

# 50 YEARS OF BAILEY'S LEMMA

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## 0. INTRODUCTION

Half a century ago, The Proceedings of the London Mathematical Society published W. N. Bailey's influential paper *Identities of the Rogers–Ramanujan type* [16]. The main result therein, which was inspired by Rogers' second proof of the Rogers–Ramanujan identities [49] (and also [48, 28, 15]), is what is now known as Bailey's lemma. To celebrate the occasion of the lemma's fiftieth birthday we present a history of Bailey's lemma in 5 chapters (or rather sections), covering (i) Bailey's work, (ii) the Bailey chain (iii) the Bailey lattice (iv) the Bailey lemma in statistical mechanics, and (v) conjugate Bailey pairs.

Due to size limitations of this paper the higher rank [42, 40, 43, 41, 14, 60] and trinomial [11, 59, 19] generalizations of the Bailey lemma will be treated at the lemma's centennial in 2049. More extensive reviews of topics (i), (ii) and (iii), can be found in [5, Sec. 3], [46] and [24], respectively.

## 1. THE BAILEY LEMMA

In an attempt to clarify Rogers' second proof [49] of the Rogers–Ramanujan identities, Bailey [16] was led to the following simple observation.

**Lemma 1.1.** *If  $\alpha = \{\alpha_L\}_{L \geq 0}, \dots, \delta = \{\delta_L\}_{L \geq 0}$  are sequences that satisfy*

$$(1.1) \quad \beta_L = \sum_{r=0}^L \alpha_r u_{L-r} v_{L+r} \quad \text{and} \quad \gamma_L = \sum_{r=L}^{\infty} \delta_r u_{r-L} v_{r+L},$$

then

$$(1.2) \quad \sum_{L=0}^{\infty} \alpha_L \gamma_L = \sum_{L=0}^{\infty} \beta_L \delta_L.$$

The proof is straightforward and merely requires an interchange of sums. Of course, in the above suitable convergence conditions need to be imposed to make the definition of  $\gamma$  and the interchange of sums meaningful.

The idea behind Bailey's lemma is clear. When trying to prove a complicated identity of the form  $\sum_L A_L = \sum_L B_L$  it is a considerable step in the right direction if one can find a dissection of this identity into two identities of the type (1.1) where  $A_L = \alpha_L \gamma_L$  and  $B_L = \beta_L \delta_L$ . Or, as Slater put it in Bailey's obituary [57],

*The root of the underlying idea . . . is that of transforming a doubly infinite series into a simply infinite and a finite series. In a geometric sense, this involves summing over a triangle instead of over a rectangle.*

In applications of his transform, Bailey chose  $u_L = 1/(q)_L$  and  $v_L = 1/(aq)_L$ , with the usual definition of the  $q$ -shifted factorial,  $(a)_\infty = (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$  and  $(a)_L = (a; q)_L = (a)_\infty / (aq^L)_\infty$  for  $L \in \mathbb{Z}$ . (Throughout, we assume that  $0 < |q| < 1$ .) With this choice, equation (1.1) becomes

$$(1.3) \quad \beta_L = \sum_{r=0}^L \frac{\alpha_r}{(q)_{L-r}(aq)_{L+r}} \quad \text{and} \quad \gamma_L = \sum_{r=L}^{\infty} \frac{\delta_r}{(q)_{r-L}(aq)_{r+L}}.$$

A pair of sequences that satisfies the first equation of (1.3) is called a Bailey pair relative to  $a$ . Similarly, the second equation defines a conjugate Bailey pair relative to  $a$ .

Still following Bailey, one can employ the  $q$ -Saalschütz summation [35, Eq. (II.12)] to establish that  $(\gamma, \delta)$  with

$$(1.4) \quad \begin{aligned} \gamma_L &= \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L} \frac{1}{(q)_{M-L}(aq)_{M+L}} \\ \delta_L &= \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_M(aq/\rho_2)_M} \frac{(aq/\rho_1\rho_2)_{M-L}}{(q)_{M-L}} \end{aligned}$$

provides a conjugate Bailey pair.

As we shall see in the next section this conjugate Bailey pair leads to the very important concept of the Bailey chain. However, Bailey missed an opportunity here and made the (mis)judgement [16, Page 4]:

*These values of  $\delta_L, \gamma_L \dots$  lead to  $\dots$  results involving only terminating basic series. We are, however, more concerned with identities of the Rogers–Ramanujan type in this paper, as the most general formulae for basic series are too involved to be of any great interest.*

Consequently, Bailey only considered the conjugate Bailey pair (1.4) when the parameter  $M$  tends to infinity. Also taking the limit  $\rho_1, \rho_2 \rightarrow \infty$  yields

$$(1.5) \quad \gamma_L = \frac{a^L q^{L^2}}{(aq)_\infty} \quad \text{and} \quad \delta_L = a^L q^{L^2},$$

which substituted into (1.2) gives

$$(1.6) \quad \frac{1}{(aq)_\infty} \sum_{L=0}^{\infty} a^L q^{L^2} \alpha_L = \sum_{L=0}^{\infty} a^L q^{L^2} \beta_L.$$

The proof of the Rogers–Ramanujan and similar such identities requires the input of suitable Bailey pairs into (1.6). For example, from Rogers' work [49] one can infer the following Bailey pair relative to 1:  $\alpha_0 = 1$  and

$$(1.7) \quad \alpha_L = (-1)^L q^{L(3L-1)/2} (1 + q^L), \quad \beta_L = \frac{1}{(q)_L}.$$

Thus one finds

$$\frac{1}{(q)_\infty} \sum_{L=-\infty}^{\infty} (-1)^L q^{L(5L-1)/2} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}.$$

The application of the Jacobi triple product identity [35, Eq. (II.28)] yields the first Rogers–Ramanujan identity [48, 49]

$$(1.8) \quad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}},$$

with the notation  $(a_1, \dots, a_k; q)_n = (a_1; q)_n \dots (a_k; q)_n$ . The second Rogers–Ramanujan identity

$$(1.9) \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}}$$

follows in a similar fashion using the Bailey pair [49]

$$(1.10) \quad \alpha_L = (-1)^L q^{L(3L+1)/2} (1 - q^{2L+1}) / (1 - q), \quad \beta_L = \frac{1}{(q)_L}$$

relative to  $q$ . By collecting a list of 96 Bailey pairs, and using (1.6) or the identity obtained from (1.4) and (1.2) by taking  $M, \rho_1 \rightarrow \infty$  and  $\rho_2 = -q^{k/2}$  (with  $k$  a small nonnegative integer) Slater compiled her famous list of 130 identities of the Rogers–Ramanujan type [55, 56]. The next two sections deal with more systematic ways of finding Bailey pairs.

## 2. THE BAILEY CHAIN

By dismissing the conjugate Bailey pair (1.4) in its finite form (i.e., with  $M$  finite, or, equivalently, with  $\rho_1$  or  $\rho_2$  of the form  $q^{-N}$ ) Bailey missed a very effective mechanism for generating Bailey pairs. Namely, if we substitute the conjugate pair (1.4) into (1.2) the resulting equation has the same form as the defining relation (1.3) of a Bailey pair. This is formalized in the following theorem due to Andrews [4, 5].

**Theorem 2.1.** *Let  $(\alpha, \beta)$  form a Bailey pair relative to  $a$ . Then so does  $(\alpha', \beta')$  with*

$$(2.1) \quad \alpha'_L = \frac{(\rho_1)_L (\rho_2)_L (aq/\rho_1 \rho_2)^L}{(aq/\rho_1)_L (aq/\rho_2)_L} \alpha_L$$

$$\beta'_L = \sum_{r=0}^L \frac{(\rho_1)_r (\rho_2)_r (aq/\rho_1 \rho_2)^r (aq/\rho_1 \rho_2)^{L-r}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)_{L-r}} \beta_r.$$

Again letting  $\rho_1, \rho_2$  tend to infinity leads to the important special case

$$(2.2) \quad \alpha'_L = a^L q^{L^2} \alpha_L \quad \text{and} \quad \beta'_L = \sum_{r=0}^L \frac{a^r q^{r^2}}{(q)_{L-r}} \beta_r,$$

which, for  $a = 1$  and  $a = q$ , was also discovered by Paule [44]. One now finds that the Bailey pairs (1.7) and (1.10) of Rogers can be obtained from the  $a = 1$  and  $a = q$  instances of the Bailey pair [4]

$$(2.3) \quad \alpha_L = (-1)^L q^{\binom{L}{2}} \frac{(1 - aq^{2L})(a)_L}{(1 - a)(q)_L}, \quad \beta_L = \delta_{L,0}$$

by application of (2.2). The Bailey pair (2.3) is an immediate consequence of the inverse Bailey transform [4]

$$(2.4) \quad \alpha_L = \frac{1 - aq^{2L}}{1 - a} \sum_{r=0}^L \frac{(-1)^{L-r} q^{\binom{L-r}{2}} (a)_{L+r}}{(q)_{L-r}} \beta_r,$$

which follows from (1.3) and [35, Eq. (II.21)] specialized to  $aq = bc$ . The iteration of (2.1) or (2.2) leads to what is known as the Bailey chain [4, 46]:

$$(\alpha, \beta) \rightarrow (\alpha', \beta') \rightarrow (\alpha'', \beta'') \rightarrow \dots$$

and thus, given a single Bailey pair, one immediately finds an infinite sequence of Bailey pairs. (To be compared with the 96 Bailey pairs collected by Slater!) As an example, iteration of (2.4) gives the Bailey pair

$$\alpha_L = (-1)^L a^{kL} q^{kL^2 + \binom{L}{2}} \frac{(1 - aq^{2L})(a)_L}{(1 - a)(q)_L}$$

$$\beta_L = \sum_{L \geq n_1 \geq \dots \geq n_{k-1} \geq 0} \frac{a^{n_1 + \dots + n_{k-1}} q^{n_1^2 + \dots + n_{k-1}^2}}{(q)_{L-n_1} (q)_{n_1-n_2} \dots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}}.$$

Substituting this into the defining relation (1.3) of a Bailey pair and letting  $L$  tend to infinity gives, for  $a = 1$  or  $a = q$ ,

$$\sum_{n_1, \dots, n_{k-1}} \frac{a^{n_1 + \dots + n_{k-1}} q^{n_1^2 + \dots + n_{k-1}^2}}{(q)_{n_1-n_2} \dots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}} = \frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} (-1)^r a^{kr} q^{kr^2 + \binom{r}{2}}.$$

Using Jacobi's triple product identity finally yields

$$(2.5) \quad \sum_{n_1, \dots, n_{k-1}} \frac{q^{n_1^2 + \dots + n_{k-1}^2 + n_i + \dots + n_{k-1}}}{(q)_{n_1-n_2} \dots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}, q^{2k+1})_\infty}{(q)_\infty},$$

where  $i = 1$  or  $i = k$ . For  $k = 2$  these are the Rogers–Ramanujan identities (1.8) and (1.9), whereas for  $k \geq 3$  they are Andrews' analytic counterpart of Gordon's partition theorem [3]. In fact, Andrews' identities are (2.5) for all  $i = 1, \dots, k$  and one concludes that the Bailey chain mechanism has failed to produce all of these. What is required is an extension of the Bailey chain known as the Bailey lattice. This will be our next topic. (Prior to the invention of the Bailey lattice Paule [44] already obtained (2.5) for all  $i$  using “ad hoc” Bailey lattice-like transformations.)

### 3. THE BAILEY LATTICE

One of the features of Theorem 2.1 is that it transforms a Bailey pair relative to  $a$  into a new Bailey pair relative to  $a$ . More generally one can of course try to transform a Bailey pair relative to  $a$  into a Bailey pair relative to  $b$ . Agarwal, Andrews and Bressoud have formulated this problem in a general setting of infinite dimensional matrices [1, 24]. Here we shall only be concerned with concrete examples of such “Bailey lattice” transformations. Since the parameter  $a$  is no longer fixed we shall write  $(\alpha(a), \beta(a))$  for a Bailey pair relative to  $a$ .

**Theorem 3.1.** Fix  $N$  a nonnegative integer and set  $b = aq^N$ . Let  $(\alpha(b), \beta(b))$  be a Bailey pair. Then so is  $(\alpha'(a), \beta'(a))$  with

$$\begin{aligned}\alpha'_L(a) &= (1 - aq^{2L})(aq)_N \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L} \\ &\quad \times \sum_{j=0}^N (-1)^j a^j q^{2Lj-j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix} \frac{(aq)_{2L-j-1}}{(aq)_{2L-j+N}} \alpha_{L-j}(b) \\ \beta'_L(a) &= \sum_{r=0}^L \frac{(\rho_1)_r(\rho_2)_r(aq/\rho_1\rho_2)^r(aq/\rho_1\rho_2)_{L-r}}{(aq/\rho_1)_L(aq/\rho_2)_L(q)_{L-r}} \beta_r(b).\end{aligned}$$

Here we have used the  $q$ -binomial coefficient defined as  $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(q^{a-b+1})_b}{(q)_b}$  for  $b \geq 0$  and 0 otherwise. A very similar result can be stated as follows.

**Theorem 3.2.** Fix  $N$  a nonnegative integer and set  $b = aq^N$ . Let  $(\alpha(b), \beta(b))$  be a Bailey pair. Then so is  $(\alpha'(a), \beta'(a))$  with

$$\begin{aligned}\alpha'_L(a) &= (1 - aq^{2L})(aq)_N \sum_{j=0}^N \frac{(\rho_1)_{L-j}(\rho_2)_{L-j}(bq/\rho_1\rho_2)^{L-j}}{(bq/\rho_1)_{L-j}(bq/\rho_2)_{L-j}} \\ &\quad \times (-1)^j a^j q^{2Lj-j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix} \frac{(aq)_{2L-j-1}}{(aq)_{2L-j+N}} \alpha_{L-j}(b) \\ \beta'_L(a) &= \sum_{r=0}^L \frac{(\rho_1)_r(\rho_2)_r(bq/\rho_1\rho_2)^r(bq/\rho_1\rho_2)_{L-r}}{(bq/\rho_1)_L(bq/\rho_2)_L(q)_{L-r}} \beta_r(b).\end{aligned}$$

The  $N = 0$  and  $N = 1$  cases of the first theorem correspond to the Bailey chain of Theorem 2.1 and the Bailey lattice of [1, Lemma 1.2], respectively. The second theorem for  $N = 0$  is again the Bailey chain whereas for  $N = 1$  it is a variation of the Bailey lattice of [51, Lemma 4.3]. Theorem 3.1 was also found by Krattenthaler and Foda [39].

First we prove Theorem 3.1. Substituting the expression for  $\alpha'(a)$  in the ‘‘primed’’ version of (1.3), transforming  $j \rightarrow r - j$  and then interchanging the order of summation, gives

$$\begin{aligned}\beta'_L(a) &= \sum_{j=0}^L \frac{(\rho_1)_j(\rho_2)_j(aq/\rho_1\rho_2)^j \alpha_j(b)}{(aq/\rho_1)_j(aq/\rho_2)_j(q)_{L-j}(bq)_{2j}(aq^{2j+1})_{L-j}} \\ &\quad \times \lim_{a_4 \rightarrow \infty} {}_8W_7(aq^{2j}; a_4, a/b, \rho_1 q^j, \rho_2 q^j, q^{-L+j}; q, abq^{L+j+2}/a_4\rho_1\rho_2),\end{aligned}$$

where we employed the conventional short-hand notation for very-well-poised basic hypergeometric series [35]. By Watson’s  ${}_8\phi_7$  transformation [35, Eq. (III.18)] (with  $a \rightarrow aq^{2j}, b \rightarrow a_4, c \rightarrow a/b, d \rightarrow \rho_1 q^j, e \rightarrow \rho_2 q^j$  and  $n \rightarrow L - j$ ) this can be simplified to

$$\beta'_L(a) = \sum_{j=0}^L \sum_{r=0}^{L-j} \frac{(\rho_1)_{j+r}(\rho_2)_{j+r}(aq/\rho_1\rho_2)^{j+r}(aq/\rho_1\rho_2)_{L-j-r} \alpha_j(b)}{(aq/\rho_1)_L(aq/\rho_2)_L(q)_{L-j-r}(q)_r(bq)_{r+2j}}.$$

Shifting  $r \rightarrow r - j$ , then interchanging sums and recalling the definition of  $\beta_r(b)$ , this indeed yields the second transformation claimed in the theorem.

The second proof proceeds in a similar manner. Substituting the expression for  $\alpha'(a)$  in the “primed” version of (1.3), transforming  $j \rightarrow r - j$  and then interchanging the order of summation, gives

$$\beta'_L(a) = \sum_{j=0}^L \frac{(\rho_1)_j (\rho_2)_j (bq/\rho_1 \rho_2)^j \alpha_j(b)}{(bq/\rho_1)_j (bq/\rho_2)_j (q)_{L-j} (bq)_{2j} (aq^{2j+1})_{L-j}} \\ \times \lim_{a_4 \rightarrow \infty} {}_6W_5(aq^{2j}, a_4, a/b, q^{-L+j}; q, bq^{L+j+2}/a_4).$$

By Rogers’  ${}_6\phi_5$  summation [35, Eq. (II.21)] (with  $a \rightarrow aq^{2j}$ ,  $b \rightarrow a_4$ ,  $c \rightarrow a/b$  and  $n \rightarrow L - j$ ) this can be simplified to

$$\beta'_L(a) = \sum_{j=0}^L \frac{(\rho_1)_j (\rho_2)_j (bq/\rho_1 \rho_2)^j \alpha_j(b)}{(bq/\rho_1)_j (bq/\rho_2)_j (q)_{L-j} (bq)_{L+j}}.$$

Using the  $q$ -Saalschütz sum [35, Eq. (II.12)] (with  $a \rightarrow \rho_1 q^j$ ,  $b \rightarrow \rho_2 q^j$ ,  $c \rightarrow bq^{2j+1}$  and  $n \rightarrow L - j$ ) this can be rewritten as

$$\beta'_L(a) = \sum_{j=0}^L \sum_{r=0}^{L-j} \frac{(\rho_1)_{j+r} (\rho_2)_{j+r} (bq/\rho_1 \rho_2)^{j+r} (bq/\rho_1 \rho_2)_{L-j-r} \alpha_j(b)}{(bq/\rho_1)_L (bq/\rho_2)_L (q)_{L-j-r} (q)_r (bq)_{r+2j}}.$$

Shifting  $r \rightarrow r - j$ , interchanging sums and recalling the definition (1.3) yields the second expression of Theorem 3.2.

To see that we are now in the position to prove (2.5) for all  $i = 1, \dots, k$  we follow [1] and take the Bailey pair of equation (2.3) with  $a = q$  as starting point. Applying (2.2)  $k - i + 1$  times, then Theorem 3.1 with  $N = 1$  and  $\rho_1, \rho_2 \rightarrow \infty$  once, and then again (2.2)  $i - 2$  times, one finds the Bailey pair  $\alpha_0 = 1$ ,

$$\alpha_L = (-1)^L q^{kL^2 + \binom{L}{2} + (k-i+1)L} (1 + q^{(2i-2k-1)L}) \\ \beta_L = \sum_{L \geq n_1 \geq \dots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \dots + n_{k-1}^2 + n_i + \dots + n_{k-1}}}{(q)_{L-n_1} (q)_{n_1-n_2} \dots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}},$$

relative to 1. Substituting this result into (1.3), letting  $L$  tend to infinity and using the triple product identity one arrives at the identities (2.5) for  $i = 2, \dots, k$ .

#### 4. THE BAILEY LEMMA IN STATISTICAL MECHANICS

In section 1 we already mentioned Slater’s famous list of 130 identities of Rogers–Ramanujan type [55, 56]. She found these identities by exploiting extensive lists of Bailey pairs (grouped from A to M) extracted from Rogers’ or Bailey’s papers [49, 16] or from known basic hypergeometric function identities. For example, the first group of Bailey pairs (all due to Rogers) reads  $\alpha_0 = 1$ ,

	$\beta_L$	$\alpha_{3L \pm 1}$	$\alpha_{3L}$
A(1)	$1/(q)_{2L}$	$-q^{(2L \pm 1)(3L \pm 1)}$	$q^{L(6L-1)} + q^{L(6L+1)}$
A(3)	$q^L/(q)_{2L}$	$-q^{2L(3L \pm 1)}$	$q^{2L(3L-1)} + q^{2L(3L+1)}$
A(5)	$q^{L^2}/(q)_{2L}$	$-q^{L(3L \pm 1)}$	$q^{L(3L-1)} + q^{L(3L+1)}$
A(7)	$q^{L^2-L}/(q)_{2L}$	$-q^{(L \pm 1)(3L \pm 1)}$	$q^{L(3L-2)} + q^{L(3L+2)}$

with  $a = 1$ , and

	$\beta_L$	$\alpha_{3L-(1\mp 1)/2}$	$\alpha_{3L+1}$
A(2)	$1/(q^2)_{2L}$	$q^{L(6L\pm 1)}$	$-q^{(2L+1)(3L+1)} - q^{(2L+1)(3L+2)}$
A(4)	$q^L/(q^2)_{2L}$	$q^{2L(3L\pm 2)}$	$-q^{2(L+1)(3L+1)} - q^{2L(3L+2)}$
A(6)	$q^{L^2}/(q^2)_{2L}$	$q^{L(3L\mp 1)}$	$-q^{L(3L+1)} - q^{(L+1)(3L+2)}$
A(8)	$q^{L^2+L}/(q^2)_{2L}$	$q^{L(3L\pm 2)}$	$-q^{(L+1)(3L+1)} - q^{L(3L+2)}$

with  $a = q$ . The Bailey pairs given in equations (1.7) and (1.10), which were used by Rogers to prove the Rogers–Ramanujan identities, are items labelled B(1) and B(3).

Remarkably, in recent work on exactly solvable lattice models in statistical mechanics identities have arisen (see [17, 30, 18, 58, 21] and references therein), which for each pair of integers  $(p, p')$ , with  $1 < p < p'$  and  $\gcd(p, p') = 1$  imply a family of Bailey pairs [31]. Moreover, many of the Bailey pairs of Rogers and Slater (as well as later pairs found in [4, 44, 1]) are included as special cases.

First we need a class of polynomials known as the one-dimensional configuration sums of the Andrews–Baxter–Forrester model [10, 32]. (See [9] for a partition theoretic interpretation of the configuration sums.) For coprime integers  $p, p'$  with  $1 \leq p < p'$ , and integers  $1 \leq b, s \leq p' - 1$ ,  $0 \leq r \leq p - 1$  and  $L \geq 0$  such that  $L + s + b$  is even, define

$$X_{r,s}^{(p,p')}(L, b) = \sum_{j \in \mathbb{Z}} \left\{ q^{j(pp'j + p'r - ps)} \left[ \frac{L+s-b}{2} - p'j \right] - q^{(pj+r)(p'j+s)} \left[ \frac{L-s-b}{2} - p'j \right] \right\}.$$

For  $r = b - \lfloor (b+1)(p' - p)/p' \rfloor$ , (with  $\lfloor x \rfloor$  the integer part of  $x$ ) the one-dimensional configuration sums are generating functions of certain sets of restricted lattice paths, and hence are polynomials with positive coefficients. This is not at all manifest from the above definition, and the identities referred to in the above claim a different, manifestly positive representation for the configuration sums. The simplest of these identities arise when  $|p'r - ps| = 1$  and  $b = s$  which we assume throughout the remainder of this section.

For the moment also assume that  $p < p' < 2p$ , and define nonnegative integers  $\nu_0, \dots, \nu_n$  by the continued fraction expansion  $p/(p' - p) = [\nu_0, \nu_1, \dots, \nu_n]$ . The integers  $n$  and  $\nu_j$ , can be used to further define  $t_m = \sum_{j=0}^{m-1} \nu_j$  ( $1 \leq m \leq n$ ) and  $d = -2 + \sum_{j=0}^n \nu_j$ . These latter numbers define a so-called fractional incidence matrix  $\mathcal{I}$  and fractional Cartan-type matrix  $B = 2I - \mathcal{I}$  (with  $I$  the  $d$  by  $d$  unit matrix) as follows

$$\mathcal{I}_{i,j} = \begin{cases} \delta_{i,j+1} + \delta_{i,j-1} & \text{for } 1 \leq i < d, i \neq t_m, \\ \delta_{i,j+1} + \delta_{i,j} - \delta_{i,j-1} & \text{for } i = t_m, 1 \leq m \leq n - \delta_{\nu_n, 2}, \\ \delta_{i,j+1} + \delta_{\nu_n, 2} \delta_{i,j} & \text{for } i = d. \end{cases}$$

When  $p' = p + 1$  the matrix  $\mathcal{I}$  has entries  $\mathcal{I}_{i,j} = \delta_{|i-j|, 1}$  ( $i, j = 1, \dots, p - 2$ ), so that  $B$  corresponds to the Cartan matrix of the Lie algebra  $A_{p-2}$ . When  $p = 2k - 1$  and  $p' = 2k + 1$  one finds  $\mathcal{I}_{i,j} = \delta_{|i-j|, 1} + \delta_{i,j} \delta_{i,k-1}$  ( $i, j = 1, \dots, k - 1$ ), so that  $B$  corresponds to the Cartan-type matrix of the tadpole graph of  $k - 1$  nodes.

Using the above definitions we have the following result [17, 30, 18, 58, 21]:

**Theorem 4.1.** *Let  $1 < p < p' < 2p$  with  $\gcd(p, p') = 1$  and let  $r(\leq p' - 1)$  and  $s(\leq p' - 1)$  satisfy  $|p'r - ps| = 1$ . Then*

$$(4.1) \quad X_{r,s}^{(p,p')}(2L, s) = \sum_{m \in 2\mathbb{Z}^d} q^{\frac{1}{4}mBm} \prod_{j=1}^d \left[ \frac{L\delta_{j,1} + \frac{1}{2}(\mathcal{I}m)_j}{m_j} \right].$$

Here we use the notation  $mBm = \sum_{j,k} m_j B_{j,k} m_k$  and  $(\mathcal{I}m)_j = \sum_k \mathcal{I}_{j,k} m_k$ . The corresponding identities for  $2p < p'$  follow simply from the symmetry

$$X_{r,s}^{(p,p')}(L, b; q) = q^{\frac{1}{4}(L^2 - (b-s)^2)} X_{b-r,s}^{(p'-p,p')}(L, b; 1/q).$$

Foda and Quano [31] used (special cases of) the above theorem and symmetry relation together with the Bailey lemma to prove conjectured  $q$ -series identities for Virasoro characters. Indeed, we can readily extract the following Bailey pairs relative to 1 [31, 22]:  $\alpha_0 = 1$ ,

$$(4.2) \quad \alpha_L = \begin{cases} q^{j(jpp'+rp'-sp)} + q^{j(jpp'-rp'+sp)} & \text{for } L = jp' > 0 \\ -q^{(jp \pm r)(jp' \pm s)} & \text{for } L = jp' \pm s > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_L = X_{r,s}^{(p,p')}(2L, s)/(q)_{2L},$$

where in the expression for  $\beta$  the representation (4.1) of  $X_{r,s}^{(p,p')}$  is taken. We note that  $(p, p') = (2, 3)$  (so that  $r = s = 1$ ) corresponds to the Bailey pair A(1) and  $(p, p') = (1, 3)$  ( $r = 0$  and  $s = 1$ ) to A(5). We also remark that  $(p, p') = (2, 5)$  ( $r = 1, s = 2$ ) is the Bailey pair [4, Eq. (5.3)].

## 5. CONJUGATE BAILEY PAIRS

We have just seen that each pair of coprime integers  $(p, p')$  labels a Bailey pair. Next we discuss some recent developments which show that a similar result holds for conjugate Bailey pairs [50, 51, 52].

First we need to introduce the string functions associated to admissible representations of the affine algebra  $A_1^{(1)}$  [38]. Again fix a pair of positive, coprime integers  $(p, p')$ . Let  $0 \leq \ell \leq p' - 2$  and let  $\Lambda_0, \Lambda_1$  denote the fundamental weights of  $A_1^{(1)}$ . Then Kac and Wakimoto showed that the  $A_1^{(1)}$  character of the admissible highest weight module of highest weight  $(p'/p - \ell - 2)\Lambda_0 + \ell\Lambda_1$  is given by a generalized Weyl–Kac formula as follows

$$\chi_\ell^{(p,p')}(z, q) = \frac{\sum_{\sigma=\pm 1} \sigma \Theta_{\sigma(\ell+1), p'}(z, q^p)}{\sum_{\sigma=\pm 1} \sigma \Theta_{\sigma, 2}(z, q)}.$$

Here  $\Theta_{n,m}$  is a classical theta function,  $\Theta_{n,m}(z, q) = \sum_{j \in \mathbb{Z} + n/2m} q^{mj^2} z^{-mj}$ . Note that when  $p > 1$  we are dealing with nonintegral highest weights. The (normalized)  $A_1^{(1)}$  string functions of level  $p'/p - 2$  are defined by the expansion

$$\chi_\ell^{(p,p')}(z, q) = q^{\frac{1}{8} - \frac{(\ell+1)^2 p}{4p'}} \sum_{m \in \mathbb{Z}} C_{m,\ell}^{(p,p')}(q) z^{-\frac{1}{2}m},$$

which immediately implies that  $C_{m,\ell}^{(p,p')}(q) = 0$  unless  $m + \ell$  is even. An explicit expression for the string functions can be derived as a double sum of Hecke indefinite



modular form type [37, 2, 52]

$$C_{m,\ell}^{(p,p')}(q) = \frac{1}{(q)_\infty^3} \left\{ \sum_{\substack{i \geq 0 \\ j \geq 0}} - \sum_{\substack{i < 0 \\ j < 0}} \right\} (-1)^i q^{\frac{1}{2}i(i+m)+p'j(pj+i)+\frac{1}{2}(\ell+1)(2pj+i)} \\ - \frac{1}{(q)_\infty^3} \left\{ \sum_{\substack{i \geq 0 \\ j > 0}} - \sum_{\substack{i < 0 \\ j \leq 0}} \right\} (-1)^i q^{\frac{1}{2}i(i+m)+p'j(pj+i)-\frac{1}{2}(\ell+1)(2pj+i)}.$$

After these preliminaries let us now return to the conjugate Bailey pair of equation (1.5) and specialize  $a = q^\eta$ , with  $\eta$  a nonnegative integer. Let us further remark the following identities ( $\ell = 0, 1$  and  $m + \ell \equiv L + \ell \equiv 0 \pmod{2}$ ):

$$C_{m,\ell}^{(1,3)}(q) = \frac{q^{\frac{1}{4}(m^2-\ell^2)}}{(q)_\infty} \quad \text{and} \quad X_{0,\ell+1}^{(1,3)}(L, 1) = q^{\frac{1}{4}(L^2-\ell)}.$$

The first result is [37, Sec. 4.6, Ex. 3] whereas the second is A(5) for  $\ell = 0$  and A(8) for  $\ell = 1$ . We thus infer that the conjugate Bailey pair (1.5) can be recast as  $\gamma_L = (q)_\eta C_{2L+\eta,\ell}^{(1,3)}(q)$  and  $\delta_L = X_{0,\ell+1}^{(1,3)}(2L + \eta, 1)$ . It now requires little imagination to conjecture the following more general result [52].

**Theorem 5.1.** *Fix integers  $1 \leq p < p'$ , and let  $\eta$  and  $\ell$  be nonnegative integers such that  $0 \leq \ell \leq p' - 2$  and  $\ell + \eta$  is even. Then*

$$(5.1) \quad \gamma_L = (q)_\eta C_{2L+\eta,\ell}^{(p,p')}(q) \quad \text{and} \quad \delta_L = X_{0,\ell+1}^{(p,p')}(2L + \eta, 1)$$

*yields a conjugate Bailey pair relative to  $a = q^\eta$ .*

The proof of this theorem relies on yet another class of conjugate Bailey pairs given by [52, Thm. 4.1]

$$(5.2) \quad \gamma_L = \frac{1}{(q)_\infty^2 (aq)_\infty} \sum_{i=1}^{\infty} (-1)^i q^{\frac{1}{2}i(i+2L+\eta)} \left\{ q^{\frac{1}{2}i(2j+\eta+1)} - q^{-\frac{1}{2}i(2j+\eta+1)} \right\} \\ \delta_L = \left[ \begin{matrix} 2L+\eta \\ L-j \end{matrix} \right] - \left[ \begin{matrix} 2L+\eta \\ L-j-1 \end{matrix} \right],$$

with  $a = q^\eta$ ,  $\eta$  an nonnegative integer and  $j$  an integer. Here we remark that, incidentally,  $\delta_L = K_{(2L-j, 1^{2j+\eta}), (1^{2L+\eta})}(q) = \tilde{K}_{(L+j+\eta, L-j), (1^{2L+\eta})}(q)$ , where  $K_{\lambda,\mu}(q)$  and  $\tilde{K}_{\lambda,\mu}(q)$  are the Kostka and cocharge Kostka polynomial, respectively. The Bailey pair (5.2) can easily be derived from the summation formula [52]

$$\sum_{r=0}^{\infty} \frac{q^r (ab)_{2r}}{(q)_r (ab)_r (aq)_r (bq)_r} = \frac{1}{(q)_\infty (aq)_\infty (bq)_\infty} \sum_{i=1}^{\infty} (-1)^{i-1} q^{\binom{i}{2}} \frac{a^i - b^i}{a - b}.$$

It is again possible to give representations of the polynomials  $X_{0,\ell+1}^{(p,p')}(L, 1)$  that are manifestly positive [52]. Treating only the simplest cases we have the following counterpart of Theorem 4.1.

**Theorem 5.2.** *Let  $1 < p < p' < 2p$  with  $\gcd(p, p') = 1$ . Then*

$$X_{0,1}^{(p,p')}(2L, 1) = q^L \sum_{m \in 2\mathbb{Z}^d} q^{\frac{1}{4}mBm + \frac{1}{2} \sum_{i=1}^n m_{\tau_i}} \prod_{j=1}^d \left[ \begin{matrix} L\delta_{j,1} - \sum_{i=1}^n \delta_{j,\tau_i} + \frac{1}{2}(\mathcal{I}m)_j \end{matrix} \right]$$

The corresponding identities for  $2p < p'$  follow from

$$X_{r,s}^{(p'-p,p')}(2L, 1; q) = q^{L(L+1)} X_{r,s}^{(p,p')}(2L, 1; 1/q).$$

Combining the Bailey pair of equation (4.2) with the conjugate Bailey pair of (5.1) and specializing some of the parameters, we find

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \left\{ q^{j(jp_1 p'_1 + r p'_1 - s p_1)} C_{2j p'_1, 0}^{(p_2, p'_2)}(q) - q^{(j p_1 + r)(j p'_1 + s)} C_{2j p'_1 + 2s, 0}^{(p_2, p'_2)}(q) \right\} \\ = \sum_{L=0}^{\infty} X_{r,s}^{(p_1, p'_1)}(2L, s) X_{0,1}^{(p_2, p'_2)}(2L, 1) / (q)_{2L}, \end{aligned}$$

with  $1 \leq p_i < p'_i$  ( $i = 1, 2$ ) and  $|p'_1 r - p_1 s| = 1$ . Here we have used the symmetry  $C_{m,\ell}^{(p,p')} = C_{-m,\ell}^{(p,p')}$ . Inserting the representations for the one-dimensional configuration sums provided by Theorems 4.1 and 5.2 this turns into a class of rather nontrivial  $q$ -series identities. For  $p_1 = 1$  or  $p_2 = 1$  the left-hand side of the above equation can be identified as a branching function of the coset pair  $(A_1^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$  at levels  $N_1 = p_1/p_1 - 2$ ,  $N_2 = p'_2/p_2 - 2$  and  $N_1 + N_2$ , respectively.

## 6. FURTHER READING

To conclude our overview of half a century of Bailey's lemma, let us mention some further papers on (or related to) the Bailey lemma that have not been mentioned in the main text. In [45], Paule gave a short operator-type proof of the special case (2.2) of the Bailey chain. Riese [47] developed the Mathematica package `Bailey` for taking (automated) walks along the Bailey lattice. He also shows how to apply his Mathematica package `qZeil` to generate Bailey pairs. The Bailey transform (1.3) and its inverse (2.4) can be formulated naturally in terms of inversion of infinite-dimensional lower-triangular matrices [1, 24], making it a special case of the generalized  $q$ -Lagrange procedure of Gessel and Stanton [36]. New types of Bailey lattice transformations which do not only change the value of  $a$  but also that of the base  $q$ , were very recently found and applied by Bressoud, Ismail and Stanton [25]. Bressoud [23] and Singh [54] have also applied conjugate Bailey pairs other than (1.4) and (1.5) of Bailey. For a special choice of parameters their conjugate pair can be shown to coincide with the  $(p, p') = (2, 3)$  case of Theorem 5.1. Andrews [6], Andrews and Hickerson [13] and Choi [27] applied the Bailey chain to prove identities for Ramanujan's mock theta functions, and Andrews [8] also used it to prove several of Ramanujan's identities for Lambert series. The Bailey lemma and its connection to  $N = 2$  supersymmetric conformal field theory was investigated by Berkovich, McCoy and Schilling [20]. For those left with the impression that the Bailey lemma is "merely" good for proving  $q$ -series identities we remark that Andrews utilized the Bailey machinery in [6, 7] to give a proof of Gauss' theorem that every integer can be written as the sum of three triangular numbers and that Andrews, Dyson and Hickerson used Bailey's lemma in the context of algebraic number theory [12]. Finally we mention that a special case of the Bailey chain admits an extension due to Burge [26]. This was extensively applied and further developed by Foda, Lee and Welsh [29] and by Schilling and the author [53].

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**Note added in proof.** The many recent references to the Bailey lemma listed in the bibliography show that after 50 years Bailey's lemma still is a source of inspiration. This makes it quite impossible to publish an account that can claim to be complete and up to date. Indeed, after this paper was accepted for publication further advances in connection with the lemma were reported in [33, 34, 61].

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