\sum_y t_{2y}+\sum_y t_{-2y-1}$. Using that $\Phi_{n;-2y-1}(q;q)=\Phi_{n;2y}(q;q)$ and thus $t_{-2y-1}=-t_{2y}q^{4y+1}$, this yields

$$\sum_{y\in\mathbb{Z}} q^{(2+\tau)\binom{2y+1}{2}-2y}\,\frac{1-q^{4y+1}}{1-q}\,\Phi_{n;2y}(q;q)=\frac{1}{(q^{2-\tau};q^{2-\tau})_n}, \tag{3.14b}$$

where it is noted that the summand vanishes unless $-\lfloor (n+1)/2\rfloor\leqslant y\leqslant\lfloor n/2\rfloor$. (Since both sides of (3.14a) and (3.14b) trivially vanish for negative values of n, both forms of the identity are true for all $n\in\mathbb{Z}$).

In the following we identify the identity (3.14b) with the root of the binary tree shown on page 11. By a $(k-a)$-fold application of Corollary 3.4 with $z=q$, which corresponds to taking $k-a$ downward steps along the left-most branch of the tree,

$$\begin{aligned}\sum_{y\in\mathbb{Z}} q^{(2k-2a+2+\tau)\binom{2y+1}{2}-2y}\,\frac{1-q^{4y+1}}{1-q}\,\Phi_{n;2y}(q;q) \\ =\sum_{\lambda\subseteq(n^{k-a})}\frac{1}{(q^{2-\tau};q^{2-\tau})_{\lambda_\ell}}\prod_{i=1}^{k-a}\mathcal{K}_{\lambda_{i-1};\lambda_i}(q;q),\end{aligned} \tag{3.15}$$

where $\lambda_0:=n$ and $k-a\in\mathbb{N}_0$. We now replace $\Phi_{n;2y}(q;q)$ by $\Phi_{n;2y}(1;q;q)$ and then take a steps along the tree in the south-east direction using (3.12b) with $z=q$ and $u=q^{(i-1)(4y+1)}$ in the ith step. Since

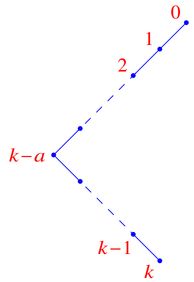
$$\mathcal{K}_{n;r}(z/q;q)=q^{-r}\mathcal{K}_{n;r}(z;q),$$

[4] These two results can be traced back to Rogers' work on the Rogers–Ramanujan identities. For example, the left-hand side of (3.13) is what Rogers denotes by $q^{-(n+1)^2}\beta_{2n+1}$ on page 316 of [73]. His equation (5) on the following page then states that $q^{-n-1}\beta_{2n+1}/(q;q)_{2n+1}=q^{n(n+1)}/(q;q)_n$.

this yields

$$\sum_{y\in\mathbb{Z}} q^{K\binom{2y+1}{2}-2(a+1)y}\,\frac{1-q^{4y+1}}{1-q}\,\Phi_{n;2y}\big(q^{a(4y+1)};q;q\big) \tag{3.16}$$
$$=\sum_{\lambda\subseteq(n^k)} \frac{1}{(q^{2-\tau};q^{2-\tau})_{\lambda_k}} \prod_{i=1}^{k} q^{-\chi(i\leqslant a)\lambda_i}\mathcal{K}_{\lambda_{i-1};\lambda_i}(q;q)$$
$$=\sum_{\lambda\subseteq(n^k)} \frac{q^{\lambda_1^2+\cdots+\lambda_k^2+\lambda_{a+1}+\cdots+\lambda_k}}{(q;q)_{n-\lambda_1}(q;q)_{\lambda_1-\lambda_2}\cdots(q;q)_{\lambda_{k-1}-\lambda_k}(q^{2-\tau};q^{2-\tau})_{\lambda_k}},$$

where a,k are integers such that $0\leqslant a\leqslant k$, and $K:=2k+2+\tau$. We note that the path along the tree we have taken is



where the labels denote the level (or distance to the root) of each vertex.

Although it is not an essential step in the proof and one can proceed by directly taking the large-n limit in (3.16), we observe that the left-hand side allows for a simplification which only requires the function $\Phi_{n;y}(q;q)$. This simplification is achieved by noting that

$$q^{K\binom{2y+1}{2}-2(a+1)y}\,\frac{1-q^{4y+1}}{1-q}\,\Phi_{n;2y}\big(q^{a(4y+1)};q;q\big)$$
$$=\sum_{y'\in\{-2y,2y-1\}} (-1)^{y'} q^{K\binom{y'+1}{2}-(a+1)y'}\,\frac{1-q^{n+y'+1}}{1-q}\,\Phi_{n;y'}(q;q).$$

Hence the left-hand side of (3.16) may also be written as

$$\sum_{y=-n-1}^{n} (-1)^{y} q^{K\binom{y+1}{2}-(a+1)y}\,\frac{1-q^{n+y+1}}{1-q}\,\Phi_{n;y}(q;q).$$

Since

$$\lim_{n\to\infty}\Phi_{n;y}(z;q)=\frac{1}{(q,zq;q)_\infty},$$

this implies that in the large-n limit

$$\frac{1}{(q;q)_\infty}\sum_{y\in\mathbb{Z}}(-1)^y q^{K\binom{y+1}{2}-(a+1)y} = \sum_{\substack{\lambda\in\mathscr{P}\\ l(\lambda)\leqslant k}} \frac{q^{\lambda_1^2+\cdots+\lambda_k^2+\lambda_{a+1}+\cdots+\lambda_k}}{(q;q)_{\lambda_1-\lambda_2}\cdots(q;q)_{\lambda_{k-1}-\lambda_k}(q^{2-\tau};q^{2-\tau})_{\lambda_k}}.$$

By the Jacobi triple product identity [40, (II.28)] the left-hand side admits the product form

$$\frac{(q^K;q^K)_\infty}{(q;q)_\infty}\,\theta\big(q^{a+1};q^K\big),$$

resulting in (1.1). ∎

4. The A_2 Bailey tree

In this section we present an A_2-analogue of the A_1 Bailey tree. This tree is three-dimensional, or parametrisable by three nonnegative integer variables, with an added layer of complexity in that the structure of the tree is not actually tree-like in the strict graph-theoretical sense. In developing our Bailey tree we once again avoid the use of Bailey pairs, although in the short Section 4.2 we briefly discuss A_2 Bailey pairs and A_2 Bailey pair inversion.

4.1. A Bailey tree for A_2

The most important definition of this section is the A_2-analogue of the rational function $\Phi_n(z;q)$, and following [8, Definition 4.2] and [89, Equation (5.1)], we let

$$\Phi_{n,m}(z,w;q) := \frac{(zwq;q)_{n+m}}{(q,zq,zwq;q)_n(q,wq,zwq;q)_m}, \tag{4.1}$$

where $n,m,r,s\in\mathbb{Z}$. This function was also considered in [31]. A first hint that (4.1) has something to do with the A_2 root system follows from the analogue of (3.2a):

$$\Phi_{n,m}\big(z^{-1},w^{-1};q^{-1}\big) = (zq)^n(wq)^m q^{n^2-nm+m^2}\Phi_{n,m}(z,w;q). \tag{4.2}$$

Here $n_1^2-n_1n_2+n_2^2=\frac{1}{2}\sum_{i,j=1}^2 n_iA_{ij}n_j$, where $(A_{ij})=(\langle\alpha_i,\alpha_j\rangle)$ is the A_2 Cartan matrix. Before we show how the function $\Phi_{n,m}(z,w;q)$ can be used to generalise all of the results of the previous section, we state the A_2-analogue of Andrews' matrix inversion (3.4). To the best of our knowledge this result is new.

Proposition 4.1. *For n,m,N,M integers such that $n\geqslant N$ and $m\geqslant M$,*

$$\sum_{r=N}^{n}\sum_{s=M}^{m} q^{n+m-r-s}\Phi_{n-r,m-s}\big(zq^{2r-s},wq^{2s-r};q\big)\Phi_{r-N,s-M}\big(zq^{2r-s},wq^{2s-r};q^{-1}\big) \tag{4.3}$$
$$= \delta_{n,N}\delta_{m,M}.$$

This inversion relation, which simplifies to (3.4) for $M=m=0$ or $w=0$, will be applied in Section 8 to prove Theorem 1.4.

Proof of Proposition 4.1. Replacing

$$(n,m,z,w)\mapsto \big(n+N,m+M,zq^{M-2N},wq^{N-2M}\big),$$

and then shifting the summation indices $(r,s)\mapsto(r+N,s+M)$, it follows that (4.3) for general N,M is equivalent to the $N=M=0$ case. The proof of (4.3) for $N=M=0$ requires Gustafson's multiple ${}_6\psi_6$ summation [44, Theorem 1.15] for the affine root system $\mathrm{A}_{r-1}^{(1)}$:

$$\sum_{y\in Q}\prod_{1\leqslant i<j\leqslant r}\frac{x_iq^{y_i}-x_jq^{y_j}}{x_i-x_j}\prod_{i,j=1}^{r}\frac{(a_jx_i/x_j;q)_{y_i}}{(b_jx_i/x_j;q)_{y_i}} \\ =\frac{(Bq^{1-r},q/A;q)_\infty}{(q,Bq^{1-r}/A;q)_\infty}\prod_{i,j=1}^{r}\frac{(qx_i/x_j,x_ib_j/a_ix_j;q)_\infty}{(x_ib_j/x_j,x_iq/a_ix_j;q)_\infty}, \tag{4.4}$$

where $A:=a_1\cdots a_r$, $B:=b_1\cdots b_r$ and $\max\{|q|,|Bq^{1-r}/A|\}<1$. Assuming $r\geqslant 3$ and specialising

$$(a_1,\dots,a_r)=(q^{-n},c_2,\dots,c_{r-1},1),\quad (b_1,\dots,b_r)=(q,c_2,\dots,c_{r-1},q^{m+1}),$$

(4.4) yields

$$\sum_{y\in Q}\prod_{1\leqslant i<j\leqslant r}\frac{x_iq^{y_i}-x_jq^{y_j}}{x_i-x_j}\prod_{i=1}^{r}\frac{(q^{-n}x_i/x_1,x_i/x_r;q)_{y_i}}{(qx_i/x_1,q^{m+1}x_i/x_r;q)_{y_i}}=0$$

for $|q|<1$ and $n+m>r-3$. The summand on the left vanishes unless $0\leqslant y_1\leqslant n$ and $0\leqslant -y_r\leqslant m$. Since $y\in Q$, this implies that for $r=3$ the summand has finite support, making the condition $|q|<1$ redundant. Then replacing $(y_1,y_2,y_3)\mapsto(r,s-r,-s)$ and $(x_1,x_2,x_3)\mapsto(zw,w,1)$, the identity (4.3) for $N=M=0$ and $(n,m)\neq(0,0)$ follows. Since the $(n,m)=(0,0)$ case trivially holds, we are done. ∎

Apart from $\Phi_{n,m}(z,w;q)$ we also need the function

$$\mathcal{K}_{n,m;r,s}(z,w;q):=\frac{z^rw^sq^{r^2-rs+s^2}}{(q;q)_{n-r}(q;q)_{m-s}}.$$

Then the A_2 Bailey lemma of [8, Theorem 4.3] is equivalent to the following reproducing identity for $\Phi_{n,m}(z,w;q)$, see also [89, Theorem 5.1] or [31, Corollary 7.9].

Theorem 4.2 (A_2 Bailey chain). *For n,m nonnegative integers,*

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\mathcal{K}_{n,m;r,s}(z,w;q)\Phi_{r,s}(z,w;q)=\Phi_{n,m}(z,w;q). \tag{4.5}$$

Since

$$\Phi_{n,m}(z,0;q)=\frac{\Phi_n(z;q)}{(q;q)_m}\quad\text{and}\quad \mathcal{K}_{n,m;r,s}(z,0;q)=\delta_{s,0}\frac{\mathcal{K}_{n;r}(z;q)}{(q;q)_m}, \tag{4.6}$$

Theorem 4.2 for $w=0$ simplifies to Lemma 3.1. It also simplifies to this lemma for $m=0$. We further remark that (4.5) holds for all integers n,m, with both sides vanishing trivially unless $n,m\geqslant 0$. The proof of (4.5) presented below replicates the second part of the proof of [8, Theorem 4.3]. For an alternative approach using Hall–Littlewood polynomials the reader is referred to [89].

Proof. Denote the double sum on the left of (4.5) by $\phi_{n,m}(z,w;q)$. Then

$$\phi_{n,m}(z,w;q)=\frac{1}{(q;q)_m}\sum_{r=0}^{n}\frac{z^r q^{r^2}}{(q;q)_{n-r}(q,zq;q)_r}\,{}_2\phi_2\left[\begin{matrix}zwq^{r+1},q^{-m}\\ wq,zwq\end{matrix};q,wq^{m-r+1}\right].$$

By a limiting case of [40, Equation (III.9)],

$${}_2\phi_2\left[\begin{matrix}a,q^{-n}\\ b,c\end{matrix};q,\frac{bcq^n}{a}\right]=\frac{1}{(c;q)_n}\,{}_2\phi_1\left[\begin{matrix}b/a,q^{-n}\\ b\end{matrix};q,cq^n\right].$$

Applying this with $(n,a,b,c)\mapsto(m,zwq^{r+1},zwq,wq)$ yields

$$\begin{aligned}\phi_{n,m}(z,w;q)&=\frac{1}{(q,wq;q)_m}\sum_{r=0}^{n}\frac{z^r q^{r^2}}{(q;q)_{n-r}(q,zq;q)_r}\,{}_2\phi_1\left[\begin{matrix}q^{-r},q^{-m}\\ zwq\end{matrix};q,wq^{m+1}\right]\\&=\sum_{s=0}^{n}\sum_{r=s}^{n}\frac{z^r w^s q^{\binom{r-s}{2}+r^2}}{(q;q)_{n-r}(q;q)_{m-s}(zq;q)_r(q;q)_{r-s}(q,zwq;q)_s}.\end{aligned}$$

After shifting $r\mapsto r+s$ this gives

$$\phi_{n,m}(z,w;q)=\frac{1}{(wq;q)_m}\sum_{s=0}^{n}\frac{(wz)^s q^{s^2}}{(q;q)_{n-s}(q;q)_{m-s}(q,zq,zwq;q)_s}\,{}_1\phi_1\left[\begin{matrix}q^{-(n-s)}\\ zq^{s+1}\end{matrix};q,zq^{n+1}\right].$$

Finally, by (3.8) with $(z,n)\mapsto(zq^s,n-s)$,

$$\phi_{n,m}(z,w;q)=\frac{1}{(q,zq;q)_n(q,wq;q)_m}\,{}_2\phi_1\left[\begin{matrix}q^{-n},q^{-m}\\ zwq\end{matrix};q,wzq^{n+m+1}\right]=\Phi_{n,m}(z,w;q),$$

where the final equality follows from the q-Chu–Vandermonde summation [40, Equation (II.7)]. ∎

Generalising the proof of Corollary 3.2 to the rank-two setting in the obvious manner gives the following multisum representation for $\Phi_{n,m}(z,w;q)$, see also [89, Corollary 3.4].

Corollary 4.3. *We have*

$$\Phi_{n,m}(z,w;q)=\sum_{\lambda,\mu\in\mathscr{P}}\prod_{i\geqslant 1}\frac{z^{\lambda_i}w^{\mu_i}q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2}}{(q;q)_{\lambda_{i-1}-\lambda_i}(q;q)_{\mu_{i-1}-\mu_i}},$$

where $\lambda_0:=n$ *and* $\mu_0:=m$.

Next we will generalise the A_1 Bailey tree of Lemma 3.3. This requires a suitable u,v-generalisation $\Phi_{n,m}(u,v;z,w;q)$ of $\Phi_{n,m}(z,w;q)$ such that

$$\Phi_{n,m}(1,1;z,w;q)=\Phi_{n,m}(z,w;q),\tag{4.7a}$$

$$\Phi_{n,m}(z^{-1},w^{-1};z,w;q)=q^n\Phi_{n,m}(z,w;q)\tag{4.7b}$$

and

$$\Phi_{n,m}(u,v;z,0;q)=\frac{\Phi_n(u;z;q)}{(q;q)_m}.\tag{4.8}$$

We begin by noting that the decomposition (3.3) for $u = 1$ follows from the relation $1 - z = (1 - cz) - z(1 - c)$ for $c = q^n$. This readily generalises to the 4-term relation

$$\begin{aligned}(1-z)&(1-w)(1-zw)(1-cdzw)\\ &= (1-cz)(1-w)(1-czw)(1-dzw) - z(1-c)(1-dw)(1-zw)(1-czw)\\ &\quad + zw^2(1-c)(1-d)(1-z)(1-cz),\end{aligned}$$

which for $(c, d) \mapsto (q^n, q^m)$ implies

$$\begin{aligned}\Phi_{n,m}(z,w;q) = \Phi_{n,m}(z/q,w;q) &- \frac{z}{(z;q)_2}\,\Phi_{n-1,m}(zq,w/q;q)\\ &+ \frac{zw^2}{(w,zw;q)_2}\,\Phi_{n-1,m-1}(z,wq;q).\end{aligned} \tag{4.9}$$

Generalising this to include parameters u and v, we define

$$\begin{aligned}\Phi_{n,m}(u,v;z,w;q) := \Phi_{n,m}(z/q,w;q) &- \frac{uz}{(z;q)_2}\,\Phi_{n-1,m}(zq,w/q;q)\\ &+ \frac{uvzw^2}{(w,zw;q)_2}\,\Phi_{n-1,m-1}(z,wq;q),\end{aligned} \tag{4.10}$$

which obviously satisfies (4.7a). After clearing denominators, the relation (4.7b) is a consequence of the 4-term relation

$$\begin{aligned}c(1-z)&(1-w)(1-zw)(1-cdzw)\\ &= (1-cz)(1-w)(1-czw)(1-dzw) - (1-c)(1-dw)(1-zw)(1-czw)\\ &\quad + w(1-c)(1-d)(1-z)(1-cz)\end{aligned}$$

for $(c, d) \mapsto (q^n, q^m)$. Finally, the relation (4.8) follows from (3.3) and (4.6). Most importantly, $\Phi_{n,m}(u, v; z, w; q)$ satisfies the following generalisation of Lemma 3.3.

Theorem 4.4 (A_2 Bailey tree, part I). *For n, m nonnegative integers,*

$$\sum_{r=0}^{n}\sum_{s=0}^{m} \mathcal{K}_{n,m;r,s}(z,w;q)\Phi_{r,s}(1,1;z,w;q) = \Phi_{n,m}(1,1;z,w;q) \tag{4.11a}$$

and

$$\sum_{r=0}^{n}\sum_{s=0}^{m} \mathcal{K}_{n,m;r,s}(z/q,w;q)\Phi_{r,s}(u,v;z,w;q) = \Phi_{n,m}(uz,vw;z,w;q). \tag{4.11b}$$

By (4.7a) the first claim is of course a restatement of the A_2 Bailey chain. We also remark that for $m = 0$ or for $w = 0$ the theorem simplifies to Lemma 3.3.

The A_2 Bailey tree as stated can only prove Theorems 1.2 and 1.4 for $b = 0$ (or, by symmetry, $a = 0$) and we also need a Bailey-type transformation for a four-parameter generalisation of $\Phi_{n,m}(z, w; q)$ involving the function $\mathcal{K}_{n,m;r,s}(z/q, w/q; q)$. This missing part of the A_2 Bailey tree will be discussed later.

Proof of Theorem 4.4. Each side of (4.11b) is a polynomial in u and v of the form $A + Bu + Cuv$. As in the A_1 case, the constant term of the identity corresponds to (4.5) with z replaced by z/q. Next, up to an overall factor of $-z/(z;q)_2$, the coefficient of u in (4.11b) is

$$\sum_{r=1}^{n}\sum_{s=0}^{m}\mathcal{K}_{n,m;r,s}(z/q,w;q)\Phi_{r-1,s}(zq,w/q;q)=z\Phi_{n-1,m}(zq,w/q;q),$$

where we have used that $\Phi_{r-1,s}$ vanishes for $r=0$. Shifting the summation index $r\mapsto r+1$ and using that

$$\mathcal{K}_{n,m;r+1,s}(z/q,w;q)=z\mathcal{K}_{n-1,m;r,s}(zq,w/q;q),$$

yields (4.5) with (n,z,w) replaced by $(n-1,zq,w/q)$. Finally, up to an overall factor of $zw^2/(z,zw;q)_2$, the coefficient of uv in (4.11b) is given by

$$\sum_{r=1}^{n}\sum_{s=1}^{m}\mathcal{K}_{n,m;r,s}(z/q,w;q)\Phi_{r-1,s-1}(z,wq;q)=zw\Phi_{n-1,m-1}(z,wq;q).$$

After shifting $(r,s)\mapsto(r+1,s+1)$ and using that

$$\mathcal{K}_{n,m;r+1,s+1}(z/q,w;q)=zw\mathcal{K}_{n-1,m-1;r,s}(z,wq;q),$$

this yields (4.5) with $(n,m,w)\mapsto(n-1,m-1,wq)$. ∎

To prove Conjecture 1.1 we need the A_2-analogues of Corollaries 3.4 and 3.5. Unlike the A_1 case, where we used a single integer parameter y to parametrise the A_1 root lattice, for A_2 we adopt the notation (2.1a) for $r=3$. That is, for $y=(y_1,y_2,y_3)\in Q$ and $y_{ij}:=y_i-y_j$, we define

$$\begin{aligned}\Phi_{n,m;y}(z,w;q)&:=\frac{\Phi_{n-y_1,m-y_1-y_2}(zq^{y_{12}},wq^{y_{23}};q)}{(zq;q)_{y_{12}}(wq;q)_{y_{23}}(zwq;q)_{y_{13}}}\qquad(4.12)\\&=\frac{(zwq;q)_{n+m}}{(q;q)_{n-y_1}(zq;q)_{n-y_2}(zwq;q)_{n-y_3}(q;q)_{m+y_3}(wq;q)_{m+y_2}(zwq;q)_{m+y_1}}.\end{aligned}$$

Clearly, $\Phi_{n,m;(0,0,0)}(z,w;q)=\Phi_{n,m}(z,w;q)$ and $\Phi_{n,m;y}(z,w;q)$ vanishes unless $n-y_1\geqslant 0$ and $m+y_3=m-y_1-y_2\geqslant 0$. Moreover, $\Phi_{n,m;(y_1,-y_1,0)}(z,0;q)=\Phi_{n;y_1}(z;q)/(q;q)_m$.

Corollary 4.5. *For $n,m\in\mathbb{Z}$ and $y=(y_1,y_2,y_3)\in Q$,*

$$\sum_{r=y_1}^{n}\sum_{s=y_1+y_2}^{m}\mathcal{K}_{n,m;r,s}(z,w;q)\Phi_{r,s;y}(z,w;q)=z^{y_1}w^{y_1+y_2}q^{\frac{1}{2}(y_1^2+y_2^2+y_3^2)}\Phi_{n,m;y}(z,w;q).\qquad(4.13)$$

Proof. Replacing

$$(n,m,z,w)\mapsto(n-y_1,m-y_1-y_2,zq^{y_{12}},wq^{y_{23}})\qquad(4.14)$$

in (4.5), shifting the summation indices $(r,s)\mapsto(r-y_1,m-y_1-y_2)$ and using

$$\mathcal{K}_{n-y_1,m-y_1-y_2;r-y_1,s-y_1-y_2}(zq^{y_{12}},wq^{y_{23}};q)=z^{-y_1}w^{-y_1-y_2}q^{-\frac{1}{2}(y_1^2+y_2^2+y_3^2)}\mathcal{K}_{n,m;r,s}(z,w;q)\qquad(4.15)$$

as well as definition (4.12), the claim follows. ∎

In much the same way we define

$$\Phi_{n,m;y}(u,v;z,w;q) := \frac{\Phi_{n-y_1,m-y_1-y_2}(u,v;zq^{y_{12}},wq^{y_{23}};q)}{(zq;q)_{y_{12}}(wq;q)_{y_{23}}(zwq;q)_{y_{13}}}, \tag{4.16}$$

so that $\Phi_{n,m;(y_1,-y_1,0)}(u,v;z,0;q) = \Phi_{n;y_1}(u;z;q)/(q;q)_m$. Equation (4.7a) implies the simplification

$$\Phi_{n,m;y}(1,1;z,w;q) = \Phi_{n,m;y}(z,w;q), \tag{4.17}$$

which yields the first of the identities in the next lemma. The second result follows in an analogous manner as Corollary 4.5, and we omit the proof.

Corollary 4.6. *For $n,m \in \mathbb{Z}$ and $y = (y_1,y_2,y_3) \in Q$,*

$$\begin{aligned}\sum_{r=y_1}^{n}\sum_{s=y_1+y_2}^{m} &\mathcal{K}_{n,m;r,s}(z,w;q)\Phi_{r,s;y}(1,1;z,w;q) \\ &= z^{y_1}w^{y_1+y_2}q^{\frac{1}{2}(y_1^2+y_2^2+y_3^2)}\Phi_{n,m;y}(1,1;z,w;q)\end{aligned} \tag{4.18a}$$

and

$$\begin{aligned}\sum_{r=y_1}^{n}\sum_{s=y_1+y_2}^{m} &\mathcal{K}_{n,m;r,s}(z/q,w;q)\Phi_{r,s;y}(u,v;z,w;q) \\ &= z^{y_1}w^{y_1+y_2}q^{\frac{1}{2}(y_1^2+y_2^2+y_3^2)-y_1}\Phi_{n,m;y}(uzq^{y_{12}},vwq^{y_{23}};z,w;q).\end{aligned} \tag{4.18b}$$

As mentioned previously, our A_2 Bailey tree is not yet complete. Conjecture 1.1 and Theorem 1.4 contain three integer parameters a, b and k. Theorem 4.4, however, is restricted to paths along the Bailey tree of the form shown on page 14. Since such paths can be described by two parameters, something is still missing. The reason for deferring the treatment of the missing part of the A_2 Bailey tree till now is that it uses most of the previously-defined functions and is less intuitive than what has been discussed so far.

For $n,m \in \mathbb{Z}$ and $\rho := (1,2,3)$, define

$$\begin{aligned}&\Phi_{n,m}(u,v;c,d;z,w;q) \\ &:= \sum_{\sigma\in S_3}\operatorname{sgn}(\sigma)\,(uz)^{\sigma_1-1}\Big(\frac{v}{d}\Big)^{\chi(\sigma_3=1)}\Big(\frac{c}{u}\Big)^{\chi(\sigma_1=3)}(dw)^{3-\sigma_3}\Phi_{n,m;\sigma-\rho}(z/q,w/q;q).\end{aligned} \tag{4.19}$$

Since the summand contains the factors $(q;q)_{n-\sigma_1+1}$ and $(q;q)_{m+\sigma_3-3}$ in the denominator, the function $\Phi_{n,m}(u,v;c,d;z,w;q)$ vanishes unless $n,m \geqslant 0$. If $n=m=0$ then only the term $\sigma=\rho$ contributes to the sum so that $\Phi_{0,0}(u,v;c,d;z,w;q)=1$. By replacing $\sigma=(\sigma_1,\sigma_2,\sigma_3)\mapsto(4-\sigma_3,4-\sigma_2,4-\sigma_1)$, and using that $\Phi_{n,m;(y_1,y_2,y_3)}(z,w;q) = \Phi_{m,n;-(y_3,y_2,y_1)}(w,z;q)$ and $\sigma_1+\sigma_2+\sigma_3=6$, it may also be seen that

$$\Phi_{n,m}(u,v;c,d;z,w;q) = \Phi_{m,n}(d,c;v,u;w,z;q). \tag{4.20}$$

Before proving a number of important properties of $\Phi_{n,m}(u,v;c,d;z,w;q)$, including a Bailey-type transformation, we remark that in the $n,m\to\infty$ limit an important special case of this function is essentially a Schur function [63].

Lemma 4.7. *Let $z := x_1/x_2$, $w := x_2/x_3$ and for a, b nonnegative integers, let $\nu := (a+b+2, b+1, 0)$. Then*

$$\lim_{n,m\to\infty} \Phi_{n,m}(z^a, w^a; z^b, w^b; z, w; q) = \frac{1}{(q, q, z, w, zw/q; q)_\infty} \det_{1\leqslant i,j\leqslant 3} \big(x_i^{\nu_i-\nu_j}\big) \tag{4.21}$$

Proof. The large-n, m limit of $\Phi_{n,m;\sigma-\rho}(z/q, w/q; q)$ gives the infinite product on the right of (4.21). Moreover, for $(u, v, c, d) = (z^a, w^a; z^b, w^b)$ the sum over S_3 in the definition of (4.19) becomes

$$\sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\, z^{(a+1)(\sigma_1-1)-(a-b)\chi(\sigma_1=3)} w^{(a-b)\chi(\sigma_3=1)-(b+1)(\sigma_3-3)} = \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma) \prod_{i=1}^{3} x_i^{\nu_i-\nu_{\sigma_i}},$$

which is the determinant in the numerator. ∎

Unlike $\Phi_{n,m}(u, v; z, w; q)$, which simplifies to $\Phi_{n,m}(z, w; q)$ for $u = v = 1$, the function $\Phi_{n,m}(u, v; c, d; z, w; q)$ for $c = d = 1$ does not simplify to $\Phi_{n,m}(u, v; z, w; q)$. Instead a simple linear combination of $\Phi_{n,m}(u, v; z, w; q)$ and $\Phi_{n,m}(u/z, v/w; z, w; q)$ arises.

Lemma 4.8. *For $n, m \in \mathbb{Z}$,*

$$\Phi_{n,m}(u, v; 1, 1; z, w; q) = \frac{\Phi_{n,m}(u, v; z, w; q) - zwq^{m-1}\Phi_{n,m}(u/z, v/w; z, w; q)}{1 - zwq^{-1}}. \tag{4.22}$$

By (4.7), the $u = v = 1$ case of the lemma simplifies to

$$\Phi_{n,m}(1, 1; 1, 1; z, w; q) = \frac{1 - zwq^{n+m-1}}{1 - zwq^{-1}}\, \Phi_{n,m}(z, w; q). \tag{4.23}$$

Proof. Both sides of (4.22) are polynomials in u and v. Equating like coefficients using (4.10), the claim splits into three separate equations. After normalisation these are

$$\frac{1 - zwq^{m-1}}{1 - zwq^{-1}}\, \Phi_{n,m}(z/q, w; q) = \Phi_{n,m;(0,0,0)}(z/q, w/q; q) - w\Phi_{n,m;(0,1,-1)}(z/q, w/q; q),$$

$$\begin{aligned}\frac{1 - wq^{m-1}}{1 - zwq^{-1}}\, &\Phi_{n-1,m}(zq, w/q; q)\\ &= (z; q)_2\Big(\Phi_{n,m;(1,-1,0)}(z/q, w/q; q) - zw\Phi_{n,m;(2,-1,-1)}(z/q, w/q; q)\Big),\end{aligned}$$

and

$$\begin{aligned}\frac{1 - q^{m-1}}{1 - zwq^{-1}}\, &\Phi_{n-1,m-1}(z, wq; q)\\ &= (w, zw; q)_2\Big(\Phi_{n,m;(1,1,-2)}(z/q, w/q; q) - z\Phi_{n,m;(2,0,-2)}(z/q, w/q; q)\Big),\end{aligned}$$

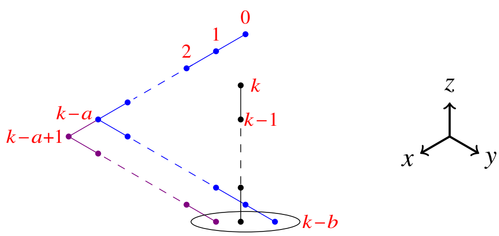
corresponding to the coefficients of u^0v^0, u^1v^0 and u^1v^1 respectively. By the definitions of $\Phi_{n,m}(z, w; q)$ and $\Phi_{n,m;y}(z, w; q)$ given in (4.1) and (4.12), all three equations are readily verified. ∎

The missing part of the A_2 Bailey tree can now be stated as follows.

Theorem 4.9 (A_2 Bailey tree, part II). *For n, m nonnegative integers,*

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\mathcal{K}_{n,m;r,s}(z/q,w/q;q)\Phi_{r,s}(u,v;c,d;z,w;q)=\Phi_{n,m}(uz,vw;cz,dw;z,w;q). \tag{4.24}$$

Once again consider the tree on page 11. In view of Lemma 4.8, we can first apply the Bailey tree of Theorem 4.4, starting at the root and taking $k-a$ south-west steps followed by $a-b$ south-east steps. This gives the same path along the tree as shown on page 14 but with $(k,a)\mapsto(k-b,a-b)$, so that the final vertex is labelled $k-b$. Next we can repeat the above but with a replaced by $a-1$, resulting in the path along the tree shown on page 14 with $(k,a)\mapsto(k-b,a-b-1)$, so that the final vertex is once again labelled $k-b$. As a third and final step we can then take a linear combination of the pair of identities represented by the two vertices labelled $k-b$ and take a further b steps using part II of the Bailey tree. If we think of south-east steps as unit steps in $\mathbb{R}^3$ in the positive x-direction, south-west steps as steps in the positive y-direction and the final b steps as steps in the positive z-direction, the above procedure can be represented by the three-dimensional diagram



where the central black vertex in the encircled region represents the appropriate linear combination of the violet and blue vertices labelled $k-b$.

Proof of Theorem 4.9. Denote the left-hand side of (4.24) by $\phi_{n,m}$. By (4.19) and an interchange in the order of the sums over r,s and over σ,

$$\phi_{n,m}=\sum_{\sigma\in S_3}\bigg(\operatorname{sgn}(\sigma)\,(uz)^{\sigma_1-1}(v/d)^{\chi(\sigma_3=1)}(c/u)^{\chi(\sigma_1=3)}(dw)^{3-\sigma_3}\\ \times\sum_{r=0}^{n}\sum_{s=0}^{m}\mathcal{K}_{n,m;r,s}(z/q,w/q;q)\Phi_{r,s;\sigma-\rho}(z/q,w/q;q)\bigg).$$

We now use that $\Phi_{r,s;\sigma-\rho}(z/q,w/q;q)=0$ unless $r-\sigma_1+1\geqslant 0$ and $s+\sigma_3-3\geqslant 0$ to change the lower bounds on the sums over r and s to σ_1-1 and $3-\sigma_3$ respectively. Since Corollary 4.5 for $y=\sigma-\rho$ simplifies to

$$\sum_{r=\sigma_1-1}^{n}\sum_{s=3-\sigma_3}^{m}\mathcal{K}_{n,m;r,s}(z,w;q)\Phi_{r,s;\sigma-\rho}(z,w;q)=(zq)^{\sigma_1-1}(wq)^{3-\sigma_3}\Phi_{n,m;\sigma-\rho}(z,w;q),$$

it follows that

$$\begin{aligned}\phi_{n,m} &= \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\,(uz^2)^{\sigma_1-1}\Big(\frac{v}{d}\Big)^{\chi(\sigma_3=1)}\Big(\frac{c}{u}\Big)^{\chi(\sigma_1=3)}(dw^2)^{3-\sigma_3}\Phi_{n,m;\sigma-\rho}(z/q,w/q;q)\\ &= \Phi_{n,m}(uz,vw;cz,dw;z,w;q).\end{aligned}$$

■

To conclude the section we define y-analogue of (4.19):

$$\Phi_{n,m;y}(u,v;c,d;z,w;q) := \frac{\Phi_{n-y_1,m-y_1-y_2}(u,v;c,d;zq^{y_{12}},wq^{y_{23}};q)}{(z;q)_{y_{12}}(w;q)_{y_{23}}(zw/q;q)_{y_{13}}}, \tag{4.25}$$

where $y=(y_1,y_2,y_3)\in Q$. It follows from Lemma 4.7 that for $z:=x_1/x_2$ and $w:=x_2/x_3$,

$$\begin{aligned}&\lim_{n,m\to\infty}\Phi_{n,m;y}\big((zq^{y_{12}})^a,(wq^{y_{23}})^a;(zq^{y_{12}})^b,(wq^{y_{23}})^b;z,w;q\big) \\ &\qquad= \frac{1}{(q,q,z,w,zw/q;q)_\infty}\det_{1\leqslant i,j\leqslant 3}\big((x_iq^{y_i})^{\nu_i-\nu_j}\big),\end{aligned}\tag{4.26}$$

where $\nu=(a+b+2,b+1,0)$. Furthermore, noting the minor difference in denominators on the right of (4.16) and (4.25), it follows that the special case of Lemma 4.8 given in (4.23) admits the y-generalisation

$$\Phi_{n,m;y}(1,1;1,1;z,w;q) = \frac{1-zwq^{n+m-1}}{1-zwq^{-1}}\,\Delta_y(z,w;q)\Phi_{n,m;y}(z,w;q), \tag{4.27}$$

where

$$\Delta_y(z,w;q) := \frac{(1-zq^{y_{12}})(1-wq^{y_{23}})(1-zwq^{y_{13}})}{(1-z)(1-w)(1-zw)}. \tag{4.28}$$

Finally, the y-analogues of Lemma 4.8 and Theorem 4.9 follow from (4.16) and (4.15) respectively,

Corollary 4.10. *For $n,m\in\mathbb{Z}$ and $y=(y_1,y_2,y_3)\in Q$,*

$$\begin{aligned}&\Phi_{n,m;y}(u,v;1,1;z,w;q) = \Delta_y(z,w;q) \\ &\qquad\times\frac{\Phi_{n,m;y}(u,v;z,w;q) - zwq^{m+y_1-1}\Phi_{n,m;y}(uq^{-y_{12}}/z,vq^{-y_{23}}/w;z,w;q)}{1-zwq^{-1}}.\end{aligned}\tag{4.29}$$

Corollary 4.11. *For $n,m\in\mathbb{Z}$ and $y=(y_1,y_2,y_3)\in Q$,*

$$\begin{aligned}&\sum_{r=y_1}^{n}\sum_{s=y_1+y_2}^{m}\mathcal{K}_{n,m;r,s}(z/q,w/q;q)\Phi_{r,s;y}(u,v;c,d;z,w;q)\\ &\qquad= z^{y_1}w^{y_1+y_2}q^{\frac12(y_1^2+y_2^2+y_3^2)-2y_1-y_2}\Phi_{n,m;y}(uzq^{y_{12}},vwq^{y_{23}};czq^{y_{12}},dwq^{y_{23}};z,w;q).\end{aligned}$$

4.2. A_2 Bailey pairs

In this section we briefly discuss the notion of A_2 Bailey pairs. We will not, however, translate all of the various A_2 Bailey transformations of Section 4.1 in terms of such pairs. Throughout, (2.1a) is used for $r = 3$.

Let

$$\alpha(z,w;q) = \big(\alpha_y(z,w;q)\big)_{y\in Q_+},$$
$$\beta(z,w;q) = \big(\beta_{n,m}(z,w;q)\big)_{n,m\in\mathbb{N}_0}$$

be a pair of sequences such that

$$\beta_{n,m}(z,w;q) = \sum_{y\in Q_+} \Phi_{n,m;y}(z,w;q)\alpha_y(z,w;q). \tag{4.30}$$

Then we say that $(\alpha(z,w;q),\beta(z,w;q))$ is an A_2 Bailey pair relative to z,w. Note that in the above definition only those $y\in Q_+$ contribute to the sum on the right for which $y_1 \leqslant n$ and $y_1+y_2\leqslant m$. Definition (4.30) is not the same as the one adopted in [8], where Q_{++} was used instead of Q_+. Further define

$$\Psi_{y;r,s}(z,w;q) := q^{r+s-y_{13}}(zq;q)_{y_{12}}(wq;q)_{y_{23}}(zwq;q)_{y_{13}} \\ \times \Phi_{y_1-r,y_1+y_2-s}\big(zq^{y_{12}},wq^{y_{23}};q^{-1}\big),$$

for $r,s\in\mathbb{N}_0$ and $y\in Q_+$. Recalling (4.12), the A_2 inversion relation (4.3) may then be written as

$$\sum_{y\in Q_+} \Phi_{n,m;y}(z,w;q)\Psi_{y;N,M}(z,w;q) = \delta_{n,N}\delta_{m,M},$$

for $n,m,N,M\in\mathbb{N}_0$. Since $\Phi_{n,m;y}(z,w;q)$ vanishes unless $n\geqslant y_1$ and $m\geqslant y_1+y_2$ and $\Psi_{y;N,M}(z,w;q)$ vanishes unless $y_1\geqslant N$ and $y_1+y_2\geqslant M$, the summand on the left is only supported on $y\in Q_+$ such that $N\leqslant y_1\leqslant n$ and $M\leqslant y_1+y_2\leqslant m$. Similarly, it follows that for $y,Y\in Q_+$,

$$\sum_{r,s\in\mathbb{N}_0} \Psi_{y;r,s}(z,w;q)\Phi_{r,s;Y}(z,w;q) = \delta_{y,Y},$$

with summand supported on $Y_1\leqslant r\leqslant y_1$ and $Y_1+Y_2\leqslant s\leqslant y_1+y_2$. The relation (4.30) is thus invertible, so that

$$\alpha_y(z,w;q) = \sum_{r,s\in\mathbb{N}_0} \Psi_{y;r,s}(z,w;q)\beta_{r,s}(z,w;q). \tag{4.31}$$

This in turn gives rise to what may be called the A_2 unit Bailey pair:

$$\alpha_y(z,w;q) = \Psi_{y;0,0}(z,w;q) \quad\text{and}\quad \beta_{n,m}(z,w;q) = \delta_{n,0}\delta_{m,0}. \tag{4.32}$$

If $\alpha_n(z;q) := \alpha_{(n,-n,0)}(z,0;q)$ and $\beta_n(z;q) := \beta_{n,0}(z,0;q)$, then (4.30) for $m=w=0$ and (4.32) for $y=(n,-n,0)$ and $w=0$ correspond to (3.5a) and (3.5b) respectively.

For a number of A_2 Bailey pairs, such as the unit Bailey pair (4.33) or pairs that follow from the unit Bailey pair by application of (4.13), the definition (4.30) is perfectly useable. However, the explicit form of many A_2 Bailey pairs is rather unwieldy, making the definition not particularly practical. A good example is the Bailey pair implied by the identity (5.2) of the next section. This identity corresponds to the root of the tree of identities on which our proof of the Kanade–Russell conjecture is based. It is quite artificial, and not at all enlightening, to write the left-hand side of (5.2) as a sum over Q_+ — which is necessary in order to read off α_y — instead of Q.

5. Proof of the Kanade–Russell conjecture

Before we can apply the A_2 Bailey tree to prove Conjecture 1.1 we need a suitable identity playing the role of root in the Bailey tree. This root identity is given by the A_2-analogue of (3.14b). Before stating the actual result, we note that for $n, m \in \mathbb{Z}$ and $y = (y_1, y_2, y_3) \in Q$,

$$\Phi_{n,m;y}(q, q; q) := \lim_{z,w \to 1} \Phi_{n,m;y}(zq, wq; q) = \frac{(q;q)_1^2 (q;q)_2}{(q;q)_{n+m+2}^2} \prod_{i=1}^{3} \begin{bmatrix} n+m+2 \\ n-y_i+i-1 \end{bmatrix}, \tag{5.1}$$

which vanishes unless $i - m - 3 \leqslant y_i \leqslant n + i - 1$ for all $1 \leqslant i \leqslant 3$. The reason $\Phi_{n,m;y}(q, q; q)$ is defined as a limit is that the term $(zwq^3; q)_{n+m}$ in the numerator of $\Phi_{n,m;y}(zq, wq; q)$ has a simple pole at $zw = 1$ if $n + m + 2 < 0$ (for $n + m + 2 \geqslant 0$ the function $\Phi_{n,m;y}(z, w; q)$ is regular at $z = w = 1$). This pole has zero residue and the above expression on the right arises. Moreover, it follows from the above inequalities for the y_i that the only instances where $\Phi_{n,m;y}(q, q; q)$ is nonvanishing for $\min\{n, m\} < 0$ correspond to $y = (-1, 0, 1)$ and $\min\{n, m\} = -1$. This in particular implies that if t is an integer greater than 1, then $\Phi_{n,m;ty}(q, q; q)$ vanishes if $(n, m) \notin \mathbb{N}_0^2$.

Recall definition (4.28) of Δ_y.

Proposition 5.1. *Let* $n, m \in \mathbb{N}_0$, $\tau \in \{-1, 0, 1\}$ *and*

$$g_{n,m;\tau}(q) := \frac{q^{\tau(\tau-1)nm}}{(q, q^2; q)_{n+m}} \begin{bmatrix} n+m \\ n \end{bmatrix}_p,$$

where $p = q$ *if* $\tau^2 = 1$ *and* $p = q^3$ *if* $\tau = 0$. *Then*

$$\sum_{y \in Q} \Phi_{n,m;3y}(q, q; q) \Delta_{3y}(q, q; q) \prod_{i=1}^{3} q^{3(3+\tau)\binom{y_i}{2} - \tau i y_i} = g_{n,m;\tau}(q). \tag{5.2}$$

The above definition of $g_{n,m;\tau}(q)$ is the same as (1.3) of the introduction.

Proof. Recall that $y_{ij} := y_i - y_j$. The identity (5.2) for $\tau = 1$ is a bounded form of the A_2-analogue of Euler's pentagonal number theorem, stated in [8, page 692] in the form

$$\sum_{y\in Q}\prod_{1\leqslant i<j\leqslant 3}(1-q^{3y_{ij}+j-i})\prod_{i=1}^{3}q^{12\binom{y_i}{2}-iy_i}\begin{bmatrix}n+m+2\\ n-3y_i+i-1\end{bmatrix}$$
$$=(1-q^{n+m+1})(1-q^{n+m+2})^2\begin{bmatrix}n+m\\ n\end{bmatrix}, \tag{5.3}$$

for $n,m\in\mathbb{N}_0$. The proof of (5.3) given in [8] is very involved. First an identity for what are known as supernomial coefficients is established (the $\ell=0$ case of [8, Equation (5.3)]). This is then transformed using an A_2 Bailey lemma for supernomial coefficients, resulting in [8, Equation (5.15)]. Using the determinant evaluation (5.5) below, this finally yields (5.3). In the appendix we present a much simpler proof which implies that (5.3) arises by taking the constant term with respect to z in the $r=3$ instance of the identity

$$\sum_{y_1,\dots,y_r\in\mathbb{Z}}\prod_{1\leqslant i<j\leqslant r}(1-q^{ry_{ij}+j-i})\prod_{i=1}^{r}(-1)^{ry_i}z^{y_i}q^{\binom{r+1}{2}y_i^2-iy_i}\begin{bmatrix}n+m+r-1\\ n-ry_i+i-1\end{bmatrix}$$
$$=\biggl(\prod_{i=1}^{r-1}(1-q^{n+m+i})^i\biggr)\sum_{k=-m}^{n}(-1)^kz^kq^{(r+1)\binom{k}{2}}\begin{bmatrix}n+m\\ n-k\end{bmatrix}. \tag{5.4}$$

This last result is a polynomial analogue of the classical theta function identity

$$\det_{1\leqslant i,j\leqslant r}\Bigl(q^{i(i-j)}\theta\bigl(z(-q^{-i})^{r+1}q^{rj+\binom{r+1}{2}};q^{r(r+1)}\bigr)\Bigr)=\frac{(q^{r+1};q^{r+1})_\infty(q;q)_\infty^{r-1}}{(q^{r(r+1)};q^{r(r+1)})_\infty^r}\,\theta(z;q^{r+1}).$$

Replacing $q\mapsto 1/q$ in (5.2) for $\tau=-1$, and then using $\begin{bmatrix}n+m\\ n\end{bmatrix}_{1/q}=q^{-nm}\begin{bmatrix}n+m\\ n\end{bmatrix}$ as well as

$$\Phi_{n,m;y}(z^{-1},w^{-1};q^{-1})=z^{n+2y_1}w^{m-2y_3}q^{n^2-nm+m^2+n+m+\sum_{i=1}^3y_i^2}\,\Phi_{n,m;y}(z,w;q)$$

where $y\in Q$, gives (5.2) for $\tau=-1$.

Finally, according to [42, Equation (6.18)],

$$\sum_{y\in rQ}\det_{1\leqslant i,j\leqslant r}\biggl(q^{\binom{y_i}{2}+(j-i)(j+y_i)}\begin{bmatrix}n+m\\ n-y_i+i-j\end{bmatrix}\biggr)=\begin{bmatrix}n+m\\ n\end{bmatrix}_{q^r}.$$

By [55, page 189]

$$\det_{1\leqslant i,j\leqslant r}\biggl(q^{(j-i)(j+i+b_i)}\begin{bmatrix}n+m\\ n-b_i-j\end{bmatrix}\biggr)$$
$$=\prod_{1\leqslant i<j\leqslant r}(1-q^{b_i-b_j})\prod_{i=1}^{r}\frac{1}{(q^{n+m+i};q)_{r-i}}\begin{bmatrix}n+m+r-1\\ n-b_i-1\end{bmatrix} \tag{5.5}$$

for $n,m,b_1,\dots,b_r\in\mathbb{Z}$, this can be recast as

$$\sum_{y\in rQ}\prod_{1\leqslant i<j\leqslant r}(1-q^{y_{ij}+j-i})\prod_{i=1}^{r}q^{\binom{y_i}{2}}\begin{bmatrix}n+m+r-1\\ n-y_i+i-1\end{bmatrix}=\begin{bmatrix}n+m\\ n\end{bmatrix}_{q^r}\prod_{i=1}^{r-1}(1-q^{n+m+i})^i. \tag{5.6}$$

For $r=3$ this is (5.2) with $\tau=0$. ∎

Remark 5.2. Although (5.2) is the natural A_2-analogue of (3.14b), there is a notable difference between the $\tau=1$ instances of these identities. From (3.5b) we may infer what is known as the A_1 unit Bailey pair:

$$\alpha_n(z;q)=\frac{1-zq^{2n}}{1-z}\,(-1)^n q^{\binom{n}{2}} \quad\text{and}\quad \beta_n(z;q)=\delta_{n,0}.$$

By (3.10a) this yields[5]

$$\sum_{r=0}^{n}\frac{1-zq^{2r}}{1-z}\,(-1)^r q^{\binom{r}{2}}\Phi_{n;r}(z;q)=\delta_{n,0}.$$

Applying Corollary (3.4) then gives

$$\sum_{r=0}^{n}\frac{1-zq^{2r}}{1-z}\,(-1)^r z^r q^{3\binom{r}{2}+r}\Phi_{n;r}(z;q)=\frac{1}{(q;q)_n},$$

which for $z=q$ is the same as (3.14a) (and hence (3.14b)) for $\tau=1$. The $\tau=1$ case of (5.2), however, does *not* follow from the A_2 unit Bailey pair (4.33). Indeed, the once-iterated A_2 unit Bailey pair gives the $k=a$ case of (8.6), which has $1/((q;q)_n(q;q)_m)$ as right-hand side, not $g_{n,m;1}(q)$. Instead, (5.2) for $\tau=1$ follows from the A_2 unit Bailey pair for supernomial coefficients, see [8, 87].

Equipped with the identity (5.2) we can prove Conjecture 1.1. The essence of the proof is encoded in the diagram on page 22, where the root identity corresponds to the vertex labelled 0 and the Kanade–Russell conjecture (1.5) corresponds to the vertex labelled k.

Proof of Conjecture 1.1. In view of the discussion regarding $\Phi_{n,m}(q,q;q)$ at the beginning of this section, if in Corollary 4.5 we restrict n,m to nonnegative integers and replace $y\mapsto ty$ for $y\in Q$ where t is an integer greater than 1, then the resulting transformation can be written as

$$\sum_{r=0}^{n}\sum_{s=0}^{m}\mathcal{K}_{n,m;r,s}(q,q;q)\Phi_{r,s;ty}(q,q;q)=\Phi_{n,m;ty}(q,q;q)\prod_{i=1}^{3}q^{t^2\binom{y_i}{2}-tiy_i}. \tag{5.7}$$

Indeed, the summand on the left vanishes unless $r\geqslant\max\{0,ty_1\}$ and $s\geqslant\max\{0,ty_1+ty_2\}$ so that (5.7) is consistent with (4.13). (The transformation (5.7) fails for $n,m\in\mathbb{N}_0$ and $t=1$ when $y=(-1,0,1)$, requiring a lower bound of -1 in the summations over r and s instead of 0.)

Now let a,k be integers such that $a\leqslant k$. (Initially only $k-a$ is required to be a nonnegative integer, but there is no loss of generality in assuming integrality of a and k from

[5] Alternatively, this follows after specialising $N=0$ in (3.4).

the outset.) Then, by a $(k-a)$-fold application of (5.7) with $t=3$, the root identity (5.2) transforms into

$$\sum_{\substack{\lambda\subseteq(n^{k-a})\\ \mu\subseteq(m^{k-a})}} g_{\lambda_{k-a},\mu_{k-a};\tau}(q)\prod_{i=1}^{k-a}\mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(q,q;q) \\ =\sum_{y\in Q}\Phi_{n,m;3y}(q,q;q)\Delta_{3y}(q,q;q)\prod_{i=1}^{3} q^{3(K-3a)\binom{y_i}{2}-(K-3a-3)iy_i}, \tag{5.8}$$

where $\lambda_0:=n$, $\mu_0:=m$, $n,m\in\mathbb{N}_0$ and $K:=3k+3+\tau$. This is the identity represented by the vertex labelled $k-a$ in the diagram on page 22. For later reference we note that by (4.27) the above may also be stated as

$$\frac{1-q^{n+m+1}}{1-q}\sum_{\substack{\lambda\subseteq(n^{k-a})\\ \mu\subseteq(m^{k-a})}} g_{\lambda_{k-a},\mu_{k-a};\tau}(q)\prod_{i=1}^{k-a}\mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(q,q;q) \\ =\sum_{y\in Q}\Phi_{n,m;3y}(1,1;1,1;q,q;q)\prod_{i=1}^{3} q^{3(K-3a)\binom{y_i}{2}-(K-3a-3)iy_i}. \tag{5.9}$$

In the remainder of the proof we will use the shorthand

$$Z_t:=q^{ty_{12}+1}\quad\text{and}\quad W_t:=q^{ty_{23}+1},$$

where t is an integer greater than 1. We then make the simultaneous substitutions

$$(u,v,z,w,y)\mapsto\big(Z_t^{\ell-1},W_t^{\ell-1},q,q,ty\big)$$

in (4.18b) for $n,m\in\mathbb{N}_0$. By

$$\mathcal{K}_{n,m;r,s}(az,bw;q)=a^rb^s\mathcal{K}_{n,m;r,s}(z,w;q), \tag{5.10}$$

for $z=w=q$ and $(a,b)=(1/q,1)$, this yields

$$\sum_{r=0}^{n}\sum_{s=0}^{m} q^{-r}\mathcal{K}_{n,m;r,s}(q,q;q)\Phi_{r,s;ty}\big(Z_t^{\ell-1},W_t^{\ell-1};q,q;q\big) \\ =q^{-ty_3+t^2\sum_{i=1}^3\binom{y_i}{2}}\Phi_{n,m;ty}\big(Z_t^{\ell},W_t^{\ell};q,q;q\big). \tag{5.11}$$

Here we have once again used that for $t\geqslant 2$ the lower bounds on the sums over r and s may be simplified to 0. Using (4.17) to replace $\Phi_{n,m;3y}(q,q;q)$ by $\Phi_{n,m;3y}(1,1;q,q;q)$ in the summand on the right of (5.8), and then applying (5.11) with $t=3$ a total of $a-b$ times, first with $\ell=1$, then $\ell=2$ and so on, we obtain

$$\sum_{\substack{\lambda\subseteq(n^{k-b})\\ \mu\subseteq(m^{k-b})}} g_{\lambda_{k-b},\mu_{k-b};\tau}(q)\prod_{i=1}^{k-b} q^{-\chi(i\leqslant a-b)\lambda_i}\mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(q,q;q) \\ =\sum_{y\in Q}\Phi_{n,m;3y}\big(Z_3^{a-b},W_3^{a-b};q,q;q\big)\Delta_{3y}(q,q;q)\prod_{i=1}^{3} q^{3(K-3b)\binom{y_i}{2}-Kiy_i-3v_iy_i}, \tag{5.12}$$

for integers $b \leqslant a \leqslant k$ and $n, m \in \mathbb{N}_0$, where $\nu := (a+b+2, b+1, 0)$. To express the summand on the right in terms of the partition ν we have used that $(a-b)y_3 - a\sum_{i=1}^3 iy_i = \sum_{i=1}^3 (\nu_i + i)y_i$ for $y \in Q$. The identity (5.12) is represented by the right-most vertex labelled $k-b$ in the diagram on page 22, and by abuse of notation will be denoted in the following as I_a. Similarly, the identity corresponding to the left-most vertex labelled $k-b$ in the diagram on page 22 will be denoted by I_{a-1}, since is follows from I_a by the substitution $a \mapsto a-1$. It then follows from Corollary 4.10 with

$$(u, v, z, w, y) \mapsto \big(Z_3^{a-b}, W_3^{a-b}, q, q, 3y\big)$$

that $(I_a - q^{m+1} I_{a-1})/(1-q)$ is given by

$$\sum_{\substack{\lambda \subseteq (n^{k-b}) \\ \mu \subseteq (m^{k-b})}} g_{\lambda_{k-b},\mu_{k-b};\tau}(q)\, \frac{1-q^{m+\lambda_{a-b}+1}}{1-q} \prod_{i=1}^{k-b} q^{-\chi(i \leqslant a-b)\lambda_i} \mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(q,q;q) \tag{5.13}$$

$$= \sum_{y \in Q} \Phi_{n,m;3y}\big(Z_3^{a-b}, W_3^{a-b}; 1, 1; q, q; q\big) \prod_{i=1}^{3} q^{3(K-3b)\binom{y_i}{2} - Kiy_i - 3\nu_i y_i}.$$

Since this is a linear combination of I_a and I_{a-1}, we should now restrict the parameters to $b < a \leqslant k$. However, since $\lambda_0 := n$, the identity (5.13) for $b = a$ simplifies to (5.9). Hence (5.13), which corresponds to the central vertex in the encircled region of the diagram on page 22, holds for all $b \leqslant a \leqslant k$.

In our third and final application of the A_2 Bailey tree, we carry out the substitutions

$$(u, v, c, d, z, w, y) \mapsto \big(uZ_t^{\ell-1}, vW_t^{\ell-1}, Z_t^{\ell-1}, W_t^{\ell-1}, q, q, ty\big)$$

in Corollary 4.11. By (5.10) for $z = w = q$ and $a = b = 1/q$, this gives

$$\sum_{r=0}^{n} \sum_{s=0}^{m} q^{-r-s} \mathcal{K}_{n,m;r,s}(q,q;q) \Phi_{r,s;ty}\big(uZ_t^{\ell-1}, vW_t^{\ell-1}; Z_t^{\ell-1}, W_t^{\ell-1}; q, q; q\big)$$

$$= q^{t^2 \sum_{i=1}^3 \binom{y_i}{2}} \Phi_{n,m;ty}\big(uZ_t^{\ell}, vW_t^{\ell}; Z_t^{\ell}, W_t^{\ell}; q, q; q\big),$$

for $n, m \in \mathbb{N}_0$ and $t \geqslant 2$. This transformation is applied to (5.13) a total of b times, with t, u, v fixed as

$$(t, u, v) = \big(3, Z_3^{a-b}, W_3^{a-b}\big),$$

and $\ell = 1$ in the first application, $\ell = 2$ in the second application and so on. As a result,

$$\sum_{\substack{\lambda \subseteq (n^{k}) \\ \mu \subseteq (m^{k})}} g_{\lambda_k,\mu_k;\tau}(q)\, \frac{1-q^{\lambda_a+\mu_b+1}}{1-q} \prod_{i=1}^{k} q^{-\chi(i \leqslant a)\lambda_i - \chi(i \leqslant b)\mu_i} \mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(q,q;q) \tag{5.14}$$

$$= \sum_{y \in Q} \Phi_{n,m;3y}\big(Z_3^{a}, W_3^{a}; Z_3^{b}, W_3^{b}; q, q; q\big) \prod_{i=1}^{3} q^{3K\binom{y_i}{2} - Kiy_i - 3\nu_i y_i},$$

which is a rational function analogue of (1.5), and corresponds to the vertex labelled k in the diagram on page 22. Although it suffices to prove (1.5) for $0 \leqslant b \leqslant a \leqslant k$, we note that the a,b-symmetry that is manifest in (1.5) is also satisfied by (5.14) thanks to (4.20). Hence (5.14) holds for all $0 \leqslant a, b, \leqslant k$. Specifically, from (4.20) the a,b-symmetry follows by making the simultaneous substitutions $(a,b,n,m) \mapsto (b,a,m,n)$ (so that $v = (a+b+2, b+1, 0) \mapsto (a+b+2, a+1, 0)$) and by then changing the summation indices $(\lambda,\mu) \mapsto (\mu,\lambda)$ on the left and $(y_1,y_2,y_3) \mapsto (-y_3,-y_2,-y_1)$ on the right.

It remains to be shown that (1.5) simplifies to the Kanade–Russell conjecture in the large-n,m limit. By (4.26) with $(y,x_i) \mapsto (ty, q^{-i})$ (so that $(z,w) \mapsto (q,q)$),

$$\lim_{n,m\to\infty} \Phi_{n,m;ty}\big(Z_t^a, W_t^a; Z_t^b, W_t^b; q, q; q\big) = \frac{1}{(q;q)_\infty^5} \det_{1\leqslant i,j\leqslant 3}\Big(q^{(ty_i-i)(v_i-v_j)}\Big).$$

The limit of (5.14) is thus given by

$$\sum_{\substack{\lambda,\mu\in\mathscr{P}\\ l(\lambda),\lambda(\mu)\leqslant k}} \frac{1-q^{\lambda_a+\mu_b+1}}{1-q}\, \frac{\prod_{i=1}^k q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2+\chi(i>a)\lambda_i+\chi(i>b)\mu_i}}{\prod_{i=1}^{k-1}(q;q)_{\lambda_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}}\, g_{\lambda_k,\mu_k;\tau}(q) \tag{5.15}$$
$$= \frac{1}{(q;q)_\infty^3} \sum_{y\in Q} \det_{1\leqslant i,j\leqslant 3}\Big(q^{3K\binom{y_i}{2}-Kiy_i-(3y_i+j-i)v_j}\Big),$$

where $v := (a+b+2, b+1, 0)$, as before. The remaining task of writing the right-hand side in product-form can easily be carried out for arbitrary rank, and in the following we consider

$$A_{v;k}(q) := \sum_{y\in Q} \det_{1\leqslant i,j\leqslant r}\Big(q^{rK\binom{y_i}{2}-Kiy_i-(ry_i+j-i)v_j}\Big),$$

for $v = (v_1,\dots,v_r)$. First we write $A_{v;k}(q)$ as a constant term and then appeal to multilinearity. Thus

$$\begin{aligned}A_{v;k}(q) &= [z^0] \sum_{y\in\mathbb{Z}^r} \det_{1\leqslant i,j\leqslant r}\Big(z^{y_i} q^{rk\binom{y_i}{2}-kiy_i+(ry_i+j-i)(j-v_j)}\Big)\\ &= [z^0] \det_{1\leqslant i,j\leqslant r}\bigg(\sum_{y\in\mathbb{Z}} z^y q^{rk\binom{y}{2}-kiy-(ry+j-i)v_j}\bigg).\end{aligned}$$

Interchanging rows and columns (i.e., replacing $(i,j) \mapsto (j,i)$), negating y and using the fact that we are taking the constant term with respect to z, this leads to

$$\begin{aligned}A_{v;k}(q) &= [z^0] \det_{1\leqslant i,j\leqslant r}\bigg(\sum_{y\in\mathbb{Z}} z^y q^{rk\binom{y}{2}+kiy+ryv_i+(j-i)(ky+v_i)}\bigg)\\ &= \sum_{y\in Q} \det_{1\leqslant i,j\leqslant r}\Big(q^{(j-i)(ky_i+v_i)}\Big) \prod_{i=1}^r q^{rk\binom{y_i}{2}+kiy_i+rv_iy_i}.\end{aligned}$$

Applying the Vandermonde determinant

$$\det_{1\leqslant i,j\leqslant r}\big(x_i^{j-i}\big) = \prod_{1\leqslant i<j\leqslant r}(1-x_i/x_j)$$

this gives

$$A_{\nu;k}(q)=\sum_{y\in Q}\prod_{i=1}^{r} q^{rk\binom{y_i}{2}+kiy_i+r\nu_i y_i}\prod_{1\leqslant i<j\leqslant r}\big(1-q^{ky_{ij}+\nu_i-\nu_j}\big).$$

By the $A_{r-1}^{(1)}$ Macdonald identity [62]

$$\sum_{y\in Q}\prod_{i=1}^{r} x_i^{ry_i} q^{r\binom{y_i}{2}+iy_i}\prod_{1\leqslant i<j\leqslant r}\big(1-q^{y_{ij}}x_i/x_j\big)=(q;q)_\infty^{r-1}\prod_{1\leqslant i<j\leqslant r}\theta(x_i/x_j;q) \tag{5.16}$$

with $(q,x_i)\mapsto(q^k,q^{\nu_i})$, this results in the product form

$$A_{\nu;k}(q)=(q^k;q^k)_\infty^{r-1}\prod_{1\leqslant i<j\leqslant r}\theta\big(q^{\nu_i-\nu_j};q^k\big).$$

Taking $r=3$, $\nu=(a+b+2,b+1,0)$ and $k=K$, yields

$$\frac{(q^K;q^K)_\infty^2}{(q;q)_\infty^3}\prod_{1\leqslant i<j\leqslant 3}\theta\big(q^{a+1},q^{b+1},q^{a+b+2};q^K\big)$$

for the right-hand side of (5.15). ∎

6. Below-the-line identities

As in Conjecture 1.1, fix the modulus K as $K=3k+\tau+3$ for k a nonnegative integer and $\tau\in\{-1,0,1\}$. In the introduction immediately preceding the conjecture, we remarked that there should be an ASW-type identity for all nonnegative integers a,b such that $a+b\leqslant K-3$, with product side given by[6]

$$\frac{(q^K;q^K)_\infty^2}{(q;q)_\infty^3}\,\theta\big(q^{a+1},q^{b+1},q^{a+b+2};q^K\big),$$

Without loss of generality assuming that

$$0\leqslant b\leqslant a\leqslant K-a-b-3, \tag{6.1}$$

this corresponds to

$$\binom{k+2}{2}-\delta_{\tau,-1}+\Big\lfloor\frac{(k+\tau)^2}{4}\Big\rfloor$$

distinct ASW-type identities. Hence in the Kanade–Russell conjecture roughly one third of all cases is missing, counted by the above floor function. In their paper, Kanade and Russell adopt a certain diagrammatic arrangement for the triples $(K-a-b-3,a,b)$

[6]For $\tau=-1$ this rules out $(a,b)=(k,k)$, which as discussed in the introduction gives the same product as $(a,b)=(k,k-1)$ albeit a slightly different multisum according to (1.5).

with fixed K, leading them to refer to the missing identities as the 'below-the-line' cases. Equivalently, this corresponds to (6.1) with $a > k$ (and thus $b \leqslant k+\tau-2$). If $k=1$ this forces $\tau=1$, in which case there is the single below-the-line solution: $(a,b)=(2,0)$. By solving the modulus-7 Corteel–Welsh equations [28], Kanade and Russell found the missing multisum, resulting in

$$\sum_{\lambda_1,\mu_1=0}^{\infty} \frac{1-q^{2\lambda_1-\mu_1}}{1-q} \frac{q^{\lambda_1^2-\lambda_1\mu_1+\mu_1^2-\lambda_1+\mu_1}}{(q;q)_{\lambda_1}(q;q)_{\mu_1}(q^2;q)_{\lambda_1+\mu_1}} = \frac{(q^7;q^7)_\infty}{(q;q)_\infty^3}\,\theta(q,q^3,q^3;q^7).$$

In general, however, no explicit such multisum-forms for below-the-line values of a and b are known. The exception is $\tau=0$, in which case Kanade and Russell observed that if

$$\Theta_{a,b;k}(q) := \theta(q^{a+1},q^{b+1},q^{a+b+2};q^{3k+3}),$$

then Weierstrass' three-term relation [40, page 61] implies,

$$\Theta_{a,b;k}(q) = \Theta_{2k-a,a+b-k;k} - q^{b+1}\Theta_{2k-a-b-1,a-k-1;k}(q).$$

Importantly, for $\tau=0$ and fixed $k\geqslant 2$, the below-the-line values of (a,b) satisfy $k<a\leqslant \lfloor 3k/2\rfloor$ and $0\leqslant b\leqslant 3k-2a$. Assuming such a,b and defining $(a',b') := (2k-a,a+b-k)$ and $(a'',b'') := (2k-a-b-1,a-k-1)$, it follows that $0<b'\leqslant a'\leqslant \lceil k/2\rceil$ and $0\leqslant b''\leqslant a'\leqslant k-2$. This implies the following theorem covering all of the below-the-line cases. For integers a,b,k such that $0\leqslant a,b\leqslant k$, let

$$\mathcal{F}_{a,b;k}(q) := \sum_{\substack{\lambda_1\geqslant\cdots\geqslant\lambda_k\geqslant 0\\ \mu_1\geqslant\cdots\geqslant\mu_k\geqslant 0}} \frac{1-q^{\lambda_a+\mu_b+1}}{1-q} \frac{q^{\sum_{i=1}^k(\lambda_i^2-\lambda_i\mu_i+\mu_i^2)+\sum_{i=a+1}^k\lambda_i+\sum_{i=b+1}^k\mu_i}}{\prod_{i=1}^{k-1}(q;q)_{\lambda_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}}\, g_{\lambda_k,\mu_k;0}(q),$$

where $q^{\lambda_0}=q^{\mu_0}:=0$.

Theorem 6.1. *Let a,b,k be integers such that $2\leqslant k<a\leqslant\lfloor 3k/2\rfloor$ and $0\leqslant b\leqslant 3k-2a$. Then*

$$\mathcal{F}_{2k-a,a+b-k;k}(q) - q^{b+1}\mathcal{F}_{2k-a-b-1,a-k-1;k}(q) = \frac{(q^K;q^K)_\infty^2}{(q;q)_\infty^3}\,\theta(q^{a+1},q^{b+1},q^{a+b+2};q^K),$$

where $K:=3k+3$.

This was first stated in [51] as a conditional result, depending on the validity of Conjecture 1.1. By Theorem 1.2 the result is now unconditional. It remains an open problem to express the left-hand side in manifestly positive form.

7. Character identities for the $\mathcal{W}_3(3,K)$ vertex operator algebra

As explained in full detail in [90, Section 4], for $\tau\neq 0$ (so that $3\nmid K$) the q-series in (1.5) multiplied by $q^{h-c/24}(q;q)_\infty$ are characters $\chi_{a,b}^K(q)$ of the $\mathcal{W}_3(3,K)$ vertex operator algebra [30, 91] of central charge

$$c = -\frac{2(K-4)(4K-9)}{K} \tag{7.1}$$

and conformal weight

$$h_{a,b} = \frac{a^2+ab+b^2-(K-3)(a+b)}{K}.$$

That is,

$$\chi^K_{a,b}(q) = q^{h_{a,b}-c/24}\,\frac{(q^K;q^K)^2_\infty}{(q;q)^2_\infty}\,\theta\big(q^{a+1},q^{b+1},q^{a+b+2};q^K\big),$$

where a,b,K are nonnegative integers such that $K \geqslant 5$, $3 \nmid K$ and $a+b \leqslant K-3$. To obtain a multisum expression for these characters without an overall factor $(q;q)_\infty$, we need to carry out a suitable rewriting of the multisum in (1.5). This is possible by means of the next lemma, which is a limiting case of [90, Lemma 7.2].

Lemma 7.1. *For k a positive integer, m a nonnegative integer and $u=(u_1,\dots,u_{k+1}) \in \mathbb{Z}^{k+1}$ define*

$$\mathcal{F}_u(q) := \sum_{\mu_1\geqslant\cdots\geqslant\mu_k\geqslant 0} \frac{q^{\sum_{i=1}^k \mu_i(\mu_i+u_i)}}{(q)_{\mu_k+u_{k+1}}\prod_{i=1}^k (q;q)_{\mu_i-\mu_{i+1}}},$$

where $\mu_{k+1} := 0$. If

$$u_1 \leqslant u_2 \leqslant \cdots \leqslant u_{k+1},$$

then

$$\mathcal{F}_u(q) = \frac{1}{(q;q)_\infty}\sum_{\mu_1,\dots,\mu_k\geqslant 0} q^{\sum_{i=1}^k \mu_i(\mu_i+u_i)} \prod_{i=1}^k \begin{bmatrix}\mu_{i+1}+u_{i+1}-u_i\\ \mu_i\end{bmatrix}, \tag{7.2}$$

where, again, $\mu_{k+1} := 0$.

The left-hand side of (1.5) for $\tau \neq 0$ may be expressed in terms of $\mathcal{F}_u$ as

$$\sum_{\lambda_1\geqslant\cdots\geqslant\lambda_k\geqslant 0} \frac{q^{\sum_{i=1}^k \lambda_i^2+\sum_{i=a+1}^k\lambda_i}}{\prod_{i=1}^k (q;q)_{\lambda_i-\lambda_{i+1}}}\,\mathcal{F}_u(q) - \chi(ab>0)\sum_{\lambda_1\geqslant\cdots\geqslant\lambda_k\geqslant 0} \frac{q^{1+\sum_{i=1}^k \lambda_i^2+\sum_{i=a}^k\lambda_i}}{\prod_{i=1}^k (q;q)_{\lambda_i-\lambda_{i+1}}}\,\mathcal{F}_v(q),$$

where $\lambda_{k+1} := 0$,

$$u_i = \begin{cases} \chi(i>b)-\lambda_i & \text{for } 1\leqslant i<k,\\ \chi(k>b)-\tau\lambda_k & \text{for } i=k,\\ 1+\lambda_k & \text{for } i=k+1,\end{cases} \quad\text{and}\quad v_i = \begin{cases} \chi(i\geqslant b)-\sigma_i\lambda_i & \text{for } 1\leqslant i<k,\\ 1-\tau\lambda_k & \text{for } i=k,\\ 1+\lambda_k & \text{for } i=k+1.\end{cases}$$

Since for $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ the inequalities $u_i \leqslant u_{i+1}$ and $v_i \leqslant v_{i+1}$ hold for all $1 \leqslant i \leqslant k$, we may use the alternative expressions for $\mathcal{F}_u(q)$ and $\mathcal{F}_v(q)$ provided by (7.2). First, for $\tau = 1$, this yields our next theorem, where $\tilde{\chi}^K_{a,b}(q) := q^{c/24-h_{a,b}}\chi^K_{a,b}(q)$.

Theorem 7.2 ($\mathrm{A}_2^{(1)}$ Andrews–Gordon identities, I). *Let $K = 3k + 4$ for $k \geqslant 1$. Then*

$$\begin{aligned}
&\tilde{\chi}^K_{a,b}(q) \\
&= \sum_{\substack{\lambda_1,\dots,\lambda_k\geqslant 0\\ \mu_1,\dots,\mu_k\geqslant 0}} \frac{q^{\sum_{i=a+1}^k \lambda_i+\sum_{i=b+1}^k \mu_i}}{(q;q)_{\lambda_1}} \prod_{i=1}^k q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2} \begin{bmatrix} \lambda_i \\ \lambda_{i+1}\end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1}+\delta_{b,i} \\ \mu_i\end{bmatrix} \\
&\quad - \sum_{\substack{\lambda_1,\dots,\lambda_k\geqslant 0\\ \mu_1,\dots,\mu_k\geqslant 0}} \frac{q^{1+\sum_{i=a}^k \lambda_i+\sum_{i=b}^k \mu_i}}{(q;q)_{\lambda_1}} \prod_{i=1}^k q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2} \begin{bmatrix} \lambda_i \\ \lambda_{i+1}\end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1}+\delta_{b-1,i} \\ \mu_i\end{bmatrix}
\end{aligned}$$

for all $0 \leqslant a, b \leqslant k$, and

$$\tilde{\chi}^K_{k,k}(q) = \sum_{\substack{\lambda_1,\dots,\lambda_k\geqslant 0\\ \mu_1,\dots,\mu_k\geqslant 0}} \frac{1}{(q;q)_{\lambda_1}} \prod_{i=1}^k q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2} \begin{bmatrix} \lambda_i \\ \lambda_{i+1}\end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1} \\ \mu_i\end{bmatrix},$$

where $q^{\lambda_0} = q^{\mu_0} = \lambda_{k+1} := 0$ and $\mu_{k+1} := \lambda_k$.

The second, simpler expression for $\tilde{\chi}^K_{k,k}(q)$ follows by either noting that for $a = b = k$, the left-hand side of (1.5) for $\tau \neq 0$ may alternatively be recognised as

$$\sum_{\lambda_1\geqslant\cdots\geqslant\lambda_k\geqslant 0} \frac{\mathcal{F}_w(q)}{\prod_{i=1}^k (q;q)_{\lambda_i-\lambda_{i+1}}},$$

where $\lambda_{k+1} := 0$ and

$$w_i = \begin{cases} -\lambda_i & \text{for } 1 \leqslant i < k, \\ -\tau\lambda_k & \text{for } i = k, \\ \lambda_k & \text{for } k+1, \end{cases}$$

or by substituting $a = b = k$ in the expression for $\tilde{\chi}^K_{a,b}(q)$, replacing $\mu_k \mapsto \mu_k - 1$ in the second multisum and then combining the two multisums using the standard recursion for the q-binomial coefficient. The $b = 0$ case of Theorem 7.2 proves [90, Conjecture 2.8] and the $a = 0$ case proves Equation (2.7) of that same paper. Since $\tilde{\chi}^K_{a,b}(q) = \tilde{\chi}^K_{b,a}(q)$ while the right-hand side of the first character formula does not have a, b-symmetry, there are two distinct expressions for each $\mathcal{W}_3(3, K)$ character $\chi^K_{a,b}(q)$ such that $a \neq b$. The reason for viewing the above as analogues of the Andrews–Gordon identities (1.1) is that in much the same way the latter are known to be identities for characters of the Virasoro algebra $\mathrm{Vir}(2, K) = \mathcal{W}_2(2, K)$.

For $\tau = -1$ we obtain the following companion to the previous theorem.

Theorem 7.3 ($\mathrm{A}_2^{(1)}$ Andrews–Gordon identities, II). *Let $K = 3k+2$ for $k \geqslant 1$ and $0 \leqslant a \leqslant k$, $0 \leqslant b < k$. Then*

$$\begin{aligned}
&\tilde{\chi}^K_{a,b}(q) \\
&= \sum_{\substack{\lambda_1,\dots,\lambda_k \geqslant 0 \\ \mu_1,\dots,\mu_{k-1}\geqslant 0}} \frac{q^{\lambda_k^2+\sum_{i=a+1}^k \lambda_i + \sum_{i=b+1}^{k-1}\mu_i}}{(q;q)_{\lambda_1}} \prod_{i=1}^{k-1} q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2} \begin{bmatrix} \lambda_i \\ \lambda_{i+1}\end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1}+\delta_{b,i} \\ \mu_i \end{bmatrix} \\
&\quad - \sum_{\substack{\lambda_1,\dots,\lambda_k \geqslant 0 \\ \mu_1,\dots,\mu_{k-1}\geqslant 0}} \frac{q^{1+\lambda_k^2+\sum_{i=a}^k \lambda_i + \sum_{i=b}^{k-1}\mu_i}}{(q;q)_{\lambda_1}} \prod_{i=1}^{k-1} q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2} \begin{bmatrix} \lambda_i \\ \lambda_{i+1}\end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1}+\delta_{b-1,i} \\ \mu_i \end{bmatrix}
\end{aligned}$$

for $0 \leqslant a \leqslant k$, $0 \leqslant b < k$, and

$$\tilde{\chi}^K_{k,k}(q) = \sum_{\substack{\lambda_1,\dots,\lambda_k \geqslant 0 \\ \mu_1,\dots,\mu_{k-1}\geqslant 0}} \frac{q^{\lambda_k^2}}{(q;q)_{\lambda_1}} \prod_{i=1}^{k-1} q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2} \begin{bmatrix} \lambda_i \\ \lambda_{i+1}\end{bmatrix} \begin{bmatrix} \lambda_i-\lambda_{i+1}+\mu_{i+1} \\ \mu_i \end{bmatrix},$$

where $q^{\lambda_0} = q^{\mu_0} = \lambda_{k+1} := 0$ and $\mu_k := 2\lambda_k$.

This time the $b=0$ case proves [90, Conjecture 2.1] and the $a=0$ case proves Equation (2.2) of [90].

For a number of special values of k, alternative multisum expressions to those of Theorems 7.2 and 7.3 are known. In [27], Corteel, Dousse and Uncu solved the Corteel–Welsh system of equations for the two-variable generating function of three-row cylindric partitions with profile $(5-a-b, a, b)$, resulting in quadruple-sum expressions for the characters $\tilde{\chi}^8_{a,b}(q)$. For example (see [27, Theorem 1.6]),

$$\tilde{\chi}^8_{2,1}(q) = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2\end{bmatrix} \begin{bmatrix} n_1 \\ n_4\end{bmatrix} \begin{bmatrix} n_2 \\ n_3\end{bmatrix}.$$

In [32, Theorems 2.3 & 2.4], Feigin, Foda and Welsh obtained an Andrews–Gordon-type theorem for a linear combination of characters of $\mathrm{Vir}(3,3k+2)$ of central charge $c = -3k(6k-5)/(3k+2)$. For $k=4$ this yields $c=-114/7$, which coincides with the central charge of $\mathcal{W}_3(3,7)$. In this case, four of the six linear combinations considered in [32] correspond to actual $\mathcal{W}_3(3,7)$ characters. Three are also covered in Theorem 7.2 while the fourth is below-the-line in the sense of Kanade and Russell. For example, the character expression for $\tilde{\chi}^7_{1,1}(q)$ arising from $\mathrm{Vir}(3,14)$ is [32, Equation (20c)]

$$\tilde{\chi}^7_{1,1}(q) = \sum_{n_1,n_2,n_3,n_3=0}^{\infty} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2+(n_1+n_2+n_3)n_4}}{(q;q)_{n_1}(q;q)_{n_4}} \begin{bmatrix} n_1 \\ n_2\end{bmatrix} \begin{bmatrix} n_2 \\ n_3\end{bmatrix}.$$

After the substitutions

$$(n_1,n_2,n_3,n_4) \mapsto (n_1+n_3+n_4, n_3+n_4, n_4, n_2)$$

this takes the form

$$\tilde{\chi}^{7}_{1,1}(q)=\sum_{n_1,n_2,n_3,n_3=0}^{\infty}\frac{q^{\sum_{i,j=1}^{4} n_iA_{ij}n_j}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}(q;q)_{n_4}}, \tag{7.3}$$

where

$$(A_{ij})=\frac{1}{2}\begin{pmatrix}2&1&2&2\\1&2&2&3\\2&2&4&4\\2&3&4&6\end{pmatrix}.$$

At a workshop on cylindric partitions held at RICAM in 2022, Shunsuke Tsuchioka raised the question if all the $\mathrm{A}_2^{(1)}$ Andrews–Gordon identities admit alternative sum-sides of the form (7.3). Such expressions would be closer to the $\mathrm{A}_1^{(1)}$ Andrews–Bressoud–Gordon identities, where the variable change $n_i\mapsto n_i+\cdots+n_k$ for all $1\leqslant i\leqslant k$ leads to the multisum

$$\sum_{n_1,\dots,n_k\geqslant 0}\frac{q^{\sum_{i,j=1}^{k} n_iA_{ij}n_j+\sum_{i=1}^{k}(A_{ki}-A_{ai})n_i}}{(q;q)_{n_1}\dots(q;q)_{n_{k-1}}(q^{2-\tau};q^{2-\tau})_{n_k}},$$

where $(A_{ij})_{i,j=1}^{k}=(\min\{i,j\})_{i,j=1}^{k}$ is the Cartan-type matrix of the tadpole graph on k vertices. As further evidence that such a rewriting might exist for all moduli, he made a conjecture for modulus 8, complementing his own proven modulus-6 identities [83], such as

$$\begin{aligned}&\sum_{n_1^{(1)},n_2^{(1)},n_1^{(2)},n_2^{(2)}=0}^{\infty}\frac{q^{\sum_{i,j,a,b=1}^{2}A_{ia,jb}n_i^{(a)}n_j^{(b)}}}{\prod_{i,a=1}^{2}(q;q)_{n_i^{(a)}}}\\&\quad=\sum_{n,m,k,l=0}^{\infty}\frac{q^{n^2+3kn+3k^2}}{(q;q)_n(q^3;q^3)_k}\begin{bmatrix}n\\m\end{bmatrix}\begin{bmatrix}k\\l\end{bmatrix}_{q^3}=(-q;q)_\infty^2(q^2,q^4;q^6)_\infty,\end{aligned}$$

where $A=\frac{1}{2}B\otimes C$ (i.e., $A_{ia,jb}=\frac{1}{2}B_{ij}C_{ab}$) with matrices B and C given by $B=\left(\begin{smallmatrix}2&3\\3&6\end{smallmatrix}\right)$ and $C=\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. From the structure of the summands in Theorems 7.2 and 7.3 it follows relatively straightforwardly that a rewriting of the form (7.3) can be carried out for the moduli 7 and 8. For larger moduli, however, this simple method fails due to the form of the summands. By iterating the Durfee rectangle identity [7, Equation (3.3.10)]

$$\begin{bmatrix}n+m\\n+a\end{bmatrix}=\sum_{k=0}^{n}q^{k(k+a)}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k+a\end{bmatrix} \tag{7.4}$$

for $n,m\in\mathbb{N}_0$ and $a\in\mathbb{Z}$, it follows that the q-binomial coefficient admits the telescopic expansion

$$\begin{bmatrix}k_0+m\\k_0+a\end{bmatrix}=\sum_{k_0\geqslant k_1\geqslant k_2\geqslant\dots\geqslant k_r\geqslant 0}\begin{bmatrix}k_0+m-\sum_{i=0}^{r-1}k_i\\k_r+a\end{bmatrix}\prod_{i=1}^{r}q^{k_i(k_i+a)}\begin{bmatrix}k_{i-1}\\k_i\end{bmatrix}, \tag{7.5}$$

for arbitrary nonnegative integer r and integers a, k_0, m such that $k_0, m \geqslant 0$ and, if $a = -k_0$, then $m \geqslant (r-1)k_0$. If we take $r = 2$ and once more apply (7.4) with $(n, m, a) \mapsto (k_0 - k_1, m - k_0, k_1 + k_2 + a - k_0)$, this implies

$$\frac{1}{(q;q)_{m-k_0}(q;q)_{k_0}} \begin{bmatrix} k_0+m \\ k_0+a \end{bmatrix} :=$$
$$\sum_{k_1,k_2,k_3} \frac{q^{\sum_{i=1}^{3} k_i(k_i+a)+(k_1+k_2-k_0)k_3}}{(q;q)_{k_1-k_2}(q;q)_{k_2}(q;q)_{k_3}(q;q)_{k_0-k_1-k_3}(q;q)_{a+k_1+k_2+k_3-k_0}(q;q)_{m-a-k_1-k_2-k_3}},$$

for all integers a, k_0, m such that $0 \leqslant k_0 \leqslant m$. Since

$$\tilde{\chi}^{7}_{1,1}(q) = \sum_{\lambda_1,\mu_1} \frac{q^{\lambda_1^2-\lambda_1\mu_1+\mu_1^2}}{(q;q)_{\lambda_1}} \begin{bmatrix} 2\lambda_1 \\ \mu_1 \end{bmatrix}$$

and

$$\tilde{\chi}^{8}_{2,2}(q) = \sum_{\lambda_1,\lambda_2,\mu_1} \frac{q^{\lambda_1^2-\lambda_1\mu_1+\mu_1^2+\lambda_2^2}}{(q;q)_{\lambda_1-\lambda_2}(q;q)_{\lambda_2}} \begin{bmatrix} \lambda_1+\lambda_2 \\ \mu_1 \end{bmatrix},$$

we can use the above expansion with (m, k_0, a) given by $(\lambda_1, \lambda_1, \lambda_1 - \mu_1)$ and $(\lambda_1, \lambda_2, \lambda_1 - \mu_1)$ respectively. In the first case this fixes k_3 as $k_3 = \mu_1 - k_1 - k_2$. Finally, making the substitutions

$$(\lambda_1, \mu_1, k_1, k_2) \mapsto (n_1+n_2+n_3+n_4, n_2+n_3+2n_4, n_3+n_4, n_4)$$

and

$$\begin{aligned}&(\lambda_1, \mu_1, \lambda_2, k_1, k_2, k_3) \\ &\quad \mapsto (n_1+n_2+n_3+n_4+n_5+n_6, n_2+n_4+n_5+2n_6, n_3+n_4+n_5+n_6, n_5+n_6, n_6, n_4)\end{aligned}$$

yields, respectively, (7.3) and

$$\tilde{\chi}^{8}_{2,2}(q) = \sum_{n_1,\dots,n_6=0}^{\infty} \frac{q^{\sum_{i,j=1}^{6} n_i A_{ij} n_i}}{(q;q)_{n_1}\cdots(q;q)_{n_6}}, \tag{7.6}$$

for

$$(A_{ij}) = \frac{1}{2}\begin{pmatrix} 2&1&2&2&2&2 \\ 1&2&1&2&2&3 \\ 2&1&4&3&4&4 \\ 2&2&3&4&4&5 \\ 2&2&4&4&6&6 \\ 2&3&4&5&6&8 \end{pmatrix}.$$

This last result is exactly one of formulas for $\tilde{\chi}^{8}_{a,b}$ conjectured by Tsuchioka [84].

8. Character formulas for principal subspaces of $A_2^{(1)}$

Let $\mathfrak{g} = \mathfrak{sl}_r = \mathrm{A}_{r-1}$ and $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_r = \mathrm{A}_{r-1}^{(1)}$ its untwisted affinisation, i.e.,

$$\hat{\mathfrak{g}} \cong \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is the canonical central element and d a derivation, acting on the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$ as $t\frac{\mathrm{d}}{\mathrm{d}t}$, see [50, Chapter 7] for details. Fix $I := \{0,1,\dots,r-1\}$ and let $\hat{\mathfrak{h}}$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$ with basis $\{\alpha_0^\vee,\dots,\alpha_{r-1}^\vee,d\}$, where the $\alpha_i^\vee$ $(i\in I)$ are the simple coroots (so that $c=\sum_{i\in I}\alpha_i^\vee$). Let $A=(a_{ij})_{i,j=0}^{r-1}$ be the (generalised) Cartan matrix of $\hat{\mathfrak{g}}$, and fix the non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on $\hat{\mathfrak{h}}$ by setting $(\alpha_i^\vee|\alpha_j^\vee)=a_{ij}$, $(d|d)=0$, $(\alpha_0^\vee|d)=1$ and $(\alpha_i^\vee|d)=0$ otherwise. Further let $\hat{\mathfrak{h}}^*$ be the dual of the Cartan subalgebra with basis $\{\alpha_0,\dots,\alpha_{r-1},\Lambda_0\}$, where the α_i $(i\in I)$ are the simple roots and Λ_0 is the 0th fundamental weight. Denote the standard pairing between the Cartan subalgebra and its dual by $\langle\cdot,\cdot\rangle$, so that $\langle\alpha_i,\alpha_j^\vee\rangle=(\alpha_i^\vee|\alpha_j^\vee)=a_{ij}$ and $\langle\Lambda_0,\alpha_i^\vee\rangle=0$. The additional fundamental weights $\Lambda_1,\dots,\Lambda_{r-1}\in\hat{\mathfrak{h}}^*$ are fixed as $\langle\Lambda_i,\alpha_j^\vee\rangle=\delta_{ij}$ for all $i,j\in I$ and $\langle\Lambda_i,d\rangle=0$ for all $i\in I$. The level of $\lambda\in\hat{\mathfrak{h}}^*$ is defined by $\mathrm{lev}(\lambda):=\langle\lambda,c\rangle$. Hence $\mathrm{lev}(\Lambda_i)=1$ for all $i\in I$ and if $\delta:=\sum_{i\in I}\alpha_i$ is the null root, then $\mathrm{lev}(\delta)=\sum_{i,j\in I}a_{ij}=0$. Finally, let

$$P := \big\{\lambda\in\mathfrak{h}^* : \langle\lambda,\alpha_i^\vee\rangle\in\mathbb{Z} \text{ for all } i\in I\big\}$$

be the weight lattice of $\hat{\mathfrak{g}}$, and $P_+\subset P$ and $P_+^\ell\subset P_+$ the set of dominant integral weights and level-ℓ dominant integral weights respectively:

$$\begin{aligned} P_+ &= \big\{\lambda\in\mathfrak{h}^* : \langle\lambda,\alpha_i^\vee\rangle\in\mathbb{N}_0 \text{ for all } i\in I\big\} = \mathbb{N}_0\Lambda_0+\cdots+\mathbb{N}_0\Lambda_{r-1}+\mathbb{C}\delta,\\ P_+^\ell &= \big\{\lambda\in P_+ : \mathrm{lev}(\lambda)=\ell\big\}. \end{aligned}$$

A much studied class of representations of $\mathrm{A}_{r-1}^{(1)}$ are the standard or integrable highest weight modules. There is a unique such module, L_λ, for each $\lambda\in P_+ \mod \mathbb{C}\delta$. If v_λ denotes the highest weight vector of L_λ, then $\hat{\mathfrak{h}}$ acts diagonally on v_λ and $cv_\lambda=\mathrm{lev}(\lambda)v_\lambda$. The principal subspace $W_\lambda\subset L_\lambda$ is defined as [9, 35, 81][7]

$$W_\lambda := U\big(\mathfrak{n}_-\otimes\mathbb{C}[t,t^{-1}]\big)v_\lambda = U\big(\mathfrak{n}_-\otimes\mathbb{C}[t^{-1}]\big)v_\lambda,$$

where $\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ is the triangular or Cartan decomposition of $\mathfrak{g}$ and $U(\cdot)$ denotes the universal enveloping algebra. Let $f_1,\dots,f_{r-1}\in\mathfrak{g}$ denote the standard generators of $\mathfrak{n}_-$. Then the character of the principal subspace W_λ is defined as

$$\mathrm{ch}\, W_\lambda := \sum_{n,d_1,\dots,d_{r-1}\geqslant 0} \dim\big(W_{\lambda;n;d_1,\dots,d_{r-1}}\big)\,\mathrm{e}^{\lambda-\delta n-\sum_{i=1}^{r-1}d_i\alpha_i},$$

[7] There are two related but distinct definitions used in the literature, and here we follow the less standard [9]. In the original paper [81], $U\big(\mathfrak{n}_+\otimes\mathbb{C}[t,t^{-1}]\big)v_\lambda$ is used instead.

where $W_{\lambda;n;d_1,\dots,d_{r-1}} \subset W_\lambda$ is the subspace generated by those elements in $U(\mathfrak{n}_- \otimes \mathbb{C}[t^{-1}])$ of degree d_i in f_i and degree n in t^{-1}. For convenience we in the following use the normalised character

$$\operatorname{ch} W'_\lambda := \mathrm{e}^{-\lambda} \operatorname{ch} W_\lambda .$$

Ardonne, Kedem and Stone [9, Equation (6.9)][8] found an explicit expression for $\operatorname{ch} W_\lambda$ in terms of generalised Kostka polynomials [53,75]. Restricting considerations to $r = 3$, and assuming the parametrisation

$$\lambda = (k - a - b)\Lambda_0 + a\Lambda_1 + b\Lambda_2 \in P_+^k, \tag{8.1}$$

the Ardonne, Kedem and Stone character formula simplifies to [9, Equations (6.9), (6.15) & (6.16)]

$$\operatorname{ch} W'_\lambda := \sum_{\substack{\lambda,\mu\in\mathscr{P} \\ l(\lambda),l(\mu)\leqslant k}} \biggl(\bigl(1 - zwq^{\lambda_a+\mu_b-1}\bigr) \prod_{i=1}^{k} \frac{z^{\lambda_i} w^{\mu_i} q^{\lambda_i^2 - \lambda_i\mu_i + \mu_i^2 - \chi(i\leqslant a)\lambda_i - \chi(i\leqslant b)\mu_i}}{(q;q)_{\lambda_i-\lambda_{i+1}} (q;q)_{\mu_i-\mu_{i+1}}} \biggr), \tag{8.2}$$

where $q^{\lambda_0} = q^{\mu_0} := 0$ and $q := \mathrm{e}^{-\delta}$, $z := \mathrm{e}^{-\alpha_1}$, $w := \mathrm{e}^{-\alpha_2}$. The restrictions $l(\lambda), l(\mu) \leqslant k$ in the sum imply that $\lambda_{k+1} = \mu_{k+1} = 0$. By mild abuse of notation we in the remainder of this section use $\operatorname{ch} W'_\lambda$ to mean the right-hand side of (8.2) for all $0 \leqslant a, b \leqslant k$, despite the fact that for $a + b > k$ the weight λ is not dominant.

In the vacuum case, corresponding to $a = b = 0$, Feigin et al. [31, Corollary 7.8] obtained an alternative 'bosonic' expression for W_λ. This is the $a = b = 0$ case of our next theorem.

Theorem 8.1. *For a, b, k integers such that $0 \leqslant a, b \leqslant k$, let the weight λ and partition ν be given by (8.1) and $\nu = (a + b + 2, b + 1, 0)$ respectively. Then*

$$\operatorname{ch} W'_\lambda = \prod_{1\leqslant i<j\leqslant 3} \frac{1}{(x_i/x_j;q)_\infty} \times \sum_{y\in Q_+} \det_{1\leqslant i,j\leqslant 3} \bigl((x_i q^{y_i})^{\nu_i-\nu_j}\bigr) \prod_{i=1}^{3} \frac{x_i^{(k+2)y_i} q^{(k+2)\binom{y_i}{2} - \nu_i y_i} (x_i/x_3;q)_{y_i}}{(qx_i/x_1;q)_{y_i}}, \tag{8.3}$$

where $x_1/x_2 := \mathrm{e}^{-\alpha_1}$ and $x_2/x_3 := \mathrm{e}^{-\alpha_2}$.

By (8.2) this is Theorem 1.4 of the introduction.

Proof of Theorem 8.1. The main steps of the proof are the same as in the proof of the Kanade–Russell conjecture in Section 5. Key difference is the root identity to which the A_2 Bailey tree is applied, which essentially is the A_2 unit Bailey pair (4.33). Also, since the right-hand side of (8.3) does not admit a product form, this time round there is no need for the $A_2^{(1)}$ Macdonald identity in the final stages of the proof.

[8] For $\lambda = (k - a)\Lambda_0 + a\Lambda_i$, $i \in I$, the dependence on the generalised Kostka polynomials trivialises and the result is essentially due to Georgiev [41], with the caveat that he used the definition of principal subspace from [81].

For $y=(y_1,y_2,y_3)\in Q$, let

$$\Psi_y(z,w;q):=q^{-y_{13}}(zq;q)_{y_{12}}(wq;q)_{y_{23}}(zwq;q)_{y_{13}}\Phi_{y_1,y_1+y_2}\big(zq^{y_{12}},wq^{y_{23}};q^{-1}\big). \tag{8.4}$$

Point of departure for our proof is (4.3) for $N=M=0$. Identifying $(r,s)=(y_1,y_1+y_2)$ and using (4.12), this may also be written as

$$\delta_{n,0}\delta_{m,0}=\sum_{y\in Q_+}\Phi_{n,m;y}(z,w;q)\Psi_y(z,w;q), \tag{8.5}$$

where $n,m\in\mathbb{N}_0$. Since $\Phi_{n,m;y}$ vanishes unless $y_1\leqslant n$ and $y_1+y_2\leqslant m$, the sum over y in (8.5) has finite support.

As in the proof in Section 5, let a,k be integers such that $a\leqslant k$. Then, by a $(k-a+1)$-fold application of (4.13) starting with the root identity (8.5), we obtain

$$\begin{aligned}\sum_{\substack{\lambda\subseteq(n^{k-a})\\ \mu\subseteq(m^{k-a})}}&\prod_{i=1}^{k-a+1}\mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(z,w;q) \\ &=\sum_{y\in Q_+}\big(z^{y_1}w^{y_1+y_2}q^{\frac{1}{2}(y_1^2+y_2^2+y_3^2)}\big)^{k-a+1}\Phi_{n,m;y}(z,w;q)\Psi_y(z,w;q),\end{aligned} \tag{8.6}$$

where $\lambda_0:=n$, $\mu_0:=m$. Next we use (4.17) to replace $\Phi_{n,m;y}(z,w;q)$ in the summand on the right by $\Phi_{n,m;y}(1,1,z,w;q)$ and define

$$Z:=zq^{y_{12}}\quad\text{and}\quad W:=wq^{y_{23}}.$$

Then, by an $(a-b)$-fold application of (4.18b) where $(u,v)=(Z^{i-1},W^{i-1})$ in the ith step, as well as the use of (5.10) for $(a,b)=(1/q,1)$, we find

$$\begin{aligned}\sum_{\substack{\lambda\subseteq(n^{k-b})\\ \mu\subseteq(m^{k-b})}}&\prod_{i=1}^{k-b+1}q^{-\chi(i\leqslant a-b)\lambda_i}\mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(z,w;q) \\ &=\sum_{y\in Q_+}\big(z^{y_1}w^{y_1+y_2}q^{\frac{1}{2}(y_1^2+y_2^2+y_3^2)}\big)^{k-b+1}q^{-(a-b)y_1}\Phi_{n,m;y}\big(Z^{a-b},W^{a-b};z,w;q\big)\Psi_y(z,w;q),\end{aligned}$$

for integers a,b,k such that $b\leqslant a\leqslant k$. Again denoting this by I_a, it follows from Corollary 4.10 that $(I_a-zwq^{m-1}I_{a-1})/(1-zwq^{-1})$ is given by

$$\begin{aligned}\sum_{\substack{\lambda\subseteq(n^{k-b})\\ \mu\subseteq(m^{k-b})}}&\frac{1-zwq^{m+\lambda_{a-b}-1}}{1-zwq^{-1}}\prod_{i=1}^{k-b+1}q^{-\chi(i\leqslant a-b)\lambda_i}\mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(z,w;q) \\ &=\sum_{y\in Q_+}\bigg(\big(z^{y_1}w^{y_1+y_2}q^{\frac{1}{2}(y_1^2+y_2^2+y_3^2)}\big)^{k-b+1}q^{-(a-b)y_1} \\ &\qquad\times\Phi_{n,m;y}\big(Z^{a-b},W^{a-b};1,1;z,w;q\big)\frac{\Psi_y(z,w;q)}{\Delta_y(z,w;q)}\bigg).\end{aligned} \tag{8.7}$$

Once again this holds for $b \leqslant a \leqslant k$ instead of the more restricted range $b < a \leqslant k$ since (8.7) for $b = a$ simplifies to (8.6) by $\lambda_0 := n$ and (4.27). The final iterative step in our proof is a b-fold application of Corollary 4.11, where

$$(u, v, c, d) = (Z^{a-b+i-1}, W^{a-b+i-1}, Z^{i-1}, W^{i-1})$$

in the ith step. By (5.10) for $a = b = 1/q$ this yields

$$\sum_{\substack{\lambda\subseteq(n^k)\\ \mu\subseteq(m^k)}} \frac{1 - zwq^{\lambda_a+\mu_b-1}}{1 - zwq^{-1}} \prod_{i=1}^{k+1} q^{-\chi(i\leqslant a)\lambda_i-\chi(i\leqslant b)\mu_i} \mathcal{K}_{\lambda_{i-1},\mu_{i-1};\lambda_i,\mu_i}(z, w; q) \tag{8.8}$$
$$= \sum_{y\in Q_+} \biggl(\bigl(z^{y_1} w^{y_1+y_2} q^{\frac{1}{2}(y_1^2+y_2^2+y_3^2)}\bigr)^{k+1} q^{-\sum_{i=1}^3 (\nu_i+i)y_i}$$
$$\times \Phi_{n,m;y}\bigl(Z^a, W^a; Z^b, W^b; z, w; q\bigr) \frac{\Psi_y(z, w; q)}{\Delta_y(z, w; q)} \biggr),$$

where we have used that $-ay_1 - b(y_1 + y_2) = -\sum_{i=1}^3 (\nu_i + i)y_i$ for $\nu := (a + b + 2, b + 1, 0)$. As for the analogous result (5.14) in the proof of the Kanade–Russell conjecture, this holds for all $0 \leqslant a, b \leqslant k$. Specifically, making the simultaneous substitutions $(z, w, a, b, n, m) \mapsto (w, z, b, a, m, n)$, changing the summation indices $(y_1, y_2, y_3) \mapsto (-y_3, -y_2, -y_1)$ on the right and $(\lambda, \mu) \mapsto (\mu, \lambda)$ on the left, it follows from (8.8) that the both sides are invariant under the interchange of a and b.

Taking the large-n, m limit using (4.26), using definitions (4.28) and (8.4), and eliminating z and w from the right-hand side in favour of x_1, x_2, x_3, we obtain

$$\sum_{\substack{\lambda,\mu\in\mathscr{P}\\ l(\lambda),l(\mu)\leqslant k}} \bigl(1 - zwq^{\lambda_a+\mu_b-1}\bigr) \prod_{i=1}^{k} \frac{z^{\lambda_i} w^{\mu_i} q^{\lambda_i^2-\lambda_i\mu_i+\mu_i^2-\chi(i\leqslant a)\lambda_i-\chi(i\leqslant b)\mu_i}}{(q; q)_{\lambda_i-\lambda_{i+1}} (q; q)_{\mu_i-\mu_{i+1}}}$$
$$= \prod_{1\leqslant i<j\leqslant 3} \frac{1}{(x_i/x_j; q)_\infty} \sum_{y\in Q_+} \biggl(\prod_{1\leqslant i<j\leqslant 3} (x_i/x_j; q)_{y_{ij}} \prod_{i=1}^{3} x_i^{(k+1)y_i} q^{(k+1)\binom{y_i}{2}-\nu_i y_i}$$
$$\times \det_{1\leqslant i,j\leqslant 3} \bigl((x_i q^{y_i})^{\nu_i-\nu_j}\bigr) \Phi_{y_1,y_1+y_2}\bigl(x_1 q^{y_{12}}/x_2, x_2 q^{y_{23}}/x_3; q^{-1}\bigr) \biggr).$$

Since, by $(a/q; q^{-1})_n = (aq^{-n}; q)_n$,

$$\Phi_{y_1,y_1+y_2}\bigl(x_1 q^{y_{12}}/x_2, x_2 q^{y_{23}}/x_3; q^{-1}\bigr) = \prod_{1\leqslant i<j\leqslant 3} \frac{1}{(x_i/x_j; q)_{y_{ij}}} \prod_{i=1}^{3} \frac{x_i^{y_i} q^{\binom{y_i}{2}} (x_i/x_3; q)_{y_i}}{(qx_i/x_1; q)_{y_i}},$$

this gives (8.3). ∎

As mentioned in the introduction, the A_1-analogue of Theorem 8.1 was first proved by Andrews, who showed that the right hand sides of (1.7) and (1.8) both satisfy

$$Q_{k,i}(z; q) - Q_{k,i-1}(z; q) = (zq)^{i-1} Q_{k,k-i+1}(zq; q) \tag{8.9}$$

for $1 \leqslant i \leqslant k$, where $Q_{k,0} := 0$. Since both expressions satisfy the same initial conditions $Q_{k,i}(0;q) = Q_{k,i}(z;0) = 1$, this proves the equality of (1.7) and (1.8). The equation (8.9) may also be derived purely algebraically using the theory of intertwining operators for vertex operator algebras, see [23]. For general $\mathrm{A}_{r-1}^{(1)}$ this approach has only been completed fully for level-1 modules, see [22, Theorem 5.3]. Restricting to $r=3$, this yields

$$\operatorname{ch} W'_{\Lambda_0}(z,w;q) - \operatorname{ch} W'_{\Lambda_0}(zq,w;q) = zq \operatorname{ch} W'_{\Lambda_0}(zq^2,wq^{-1};q), \tag{8.10a}$$

$$\operatorname{ch} W'_{\Lambda_0}(z,w;q) - \operatorname{ch} W'_{\Lambda_0}(q,wq;q) = wq \operatorname{ch} W'_{\Lambda_0}(zq^{-1},wq^2;q), \tag{8.10b}$$

where the exponents of q in the argument of $\operatorname{ch} W'_{\Lambda_0}$ on the right are the Cartan integers of $\mathfrak{sl}_3$. Together with

$$\operatorname{ch} W'_{k\Lambda_0}(z,w;q) = \operatorname{ch} W'_{k\Lambda_1}(zq,w;q) = \operatorname{ch} W'_{k\Lambda_2}(z,wq;q)$$

for arbitrary level k and $\operatorname{ch} W'_{\Lambda_0}(0,0;q) = \operatorname{ch} W'_{\Lambda_0}(z,w;0) = 1$, this uniquely determines the characters $\operatorname{ch} W'_{\Lambda_i}$ for $0 \leqslant i \leqslant 2$. It is routine to show that the right-hand side of (8.2) for $k=1$ and $a=b=0$ satisfies (8.10). The same cannot be said for the bosonic representation

$$\begin{aligned}\operatorname{ch} W'_{\Lambda_0}(z,w;q) &= \frac{1}{(zq,wq,zwq;q)_\infty} \\ &\times \sum_{r,s=0}^{\infty} \bigg((-1)^{r+s} z^{2r} w^{2s} q^{2r^2+2s^2-2rs+\binom{r}{2}+\binom{s}{2}} \frac{(1-zq^{2r-s})(1-wq^{2s-r})(1-zwq^{r+s})}{(1-z)(1-w)(1-zw)} \\ &\qquad \times \frac{(zw;q)_r (zw;q)_s (z;q)_{r-s} (w;q)_{s-r}}{(q;q)_r (q;q)_s} \bigg),\end{aligned}$$

for which showing (8.10) holds requires a lengthy computation. It would be very interesting to extend the approach using functional equations to $\operatorname{ch} W'_\lambda(z,w;q)$ for weights of arbitrary level.

9. Outlook

An important open question is how to generalise Theorems 1.2 and 1.4 to $\mathrm{A}_{r-1}^{(1)}$ for all r.[9] As far as the A_{r-1}-analogue of the Bailey chains of Lemma 3.1 and Theorem 4.2 is concerned, things are relatively straightforward. Let $\boldsymbol{n} = (n_1,\dots,n_{r-1})$, $\boldsymbol{m} = (m_1,\dots,m_{r-1})$ be integer sequences and $\boldsymbol{z} = (z_1,\dots,z_{r-1})$ a sequence of indeterminates. In [89] the definition of the rational function $\Phi_{n,m}(z,w;q)$ was extended to A_{r-1} as:

$$\Phi_{\boldsymbol{n}}(\boldsymbol{z};q) := \sum_{\lambda^{(1)},\dots,\lambda^{(r-1)} \in \mathscr{P}} \prod_{i=1}^{r-1} \prod_{l \geqslant 1} \frac{z_i^{\lambda_l^{(i)}} q^{\frac{1}{2}\sum_{j=1}^{r-1} A_{ij} \lambda_l^{(i)} \lambda_l^{(j)}}}{(q;q)_{\lambda_{l-1}^{(i)} - \lambda_l^{(i)}}}, \tag{9.1}$$

[9] The vacuum case $a=b=0$ of Theorem 8.1 was generalised to all r in [34, Theorem 3.1] without the use of the Bailey machinery.

where $\lambda_0^{(i)} := n_i$ and where $(A_{ij})_{1\leqslant i,j\leqslant r-1}$ is the Cartan matrix of A_{r-1}. For an arbitrary sequence $\boldsymbol{a} = (a_1, \dots, a_{r-1})$, let $\bar{\boldsymbol{a}} := (a_{r-1}, \dots, a_1)$. Replacing $\lambda^{(i)}$ by $\lambda^{(r-i)}$ in (9.1) it follows that

$$\Phi_{\boldsymbol{n}}(z;q) = \Phi_{\bar{\boldsymbol{n}}}(\bar{z};q). \tag{9.2}$$

Another immediate consequence of the definition (9.1) is the A_{r-1} Bailey chain

$$\sum_{m_1=0}^{n_1} \cdots \sum_{m_{r-1}=0}^{n_{r-1}} \mathcal{K}_{\boldsymbol{n},\boldsymbol{m}}(z;q)\Phi_{\boldsymbol{m}}(z;q) = \Phi_{\boldsymbol{n}}(z;q), \tag{9.3}$$

with $\mathcal{K}_{\boldsymbol{n},\boldsymbol{m}}$ given by

$$\mathcal{K}_{\boldsymbol{n},\boldsymbol{m}}(z;q) := \prod_{i=1}^{r-1} \frac{z_i^{m_i} q^{\frac{1}{2}\sum_{j=1}^{r-1} A_{ij}m_i m_j}}{(q;q)_{n_i-m_i}}.$$

Moreover, by Hua's identity [47, Theorem 4.9] for A_{r-1},

$$\lim_{n_1,\dots,n_{r-1}\to\infty} \Phi_{\boldsymbol{n}}(z;q) = \frac{1}{(q;q)_\infty^{r-1}} \prod_{1\leqslant i<j\leqslant r} \frac{1}{(z_i\cdots z_{j-1}q;q)_\infty}. \tag{9.4}$$

The alternative expressions for $\Phi_n(z;q)$ and $\Phi_{n,m}(z,w;q)$ as given in (3.1) and (4.1) follow from Corollaries 3.2 and 4.3, or from [34, Proposition 2.2] which is based on the decomposition in the Gelfand–Zetlin basis of the Whittaker vectors for the quantum group $U_v(\mathfrak{gl}_r)$ over $\mathbb{C}(v)$. This more generally implies that

$$\begin{aligned}\Phi_{\boldsymbol{n}}(z;q) = \sum \prod_{k\geqslant 1} \Biggl(\prod_{i=1}^{r-1} \frac{(-1)^{\lambda_{k+1}^{(i)}} q^{\binom{\lambda_{k+1}^{(i)}}{2}}}{(q;q)_{\lambda_k^{(i)}-\lambda_{k+1}^{(i)}}} \prod_{1\leqslant i<j\leqslant r} \Biggl(z_{j-1}^{\lambda_{k+j-i}^{(i)}} q^{-(\lambda_{k+j-i}^{(i)}-\lambda_{k+j-i+1}^{(i)})\lambda_k^{(j)}} \\ \times \frac{1-z_i\cdots z_{j-1}q^{\lambda_{k+j-i}^{(i)}-\lambda_k^{(j)}}}{1-z_i\cdots z_{j-1}} \frac{(z_i\cdots z_{j-1};q)_{\lambda_{k+j-i+1}^{(i)}-\lambda_k^{(j)}}}{(z_i\cdots z_{j-1}q;q)_{\lambda_{k+j-i-1}^{(i)}-\lambda_k^{(j)}}} \Biggr)\Biggr),\end{aligned} \tag{9.5}$$

where the sum is over partitions $\lambda^{(1)}, \dots, \lambda^{(r)}$ such that $l(\lambda^{(i)}) \leqslant r-i$ for $1 \leqslant i \leqslant r$ (so that $\lambda^{(r)} = 0$) and $\lambda_1^{(i)} + \lambda_2^{(i-1)} + \cdots + \lambda_i^{(1)} = n_i$ for $1 \leqslant i \leqslant r-1$. For $r = 2$ this yields (3.1) and for $r = 3$ it gives

$$\begin{aligned}\Phi_{n,m}(z,w;q) &= \frac{1}{(q,zq^{1-m};q)_n(q,wq;q)_m} \\ &\quad \times {}_6W_5\bigl(zq^{-m};q^{-m}/w,q^{-n},q^{-m};q,zwq^{n+m+1}\bigr).\end{aligned} \tag{9.6}$$

By Jackson's ${}_6W_5$ summation [40, Equation (II.20)] this simplifies to (4.1). The expression (9.5) obscures the symmetry (9.2), although it can be simplified relatively easily to a $\binom{r-2}{2}$-fold multisum that is symmetric. For example, for $r = 4$ two of the three summations can be carried out to give an expression as a balanced ${}_4\phi_3$ basic hypergeometric series:

$$\begin{aligned}\Phi_{\boldsymbol{n}}(z;q) &= \frac{(z_1z_2q;q)_{n_1+n_2}(z_2z_3q;q)_{n_2+n_3}}{(q,z_1q,z_1z_2q;q)_{n_1}(q,z_2q,z_1z_2q,z_2z_3q;q)_{n_2}(q,z_3q,z_2z_3q;q)_{n_3}} \\ &\quad \times {}_4\phi_3\left[\begin{matrix} q^{-n_2}/z_2, q^{-n_1}, q^{-n_2}, q^{-n_3} \\ q^{-n_1-n_2}/z_1z_2, q^{-n_2-n_3}/z_2z_3, z_1z_2z_3q \end{matrix};q,q\right].\end{aligned}$$

Regardless of how $\Phi_{\boldsymbol{n}}(z;q)$ is expressed, it is an open problem to lift the A_{r-1} Bailey chain (9.3) to an A_{r-1} Bailey tree. It follows from the work of Ardonne, Kedem and Stone (see [9, Equation (6.16)]) that the 1 and $-q^{-1}$ in $1-zwq^{\lambda_a+\mu_b-1}$ in formula (8.2) — this factor can be traced back to the structure of the numerator of (4.22) — should be interpreted as entries of the inverse of the matrix of generalised Kostka polynomials [53,75] for $\mathfrak{sl}_3$. This suggests that the as-yet-to-be-discovered A_{r-1} Bailey tree involves the generalised Kostka polynomials for $\mathfrak{sl}_r$. Another open problem is to find the A_{r-1}-analogue of the (3.14b) and (5.2). For $y=(y_1,\dots,y_r)\in Q$, let

$$\Phi_{\boldsymbol{n};y}(z;q):=\frac{\Phi_{\boldsymbol{m}}(\boldsymbol{w};q)}{\prod_{1\leqslant i<j\leqslant r}(z_i\cdots z_{j-1}q;q)_{y_{ij}}},$$

where $m_i:=n_i-y_1-\cdots-y_i$ and $w_i:=z_iq^{y_{i,i+1}}$ for $1\leqslant i\leqslant r-1$. The problem then is to find a manifestly positive representation for the rational function $g_{\boldsymbol{n};\tau}(q)$ defined by

$$g_{\boldsymbol{n};\tau}(q):=\sum_{y\in Q}\Phi_{\boldsymbol{n};ry}(\underbrace{q,\dots,q}_{r-1\text{ times}};q)\prod_{1\leqslant i<j\leqslant r}\frac{1-q^{ry_{ij}+j-i}}{1-q^{j-i}}\prod_{i=1}^{r}q^{r(r+\tau)\binom{y_i}{2}-\tau iy_i},$$

where $\boldsymbol{n}\in\mathbb{N}_0^{r-1}$ and $\tau\in\{2-r,\dots,0,1\}$. For general r this is a very hard problem since

$$\begin{aligned}&\Phi_{n_1,\dots,n_{i-1},0,n_{i+1},\dots,n_{r-1};ry}(\underbrace{q,\dots,q}_{r-1\text{ times}};q)\\&\quad=\Phi_{n_1,\dots,n_{i-1}}(\underbrace{q,\dots,q}_{i-1\text{ times}};q)\,\Phi_{n_{i+1},\dots,n_{r-1}}(\underbrace{q,\dots,q}_{r-i-1\text{ times}};q)\prod_{i=j}^{r}\delta_{y_j,0},\end{aligned}$$

which implies that

$$\begin{aligned}&g_{n_1,\dots,n_{i-1},0,n_{i+1},\dots,n_{r-1};\tau}(q)\\&\quad=\Phi_{n_1,\dots,n_{i-1}}(\underbrace{q,\dots,q}_{i-1\text{ times}};q)\,\Phi_{n_{i+1},\dots,n_{r-1}}(\underbrace{q,\dots,q}_{r-i-1\text{ times}};q).\end{aligned}$$

For example, setting $m=0$ in (1.3) gives $g_{n,0;\tau}(q)=1/(q,q^2;q)_n=\Phi_n(q;q)$. Some properties of $g_{\boldsymbol{n};\tau}(q)$ are easily deduced for general r. From (9.4) followed by (5.16) it immediately follows that

$$\lim_{n_1,\dots,n_{r-1}\to\infty}g_{\boldsymbol{n};\tau}(q)=\begin{cases}\displaystyle\prod_{1\leqslant i<j\leqslant r}\frac{1}{(q^{j-i};q)_\infty} & \text{if } \tau=1,\\[2mm] \displaystyle\frac{(q;q)_\infty}{(q^r;q^r)_\infty}\prod_{1\leqslant i<j\leqslant r}\frac{1}{(q^{j-i};q)_\infty} & \text{if } \tau=0,\\[2mm] 0 & \text{if } \tau\in\{2-r,\dots,-1\}.\end{cases}\tag{9.7}$$

We can do slightly better for special values of τ. First we note that by [89, Equation (6.3)][10] it follows that for $r \geqslant 3$

$$\lim_{n_2,\dots,n_{r-2}\to\infty} \Phi_{\boldsymbol{n};y}(z;q) = \frac{1}{(q;q)_\infty^{r-3}} \prod_{2\leqslant i<j\leqslant r-1} \frac{1}{(z_i\cdots z_{j-1}q;q)_\infty} \\ \times \frac{(z_1\cdots z_{r-1}q;q)_{n_1+n_{r-1}}}{\prod_{i=1}^r (z_1\cdots z_{i-1}q;q)_{n_1-y_i}(z_i\cdots z_{r-1}q;q)_{n_{r-1}+y_i}},$$

generalising (9.4). Hence, for such r,

$$\begin{aligned} g_{n,m;\tau}^{(r)}(q) &:= \lim_{n_2,\dots,n_{r-2}\to\infty} g_{(n,n_2,\dots,n_{r-2},m);\tau}(q) \\ &= (q;q)_\infty \prod_{1\leqslant i<j\leqslant r-1} \frac{1}{(q^{j-i};q)_\infty} \prod_{i=1}^{r-1} \frac{1}{(q^{r-i};q)_{n+m+i}} \\ &\quad \times \sum_{y\in Q} \prod_{1\leqslant i<j\leqslant r} (1-q^{ry_{ij}+j-i}) \prod_{i=1}^r q^{r(r+\tau)\binom{y_i}{2}-\tau i y_i} \begin{bmatrix} n+m+r-1 \\ n-ry_i+i-1 \end{bmatrix}. \end{aligned}$$

By (5.6), (A.4) and $q \mapsto 1/q$ duality this may be expressed in closed form for $\tau \in \{-1,0,1\}$ as

$$g_{n,m;\tau}^{(r)}(q) = q^{\binom{\tau}{2}(r-1)nm} \begin{bmatrix} n+m \\ n \end{bmatrix}_p (q;q)_\infty \prod_{1\leqslant i<j\leqslant r-1} \frac{1}{(q^{j-i};q)_\infty} \prod_{i=1}^{r-1} \frac{1}{(q^{r-i};q)_{n+m}},$$

where $p = q$ if $\tau \in \{-1,1\}$ and $p = q^r$ if $\tau = 0$. For $r = 3$ this is (5.2), and in the limit of large n and m this gives (9.7) for $\tau \in \{-1,0,1\}$.

Appendix A. New proof of (5.3)

We begin with the following q-Pfaff–Saalschütz summation for the root system A_{r-1}:

$$\sum_{y\in\mathbb{N}_0^r} \Biggl((b,q^{-N};q)_{|y|} \prod_{1\leqslant i<j\leqslant r} \frac{x_i q^{y_i} - x_j q^{y_j}}{x_i - x_j} \prod_{i,j=1}^r \frac{(a_j x_i/x_j;q)_{y_i}}{(qx_i/x_j;q)_{y_i}} \\ \times \prod_{i=1}^r \frac{(bq^{1-N}/cx_i;q)_{|y|-y_i} q^{y_i}}{(a_i b q^{1-N}/cx_i;q)_{|y|}(cx_i;q)_{y_i}} \Biggr) = \prod_{i=1}^r \frac{(cx_i/a_i, cx_i/b;q)_N}{(cx_i, cx_i/a_i b;q)_N}, \tag{A.1}$$

where N is a nonnegative integer and $|y| := y_1 + \cdots + y_r$. It should be noted that the summand vanishes unless $|y| \leqslant N$ so that only finitely many terms contribute to the sum. The result (A.1) was first obtained in the appendix of a preliminary version of Leininger and Milne's paper [56]; an appendix that was dropped in the published version. Subsequently (A.1) was rederived and published by Bhatnagar and Schlosser, see [16, Remark 5.11].

[10] This result is stated in [89] without proof.

To obtain (5.3), we replace $q \mapsto q^r$ in (A.1) and then specialise $x_i = q^{r-i}bz/c$ and $a_i = q^{-n}$ for n a nonnegative integer. Using $\prod_{i=1}^r (aq^{r-i};q)_k = (a;q)_{rk}$, this gives

$$\prod_{i,j=1}^r \frac{(a_jx_i/x_j;q)_{y_i}}{(qx_i/x_j;q)_{y_i}} \mapsto \prod_{i=1}^r \frac{(q^{-n-i+1};q)_{ry_i}}{(q^{r-i+1};q)_{ry_i}},$$

so that the resulting summand vanishes unless $0 \leqslant ry_i \leqslant n+i-1$. Since this is independent of N, q^{-rN} may be replaced by the indeterminate d, resulting in

$$\sum_{y\in\mathbb{N}_0^r} \frac{(b,d;q^r)_{|y|}}{(dq^{1-n}/z;q)_{r|y|}} \prod_{1\leqslant i<j\leqslant r} \frac{1-q^{ry_{ij}+j-i}}{1-q^{j-i}} \prod_{i=1}^r \frac{(q^{-n-i+1};q)_{ry_i}(dq^i/z;q^r)_{|y|-y_i}q^{riy_i}}{(q^{r-i+1};q)_{ry_i}(bzq^{r-i};q^r)_{y_i}}$$
$$= \frac{(z,bz/d;q)_n}{(bz,z/d;q)_n},$$

where the reader is reminded that $y_{ij} := y_i - y_j$. Indeed, after multiplying the above identity by $d^n(z/d;q)_n$ and carrying out some standard simplifications of the q-shifted factorials involving d, it follows that both sides are polynomials in d of degree n. Since the difference between the right- and left-hand side is zero for $d = q^{-rN}$ where N is an arbitrary nonnegative integer, this difference is zero for all d. Next, if we set $b = 0$, let d tend to infinity and carry out some elementary manipulations, we find

$$\sum_{y\in\mathbb{N}_0^r} \big((-1)^r z\big)^{|y|} q^{-r\binom{|y|}{2}} \prod_{1\leqslant i<j\leqslant r} (1-q^{ry_{ij}+j-i}) \prod_{i=1}^r q^{\binom{r+1}{2}y_i^2-iy_i} \begin{bmatrix} n+r-1 \\ n-ry_i+i-1 \end{bmatrix} \tag{A.2}$$
$$= (z;q)_n \prod_{i=1}^{r-1} (1-q^{n+i})^i.$$

We now consider the sum over the y_i for fixed $|y| = m$ and carry out what in [42] is referred to as the rotation trick. That is, if u, v are the unique integers such that $m = ur + v$ for $0 \leqslant v < r$, $u \geqslant 0$, then we shift and rotate the summation indices $y_1, \dots, y_r$ as

$$y_i \mapsto \begin{cases} y_{i+v} + u & \text{for } 1 \leqslant i \leqslant r-v, \\ y_{i+v-r} + u + 1 & \text{for } r-v < i \leqslant r. \end{cases}$$

This substitution leads to the following alternative expression for the left-hand side of (A.2):

$$\sum_{m=0}^n \sum_{y\in Q} (-z)^m q^{\binom{m}{2}} \prod_{1\leqslant i<j\leqslant r} (1-q^{ry_{ij}+j-i}) \prod_{i=1}^r q^{\binom{r+1}{2}y_i^2-iy_i} \begin{bmatrix} n+r-1 \\ n-m-ry_i+i-1 \end{bmatrix}.$$

Equating coefficients of z^m with the right-hand side of (A.2) using the q-binomial theorem

$$(z;q)_n = \sum_{m=0}^n (-z)^m q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}, \tag{A.3}$$

this implies

$$\sum_{y\in Q}\prod_{1\leqslant i<j\leqslant r}(1-q^{ry_{ij}+j-i})\prod_{i=1}^{r} q^{\binom{r+1}{2}y_i^2-iy_i}\begin{bmatrix} n+r-1 \\ n-m-ry_i+i-1\end{bmatrix}=\begin{bmatrix} n\\ m\end{bmatrix}\prod_{i=1}^{r-1}(1-q^{n+i})^i. \tag{A.4}$$

Finally, replacing n by $n+m$ and specialising $r=3$ yields (5.3).

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