# CONJUGATE BAILEY PAIRS

## FROM CONFIGURATION SUMS AND FRACTIONAL-LEVEL STRING FUNCTIONS TO BAILEY'S LEMMA

ANNE SCHILLING AND S. OLE WARNAAR

Abstract. In this paper it is shown that the one-dimensional configuration sums of the solvable lattice models of Andrews, Baxter and Forrester and the string functions associated with admissible representations of the affine Lie algebra $\mathrm{A}_1^{(1)}$ as introduced by Kac and Wakimoto can be exploited to yield a very general class of conjugate Bailey pairs. Using the recently established fermionic or constant-sign expressions for the one-dimensional configuration sums, our result is employed to derive fermionic expressions for fractional-level string functions, parafermion characters and $\mathrm{A}_1^{(1)}$ branching functions. In addition, $q$-series identities are obtained whose Lie algebraic and/or combinatorial interpretation is still lacking.

## 0. Notation

Throughout the paper the following notation is used. $\mathbb{N}$ are the positive integers, $\mathbb{Z}_+$ the nonnegative integers, $\mathbb{N}_p = \{1,\dots,p\}$, $\mathbb{Z}_p = \{0,\dots,p-1\}$. For $n \in \mathbb{Z}$, $\binom{n}{2} = n(n-1)/2$.

## 1. The Bailey lemma

In an attempt to clarify Rogers' second proof [60] of the Rogers–Ramanujan identities, Bailey [19] was led to the following simple but important observation.

**Lemma 1.1.** *If $\alpha = \{\alpha_L\}_{L\geq 0},\dots,\delta = \{\delta_L\}_{L\geq 0}$ are sequences that satisfy*

$$\beta_L = \sum_{r=0}^{L} \alpha_r u_{L-r} v_{L+r} \quad \textit{and} \quad \gamma_L = \sum_{r=L}^{\infty} \delta_r u_{r-L} v_{r+L}, \tag{1.1}$$

*then*

$$\sum_{L=0}^{\infty} \alpha_L \gamma_L = \sum_{L=0}^{\infty} \beta_L \delta_L. \tag{1.2}$$

The proof is straightforward and merely requires an interchange of sums. Of course, in the above suitable convergence conditions need to be imposed to make the definition of $\gamma$ and the interchange of sums meaningful.

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In applications of his transform, Bailey chose $u_L = 1/(q)_L$ and $v_L = 1/(aq)_L$, with the usual definition of the $q$-raising factorial,

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

and

$$(a)_L = (a;q)_L = (a)_\infty/(aq^L)_\infty$$

for all $L \in \mathbb{Z}$. With this choice, equation (1.1) reads

$$\beta_L = \sum_{r=0}^{L} \frac{\alpha_r}{(q)_{L-r}(aq)_{L+r}} \tag{1.3}$$

and

$$\gamma_L = \sum_{r=L}^{\infty} \frac{\delta_r}{(q)_{r-L}(aq)_{r+L}}. \tag{1.4}$$

A pair of sequences that satisfies (1.3) is called a Bailey pair relative to $a$. Similarly, a pair satisfying (1.4) is called a conjugate Bailey pair relative to $a$.

Still following Bailey, one can employ the $q$-Saalschütz summation [43, Eq. (II.12)] to establish that $(\gamma,\delta)$ with

$$\begin{aligned}\gamma_L &= \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L}\frac{1}{(q)_{M-L}(aq)_{M+L}}\\ \delta_L &= \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_M(aq/\rho_2)_M}\frac{(aq/\rho_1\rho_2)_{M-L}}{(q)_{M-L}}\end{aligned} \tag{1.5}$$

provides a conjugate Bailey pair.

Unfortunately, Bailey outrightly rejected the above conjugate Bailey pair as too complicated to yield any results of interest and focussed on the simpler case obtained by letting $M$ go to infinity. Doing so as well as letting the indeterminates $\rho_1$ and $\rho_2$ tend to infinity yields

$$\gamma_L = \frac{a^L q^{L^2}}{(aq)_\infty} \quad \text{and} \quad \delta_L = a^L q^{L^2}, \tag{1.6}$$

which substituted into (1.2) gives

$$\frac{1}{(aq)_\infty}\sum_{L=0}^{\infty} a^L q^{L^2}\alpha_L = \sum_{L=0}^{\infty} a^L q^{L^2}\beta_L. \tag{1.7}$$

The proof of the Rogers–Ramanujan and many similar such $q$-series identities requires the input of suitable Bailey pairs into (1.7). For example, from Rogers' work [60] one can infer the following Bailey pair relative to 1: $\alpha_0 = 1$ and

$$\alpha_L = (-1)^L q^{L(3L-1)/2}(1+q^L), \qquad \beta_L = \frac{1}{(q)_L}. \tag{1.8}$$

Thus one finds

$$\frac{1}{(q)_\infty}\sum_{L=-\infty}^{\infty}(-1)^L q^{L(5L-1)/2} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}.$$

The application of the Jacobi triple product identity

$$\sum_{r=-\infty}^{\infty}(-1)^r x^r q^{\binom{r}{2}} = (x, q/x, q)_\infty \tag{1.9}$$

yields the first Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n} = \frac{1}{(q, q^4; q^5)_\infty}.$$

Here and later in the paper we employ the condensed notation

$$(a_1, \dots, a_k; q)_n = (a_1, \dots, a_k)_n = (a_1)_n \dots (a_k)_n.$$

The second Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}$$

follows in a similar fashion using the Bailey pair [60]

$$\alpha_L = (-1)^L q^{L(3L+1)/2}(1-q^{2L+1})/(1-q), \qquad \beta_L = \frac{1}{(q)_L}$$

relative to $q$. By collecting as many Bailey pairs as possible, Slater compiled a list of over a hundred Rogers–Ramanujan-type identities [65, 66]. (Apart from a few exceptions Slater either used (1.7) or the analogous identity obtained from (1.5) and (1.2) by taking $M, \rho_1 \to \infty$ and letting $\rho_2 = -q^{k/2}$ with $k$ a small nonnegative integer.)

By dismissing the conjugate Bailey pair (1.5) Bailey missed a very powerful mechanism for generating Bailey pairs. Namely, if we substitute the conjugate pair (1.5) into (1.2) the resulting equation has the same form as the defining relation (1.3) of a Bailey pair. This is formalized in the following theorem due to Andrews [10, 11].

**Theorem 1.2.** *Let $(\alpha, \beta)$ form a Bailey pair relative to $a$. Then so does $(\alpha', \beta')$ with*

$$\begin{aligned}\alpha'_L &= \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L}\alpha_L\\ \beta'_L &= \sum_{r=0}^{L}\frac{(\rho_1)_r(\rho_2)_r(aq/\rho_1\rho_2)^r(aq/\rho_1\rho_2)_{L-r}}{(aq/\rho_1)_L(aq/\rho_2)_L(q)_{L-r}}\beta_r.\end{aligned} \tag{1.10}$$

Again letting $\rho_1, \rho_2$ tend to infinity leads to the important special case

$$\alpha'_L = a^L q^{L^2}\alpha_L \quad \text{and} \quad \beta'_L = \sum_{r=0}^{L}\frac{a^r q^{r^2}}{(q)_{L-r}}\beta_r, \tag{1.11}$$

which was also discovered by Paule [57] for $a = 1$ and $a = q$.

With this last result one finds that the Bailey pair of equation (1.8) can be obtained by application of (1.11) with initial Bailey pair $\alpha_0 = 1$ and

$$\alpha_L = (-1)^L q^{\binom{L}{2}}(1+q^L), \qquad \beta_L = \delta_{L,0} \tag{1.12}$$

relative to 1. Here $\delta_{i,j}$ is the Kronecker-delta symbol. The Bailey pair (1.12) follows after setting $x=1$ in the $q$-binomial sum

$$\sum_{r=-L}^{L}(-1)^r x^r q^{\binom{r}{2}}\begin{bmatrix}2L\\L-r\end{bmatrix}=(x,q/x)_L,$$

where throughout this paper the following definition of the $q$-binomial coefficient or Gaussian polynomial is used

$$\begin{bmatrix}n\\m\end{bmatrix}=\frac{(q^{n-m+1})_m}{(q)_m}$$

for $m\in\mathbb{Z}_+$ and zero otherwise.

At this point one may wonder why Bailey and Slater put so much emphasis on finding new Bailey pairs, but contented themselves with just the single conjugate Bailey pair (1.4). After all, the defining relations (1.3) and (1.4) are very similar and it is therefore not unreasonable to expect that conjugate pairs are as important and as numerous as ordinary Bailey pairs. In fact, because Andrews' Theorem 1.2 is equivalent to equation (1.2) with conjugate Bailey pair (1.5), in modern expositions of the Bailey lemma there often is no mention of conjugate Bailey pairs and equation (1.2) at all, see e.g., Refs. [2, 10, 12, 13, 14, 17, 22, 30, 33, 40, 58, 59]. Instead, (1.10) is referred to as the Bailey lemma and in the spirit of Slater, all focus is on finding interesting Bailey pairs. These are then either iterated using (1.10) or (1.11) to yield what is called a Bailey chain, or directly substituted into (1.7). The only exception that we were able to trace in the literature is the conjugate Bailey pair ($|r|<1$)

$$\gamma_L=\frac{r^L}{(r)_\infty(aq)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(r)_k(aq^{2L+1})^k}{(q)_k}\quad\text{and}\quad\delta_L=r^L \tag{1.13}$$

which can be found in the work by Bressoud [29] and Singh [64] (and which for $r=q$ we will meet again in Section 6).

This paper intends to revive the interest in conjugate Bailey pairs. In our earlier papers [61, 62] we made a first step towards this goal by proving an infinite series of conjugate Bailey pairs generalizing (1.6). Here we develop the theory of conjugate Bailey pairs much further, exploiting the connection of Bailey's lemma with integrable systems and Lie algebras. We show that appropriate series of one-dimensional configuration sums and $\mathrm{A}_1^{(1)}$ string functions can be identified with the series $\delta$ and $\gamma$ defining a conjugate Bailey pair. Here one-dimensional configuration sums [16, 42], also known as hook-partition generating functions [15], are polynomials that have arisen in statistical mechanics and partition theory. A well-known example are the polynomials introduced by Schur [63] in his famous proof of the Rogers–Ramanujan identities. The string functions that occur are associated to the admissible representations of the affine Kac–Moody algebra $\mathrm{A}_1^{(1)}$ as introduced by Kac and Wakimoto [49, 50].

Before we carry out the above program let us attempt to give an explanation of the origin of our findings. An important notion in the theory of affine Lie algebras is that of branching functions [47]. Here we consider the branching functions $B^{N_1,N_2}$ associated to $(\mathrm{A}_1^{(1)}\oplus\mathrm{A}_1^{(1)},\mathrm{A}_1^{(1)})$ at levels $N_1$, $N_2$ and $N_1+N_2$, respectively, where $N_1$ and $N_2$ are rational numbers such that either $N_1$ or $N_2$ is a positive integer. The branching functions obey the symmetry $B^{N_1,N_2}=B^{N_2,N_1}$. Following the work of Andrews [10] and Foda and Quano [40], the infinite hierarchy of conjugate Bailey pairs of [61, 62] were used in [24] to derive $q$-series identities for the $\mathrm{A}_1^{(1)}$ branching functions. Schematically the results of [24] read as follows:

$$B^{N_1,N_2}=\sum_{L=0}^{\infty}\alpha_L^{(N_1)}\gamma_L^{(N_2)}=\sum_{L=0}^{\infty}\beta_L^{(N_1)}\delta_L^{(N_2)}, \tag{1.14}$$

where $N_1$ is rational, $N_2$ integer, $\gamma_L^{(N_2)}$ is a level-$N_2$ string function, $\beta_L^{(N_1)}$ a (normalized) one-dimensional configuration sum and $(\alpha^{(N_1)},\beta^{(N_1)})$, $(\gamma^{(N_2)},\delta^{(N_2)})$ are a Bailey and conjugate Bailey pair respectively. In the middle term of this identity the symmetry between $N_1$ and $N_2$ is not at all manifest since it involves only the (integer) level-$N_2$ string functions and not the (fractional) level-$N_1$ string functions. This suggests that there should be more general conjugate Bailey pairs such that one can also derive

$$B^{N_1,N_2}=\sum_{L=0}^{\infty}\bar{\alpha}_L^{(N_2)}\bar{\gamma}_L^{(N_1)}=\sum_{L=0}^{\infty}\bar{\beta}_L^{(N_2)}\bar{\delta}_L^{(N_1)}, \tag{1.15}$$

where now $\bar{\gamma}_L^{(N_1)}$ is a fractional-level string function, $(\bar{\alpha}^{(N_2)},\bar{\beta}^{(N_2)})$, $(\bar{\gamma}^{(N_1)},\bar{\delta}^{(N_1)})$ are a Bailey and conjugate Bailey pair, and such that, manifestly,

$$\sum_{L=0}^{\infty}\beta_L^{(N_1)}\delta_L^{(N_2)}=\sum_{L=0}^{\infty}\bar{\beta}_L^{(N_2)}\bar{\delta}_L^{(N_1)}.$$

This last equation is obviously satisfied if

$$\bar{\delta}_L^{(N_1)}=g_L\beta_L^{(N_1)} \qquad \text{and} \qquad \delta_L^{(N_2)}=g_L\bar{\beta}_L^{(N_2)}, \tag{1.16}$$

with $g_L$ independent of $N_1$ and $N_2$. Since $\beta_L^{(N_1)}$ is a (normalized) one-dimensional configuration sum we can now conclude that in the "yet to be found" conjugate Bailey pair $(\bar{\gamma}^{(N_1)},\bar{\delta}^{(N_1)})$ the sequence $\bar{\gamma}^{(N_1)}$ is a sequence of (fractional) level-$N_1$ string functions and the sequence $\bar{\delta}^{(N_1)}$ is proportional to a sequence of one-dimensional configuration sums. This is indeed in accordance with the announced results. We also note that the above discussion establishes a duality between Bailey and conjugate Bailey pairs through equation (1.16).

The remainder of the paper can be outlined as follows. In the next two sections we review the one-dimensional configuration sums of the Andrews–Baxter–Forrester models and the string functions associated with admissible representations of the affine Lie algebra $\mathrm{A}_1^{(1)}$. In Section 4 these are used to prove a very general class of conjugate Bailey pairs stated in Corollary 4.2. In Section 5 we give fermionic or constant-sign expressions for the one-dimensional configuration sums. This allows us to apply the Bailey lemma, together with our new conjugate Bailey pairs, to derive many new $q$-series results in Sections 6 and 8. In Section 6 we give fermionic formulas for the fractional-level $\mathrm{A}_1^{(1)}$ string functions and parafermion characters. In Section 8 we derive a new type of bose-fermi identities extending identities of the form (1.14) and (1.15) for the $\mathrm{A}_1^{(1)}$ branching functions by allowing for both $N_1$ and $N_2$ to be rational numbers. To put this in the right context we first present a discussion of the $\mathrm{A}_1^{(1)}$ branching functions in Section 7 proving a generalization of a theorem of Kac and Wakimoto that expresses the branching functions in terms of fractional-level string functions in accordance with (1.15).

## 2. One-dimensional configuration sums

The one-dimensional configuration sums of the Andrews–Baxter–Forrester models were introduced in several stages in Refs. [63, 9, 16, 42].

**Definition 2.1.** For integers $p, p'$ with $1 \le p < p'$, and $b, s \in \mathbb{N}_{p'-1}$, $r \in \mathbb{Z}_{p+1}$ and $L \in \mathbb{Z}_+$ such that $L+s+b$ is even, let

$$X^{(p,p')}_{r,s}(L,b;q) = X^{(p,p')}_{r,s}(L,b) \tag{2.1}$$
$$= \sum_{j\in\mathbb{Z}} \Big\{ q^{j(pp'j+p'r-ps)} \Big[{L \atop (L+s-b)/2-p'j}\Big] - q^{(pj+r)(p'j+s)} \Big[{L \atop (L-s-b)/2-p'j}\Big] \Big\}.$$

The configuration sums possess two symmetries which will be used later. From the definition it can be deduced immediately that

$$X^{(p,p')}_{r,s}(L,b) = X^{(p,p')}_{p-r,p'-s}(L,p'-b), \tag{2.2}$$

whereas

$$X^{(p,p')}_{r,s}(L,b;q) = q^{\frac{1}{4}(L^2-(b-s)^2)} X^{(p'-p,p')}_{b-r,s}(L,b;1/q) \tag{2.3}$$

follows by application of

$$\begin{bmatrix} n \\ m \end{bmatrix}_{1/q} = q^{m(m-n)} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{2.4}$$

When the parameters $p$ and $p'$ obey the additional restriction

$$\gcd(p,p') = 1 \tag{2.5}$$

the polynomials (2.1) were encountered by Forrester and Baxter [42] as the generating function of sets of restricted lattice path. Below we describe a slight extension of their result. A lattice path interpretation of the one-dimensional configuration sums $X^{(p,p')}_{r,s}(L,b;q)$ for all $1 \le p < p'$ can be found in [38].

Let $P = (x_0, \dots, x_{L+1})$ be a lattice path consisting of an ordered sequence of $L+2$ integers such that $|x_{i+1} - x_i| = 1$ for $0 \le i \le L$, $x_0 = s$, $x_L = b$, $x_{L+1} = c$ and $x_i \in \mathbb{N}_{p'-1}$ for $1 \le i \le L$. Denote the set of all such paths by $\mathcal{P}^{s,b,c}_L$. Assign a weight $|P|$ to $P \in \mathcal{P}^{s,b,c}_L$ as follows

$$|P| = \sum_{i=1}^{L} iH(x_{i-1}, x_i, x_{i+1}),$$

where

$$H(a, a\mp 1, a) = \pm \Big\lfloor \frac{a(p'-p)}{p'} \Big\rfloor \quad \text{and} \quad H(a\pm 1, a, a\mp 1) = \frac{1}{2}.$$

Here $\lfloor x \rfloor$ denotes the integer part of $x$. Forrester and Baxter studied the generating function

$$D_L(s,b,c;q) = \sum_{P \in \mathcal{P}^{s,b,c}_L} q^{|P|}$$

and proved for $c \in \mathbb{N}_{p'-1}$ that [42, Thm 2.3.1]

$$D_L(s,b,c;q) = q^{\frac{1}{4}L(c-b)(c+b-1-2r) + \frac{1}{4}(s-b)(s+b-1-2r)} X^{(p,p')}_{r,s}(L,b), \tag{2.6}$$

where $r$ is given by

$$r = \frac{b+c-1}{2} - \Big\lfloor \frac{c(p'-p)}{p'} \Big\rfloor \tag{2.7}$$

$$= \frac{b-c+1}{2} + \Big\lfloor \frac{cp}{p'} \Big\rfloor. \tag{2.8}$$

For $p' = p+1$ this result was first obtained in [16].

Later in this paper the configuration sum $X_{0,s}^{(p,p')}(L,1)$ will play a prominent role. Using the standard $q$-binomial recurrences

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$$

it readily follows that

$$\begin{bmatrix} L \\ a \end{bmatrix} - \begin{bmatrix} L \\ a-1 \end{bmatrix} = q^a \begin{bmatrix} L \\ a \end{bmatrix} - q^{L-a+1} \begin{bmatrix} L \\ a-1 \end{bmatrix}.$$

One thus finds the relation

$$X_{0,s}^{(p,p')}(L,1) = q^{\frac{1}{2}(L-s+1)} X_{1,s}^{(p,p')}(L,1). \tag{2.9}$$

The corresponding lattice path interpretation for $X_{0,s}^{(p,p')}(L,1)$ is easily found. When $p' > 2p$ it is included in the Forrester–Baxter result (2.6) since $b=1$ and $c=2$ yields $r=0$. When $p' < 2p$ we need to allow for paths with $c=0$. Then $b=1$ and, using (2.7), $r=0$. To see that the corresponding generating function is indeed

$$D_L(s,1,0;q) = q^{\frac{1}{4}s(s-1)} X_{0,s}^{(p,p')}(L,1) \tag{2.10}$$

we compute $D_L(s,1,2;q)/D_L(s,1,0;q)$. On the one hand, by the one-to-one correspondence $(s,x_2,\dots,x_{L-2},2,1,2) \leftrightarrow (s,x_2,\dots,x_{L-2},2,1,0)$ between paths in $\mathcal{P}_L^{s,1,2}$ and $\mathcal{P}_L^{s,1,0}$, and the fact that $H(2,1,2)=0$ and $H(2,1,0)=1/2$ one finds $D_L(s,1,2;q)/D_L(s,1,0;q) = q^{-L/2}$. On the other hand, by (2.6) and (2.9) we get

$$\frac{D_L(s,1,2;q)}{D_L(s,1,0;q)} = q^{\frac{1}{4}(s-1)(s-2)} \frac{X_{1,s}^{(p,p')}(L,1)}{D_L(s,1,0;q)} = q^{\frac{1}{4}s(s-1)-\frac{1}{2}L} \frac{X_{0,s}^{(p,p')}(L,1)}{D_L(s,1,0;q)}.$$

Combining the last two results clearly implies (2.10).

By the symmetry (2.2) we also need $X_{p,s}^{(p,p')}(L,p'-1)$. For $p' > 2p$ its lattice path interpretation follows again from the Forrester–Baxter result, as $b = p'-1$ and $c = p'-2$ yields $r=p$. When $p' < 2p$ we need to allow for paths with $c=p'$. Then $b=p'-1$ and, using (2.8), $r=p$. By a calculation similar to the one above it is then readily shown that $D_L(s,p'-1,p';q)$ is indeed given by (2.6).

The expressions (2.1) have also been studied extensively in the theory of partitions, see e.g. [5, 15, 26, 31, 32, 44]. Here we quote the most general result, obtained in [15]. Let $\lambda$ be a partition and $\lambda'$ its conjugate. The $(i,j)$-th node of $\lambda$ is the node (or box) in the $i$th row and $j$th column of the Ferrers diagram of $\lambda$. The $d$th diagonal of $\lambda$ is formed by the nodes with coordinates $(i,i-d)$. The hook difference at node $(i,j)$ is defined as $\lambda_i - \lambda'_j$. Theorem 1 of [15] states that the generating function of partitions $\lambda$ with at most $(L+s-b)/2$ parts, largest part not exceeding $(L-s+b)/2$, and hook differences on the $(1-r)$th diagonal at least $r-s+1$ and on the $(p-r-1)$th diagonal at most $p'-p+r-s-1$ is given by $X_{r,s}^{(p,p')}(L,b)$. Here the following two conditions apply [15], $1 \le r \le p-1$ and

$0 \le b-r \le p'-p$. When $r=0$ one has to impose the additional condition that the largest part exceeds $(L-s-b)/2$. Similarly, the case $r=p$ can be included provided one demands that the number of parts exceeds $(L+s+b)/2$.

## 3. Characters and string functions for $\mathrm{A}_1^{(1)}$

In [49, 50] Kac and Wakimoto introduced admissible highest weight representations of affine Lie algebras as generalizations of the familiar integrable highest weight representations [47]. Let $p, p'$ be integers such that $1 \le p < p'$ and $\gcd(p,p')=1$, and define

$$N = p'/p - 2$$

so that $-1 < N < 0$ for $p < p' < 2p$ and $N > 0$ for $p' > 2p$. Let $\Lambda_0$ and $\Lambda_1$ be the fundamental weights of $\mathrm{A}_1^{(1)}$. Fix an integer $\ell \in \mathbb{Z}_{p'-1}$ and let $L(\lambda)$ be an admissible $\mathrm{A}_1^{(1)}$ highest weight module of highest weight[1] $\lambda = (N-\ell)\Lambda_0 + \ell\Lambda_1$. The corresponding character is formally defined as

$$\chi_\ell^N(z,q) = \chi_\ell(z,q) = \mathrm{tr}_{L(\lambda)} q^{s_\lambda - d} z^{-\frac{1}{2}\alpha_1^\vee},$$

where $d=3$ is the dimension of $\mathrm{A}_1$, $\alpha_1^\vee$ is a simple coroot and

$$s_\lambda = -\frac{1}{8} + \frac{(\ell+1)^2}{4(N+2)}.$$

In terms of the classical theta function

$$\Theta_{n,m}(z,q) = \sum_{j \in \mathbb{Z}+n/2m} q^{mj^2} z^{-mj} \tag{3.1}$$

of degree $m$ and characteristic $n$, one can express the $\mathrm{A}_1^{(1)}$ character as

$$\chi_\ell(z,q) = \frac{\sum_{\sigma=\pm 1} \sigma \Theta_{\sigma(\ell+1),p'}(z,q^p)}{\sum_{\sigma=\pm 1} \sigma \Theta_{\sigma,2}(z,q)}. \tag{3.2}$$

In (3.1) and elsewhere in the paper we use the notation $\sum_{j \in n\mathbb{Z}+a}$ for a sum over all $j$ such that $j - a \equiv 0 \pmod{n}$.

The level-$N$ $\mathrm{A}_1^{(1)}$ string functions are defined by the expansion

$$\chi_\ell(z,q) = \sum_{m \in 2\mathbb{Z}+\ell} C_{m,\ell}^N(q) q^{\frac{m^2}{4N}} z^{-\frac{1}{2}m}, \tag{3.3}$$

and enjoy the symmetry

$$C_{m,\ell}^N = C_{-m,\ell}^N. \tag{3.4}$$

When $N$ is integer we furthermore have

$$C_{m,\ell}^N = C_{2N-m,\ell}^N = C_{N-m,N-\ell}^N \tag{3.5}$$

so that (3.3) may be put in the familiar form

$$\chi_\ell(z,q) = \sum_{\substack{0 \le m < 2N \\ m+\ell \text{ even}}} C_{m,\ell}^N(q) \Theta_{m,N}(z,q).$$

[1] Kac and Wakimoto considered the more general case $\lambda = (N-\ell)\Lambda_0 + \ell\Lambda_1 + k(N+2)(\Lambda_0 - \Lambda_1)$ with $k \in \mathbb{Z}_p$.

We derive an expression for the string functions following the approach of e.g., Refs. [45, 3]. First observe that

$$\sum_{\sigma=\pm 1} \sigma \Theta_{\sigma,2}(z,q) = q^{1/8} z^{-\frac{1}{2}} \sum_{j\in\mathbb{Z}} (-1)^j q^{\binom{j}{2}} z^j = q^{1/8} z^{-\frac{1}{2}} (z, q/z, q)_\infty$$

where in the second step Jacobi's triple product identity (1.9) has been employed. Next recall the identity

$$\frac{1}{(z,q/z)_\infty} = \frac{1}{(q)_\infty^2} \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} (-1)^{i+1} q^{\binom{i}{2}-ik} z^k,$$

which can be extracted from an expansion of the following ratio of Jacobi theta functions $\vartheta_1'(0)/\vartheta_1(u)$ in [67, §486] (see also [48, Eq. (5.26)] and [68, Eqs. (A.4), (A.5)]). Using this we find that

$$\chi_\ell(z,q) = \frac{1}{\eta^3(\tau)} \sum_{\sigma=\pm 1} \sum_{j,k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \sigma (-1)^{i+1} q^{\binom{i}{2}-ik+pp'(j+\sigma(\ell+1)/(2p'))^2} \\ \times z^{-\frac{1}{2}(2p'j-2k+\sigma(\ell+1)-1)},$$

where, as usual, $\eta(\tau) = q^{1/24}(q)_\infty$ with $q = \exp(2\pi i \tau)$. Now replace $j$ by $\sigma j$ and then $k$ by $\frac{1}{2}(2\sigma p'j - m - 1 + \sigma(\ell+1))$. This yields

$$\chi_\ell(z,q) = \frac{1}{\eta^3(\tau)} \sum_{m\in 2\mathbb{Z}+\ell} \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{N}} (-1)^i q^{\frac{1}{2}i(i+m)+pp'(j+(\ell+1)/(2p'))^2} \\ \times \Big\{ q^{\frac{1}{2}i(2p'j+\ell+1)} - q^{-\frac{1}{2}i(2p'j+\ell+1)} \Big\} z^{-\frac{1}{2}m}.$$

Comparing this with (3.3) one can extract the string functions as

$$C^N_{m,\ell}(q) = \frac{q^{\frac{(\ell+1)^2}{4(N+2)}-\frac{m^2}{4N}}}{\eta^3(\tau)} \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{N}} (-1)^i q^{\frac{1}{2}i(i+m)+jp(p'j+\ell+1)} \\ \times \Big\{ q^{\frac{1}{2}i(2p'j+\ell+1)} - q^{-\frac{1}{2}i(2p'j+\ell+1)} \Big\}. \tag{3.6}$$

We slightly extend the original definition of the string functions given in equation (3.3) by dropping the condition $\gcd(p,p') = 1$. Also normalizing for later convenience we are led to the following definition.

**Definition 3.1.** For integers $1 \le p < p'$, $m \in \mathbb{Z}$ and $\ell \in \mathbb{Z}_{p'-1}$ such that $\ell$ and $m$ have equal parity,

$$\mathcal{C}^{(p,p')}_{m,\ell}(q) = \\ \frac{1}{(q)_\infty^3} \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{N}} (-1)^i q^{\frac{1}{2}i(i+m)+jp(p'j+\ell+1)} \Big\{ q^{\frac{1}{2}i(2p'j+\ell+1)} - q^{-\frac{1}{2}i(2p'j+\ell+1)} \Big\}. \tag{3.7}$$

When $\gcd(p,p') = 1$ we also use the notation $\mathcal{C}^N_{m,\ell}(q) = \mathcal{C}^{(p,p')}_{m,\ell}(q)$, where $N = p'/p - 2$ is the level of the modified string function.

As a note of warning we remark that for a generic choice of variables the order of summation in (3.6) and (3.7) has to be strictly obeyed. We use the form (3.7) as defining relation rather than the more familiar (and computationally efficient) expression

$$\mathcal{C}^{(p,p')}_{m,\ell}(q)=\frac{1}{(q)_\infty^3}\Bigl\{\sum_{\substack{i\geq 0\\ j\geq 0}}-\sum_{\substack{i<0\\ j<0}}\Bigr\}(-1)^i q^{\frac{1}{2}i(i+m)+p'j(pj+i)+\frac{1}{2}(\ell+1)(2pj+i)}$$
$$-\frac{1}{(q)_\infty^3}\Bigl\{\sum_{\substack{i\geq 0\\ j>0}}-\sum_{\substack{i<0\\ j\leq 0}}\Bigr\}(-1)^i q^{\frac{1}{2}i(i+m)+p'j(pj+i)-\frac{1}{2}(\ell+1)(2pj+i)} \tag{3.8}$$

for later reasons. By

$$\sum_{i=-\infty}^{\infty}(-1)^i q^{\binom{i}{2}+in}=0 \quad \text{for } n\in\mathbb{Z}, \tag{3.9}$$

which is a specialization of Jacobi's triple product identity (1.9), it is straightforward to transform (3.7) into (3.8). We also note that for integer level, i.e., $p=1$ and $p'=N+2$ we can rewrite (3.8) in the neat form (by (3.9) equivalent to [37, Eq. (3.17)])

$$\mathcal{C}^{N}_{m,\ell}(q)=\frac{1}{(q)_\infty^3}\Bigl\{\sum_{\substack{j\geq 1\\ k\leq 0}}-\sum_{\substack{j\leq 0\\ k\geq 1}}\Bigr\}(-1)^{k-j} q^{\binom{k-j}{2}-Njk+\frac{1}{2}k(m-\ell)+\frac{1}{2}j(m+\ell)}.$$

To see this, make the variable changes $j\to -j$ followed by $i\to k+j-1$ in the first line and $j\to 1-k$ followed by $i\to k+j-1$ in the second line of (3.8) and use the symmetry $\mathcal{C}^N_{m,\ell}(q)=q^{(m-\ell)/2}\mathcal{C}^N_{m-N,N-\ell}(q)$.

To conclude this section we introduce the characters $e^N_{m,\ell}(q)$ of the $\mathrm{Z}_N$ parafermion algebra at rational level $N$ [3]. It was argued in [3] that these characters are realized as branching functions as follows:

$$\chi_\ell(z,q)=\sum_{m\in 2\mathbb{Z}+\ell} e^N_{m,\ell}(q)\frac{q^{\frac{m^2}{4N}}z^{-m/2}}{\eta(\tau)}$$

Comparison with (3.3) shows that

$$e^N_{\ell,m}(q)=\eta(\tau)C^N_{\ell,m}(q). \tag{3.10}$$

For integer $N$ the $e^N_{m,\ell}$ have also been shown to be branching functions of the Lie algebra pair $(\mathrm{A}^{(1)}_{2N-1},\,\mathrm{C}^{(1)}_N)$ [46].

## 4. Fractional-level conjugate Bailey pairs

This section contains the key results of this paper. In Theorem 4.1 new conjugate Bailey pairs are stated, which by Corollary 4.2 imply conjugate Bailey pairs involving the one-dimensional configurations sums and fractional-level string functions of the previous two sections.

**Theorem 4.1.** *For $\eta\in\mathbb{Z}_+$ and $j\in\mathbb{Z}$, the pair of sequences $(\gamma,\delta)$ with*

$$\begin{aligned}\gamma_L&=\frac{1}{(q)_\infty^2(aq)_\infty}\sum_{i=1}^{\infty}(-1)^i q^{\frac{1}{2}i(i+2L+\eta)}\Bigl\{q^{\frac{1}{2}i(2j+\eta+1)}-q^{-\frac{1}{2}i(2j+\eta+1)}\Bigr\}\\ \delta_L&=\begin{bmatrix}2L+\eta\\ L-j\end{bmatrix}-\begin{bmatrix}2L+\eta\\ L-j-1\end{bmatrix}\end{aligned} \tag{4.1}$$

*forms a conjugate Bailey pair relative to* $a = q^{\eta}$.

Before we prove this theorem let us first state the following corollary.

**Corollary 4.2.** *Fix integers* $1 \le p < p'$, *and let* $\eta \in \mathbb{Z}_+$ *and* $\ell \in \mathbb{Z}_{p'-1}$ *such that* $\ell + \eta$ *is even. Let* $\mathcal{C}^{(p,p')}_{m,\ell}$ *and* $X^{(p,p')}_{r,s}(L,b)$ *be defined as in* (3.7) *and* (2.1). *Then* $(\gamma, \delta)$ *with*

$$\gamma_L = (q)_\eta \mathcal{C}^{(p,p')}_{2L+\eta,\ell}(q) \qquad \text{and} \qquad \delta_L = X^{(p,p')}_{0,\ell+1}(2L+\eta, 1) \tag{4.2}$$

*forms a conjugate Bailey pair relative to* $a = q^{\eta}$.

*Proof.* Take the conjugate Bailey pair (4.1) and replace $j$ by $jp' + (\ell - \eta)/2$. Then multiply both $\gamma_L$ and $\delta_L$ by $q^{jp(jp'+\ell+1)}$ and sum $j$ over the integers. Using (3.7) and (2.1) this transforms $\gamma_L$ and $\delta_L$ of (4.1) into those of (4.2). ☐

The proof of Theorem 4.1 rests upon the following lemma.

**Lemma 4.3.** *For* $a$ *and* $b$ *indeterminates,*

$$\sum_{r=0}^{\infty} \frac{(ab)_{2r}}{(q)_r (ab)_r} \Big\{ \frac{1}{(aq)_{r-1}(bq)_r} - \frac{1}{(aq)_r (bq)_{r-1}} \Big\} = \frac{1}{(q)_\infty (aq)_\infty (bq)_\infty} \sum_{i=1}^{\infty} (-1)^i q^{\binom{i}{2}} (a^i - b^i). \tag{4.3}$$

*Proof.* The terms on the left within the curly braces can be combined to $(b - a)q^r/(aq)_r(bq)_r$. Using this as well as $(a)_\infty/(a)_r = (aq^r)_\infty$ and $(a)_{2r}/(a)_r = (aq^r)_r$, equation (4.3) can be written as

$$\sum_{r=0}^{\infty} \frac{q^r (abq^r)_r (aq^{r+1})_\infty (bq^{r+1})_\infty}{(q)_r} = \frac{1}{(q)_\infty} \sum_{i=1}^{\infty} (-1)^{i+1} q^{\binom{i}{2}} \frac{a^i - b^i}{a - b}. \tag{4.4}$$

We now use the $q$-binomial sum [7, Eq. (3.3.6)]

$$(a)_n = \sum_{k=0}^{n} (-a)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \tag{4.5}$$

as well as the limiting case

$$(a)_\infty = \sum_{k=0}^{\infty} \frac{(-a)^k q^{\binom{k}{2}}}{(q)_k},$$

to express the left-hand side of (4.4) as the following quadruple sum,

$$\sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{r} (-1)^{i+j+k} a^{i+k} b^{j+k} \, \frac{q^{\binom{i+1}{2} + \binom{j+1}{2} + \binom{k}{2} + r(i+j+k+1)}}{(q)_i (q)_j (q)_k (q)_{r-k}}.$$

After shifting $i \to i - k$, $j \to j - k$ and $r \to r + k$ this becomes

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} q^{\binom{i+1}{2} + \binom{j+1}{2}} a^i b^j \sum_{k=0}^{\min\{i,j\}} \frac{(-1)^k q^{\binom{k}{2}}}{(q)_{i-k} (q)_{j-k} (q)_k} \sum_{r=0}^{\infty} \frac{q^{r(i+j-k+1)}}{(q)_r}.$$

The sum over $r$ can readily be performed thanks to [43, Eq. (1.3.15)]

$$\sum_{r=0}^{\infty} \frac{x^r}{(q)_r} = \frac{1}{(x)_\infty}, \tag{4.6}$$

leading to

$$\frac{1}{(q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}q^{\binom{i+1}{2}+\binom{j+1}{2}}a^ib^j\sum_{k=0}^{\min\{i,j\}}\frac{(-1)^kq^{\binom{k}{2}}(q)_{i+j-k}}{(q)_{i-k}(q)_{j-k}(q)_k}.$$

The sum over $k$ yields $q^{ij}$ by the $q$-Chu–Vandermonde sum [43, Eq. (II.7)]

$$ {}_2\phi_1\Big[\begin{matrix}a,q^{-n}\\ c\end{matrix};q,\frac{cq^n}{a}\Big]=\frac{(c/a)_n}{(c)_n} \tag{4.7}$$

with $n=\min\{i,j\}$, $c=q^{-i-j}$ and $a=cq^n$, where the following standard notation for basic hypergeometric series is employed

$$ {}_{r+1}\phi_r\Big[\begin{matrix}a_1,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix};q,z\Big]=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_{r+1})_k}{(q,b_1,\dots,b_r)_k}z^k.$$

As a result we are left with

$$\frac{1}{(q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}a^ib^jq^{\binom{i+j+1}{2}}.$$

This corresponds to the right-hand side of (4.4) as

$$\begin{aligned}\sum_{i=1}^{\infty}(-1)^{i+1}q^{\binom{i}{2}}\frac{a^i-b^i}{a-b}&=\sum_{i=1}^{\infty}(-1)^{i+1}q^{\binom{i}{2}}\sum_{j=0}^{i-1}a^{i-j-1}b^j\\ &=\sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty}(-1)^{i+1}q^{\binom{i}{2}}a^{i-j-1}b^j=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}q^{\binom{i+j+1}{2}}a^ib^j.\end{aligned}$$

□

Finally we have to show that Theorem 4.1 follows from Lemma 4.3.

*Proof of Theorem 4.1.* Substitute the conjugate Bailey pair (4.1) into the defining relation (1.4). After the shift $r\to r+L$ this becomes

$$\begin{aligned}&\frac{1}{(q)_\infty^3}\sum_{i=1}^{\infty}(-1)^iq^{\binom{i}{2}}\Big\{q^{\frac12 i(\zeta-\sigma)}-q^{\frac12 i(\zeta+\sigma+2)}\Big\}\\ &\qquad=\sum_{r=0}^{\infty}\frac{1}{(q)_r(q)_{r+\zeta}}\Big\{\Big[\begin{matrix}2r+\zeta\\ r+\frac12(\zeta-\sigma-2)\end{matrix}\Big]-\Big[\begin{matrix}2r+\zeta\\ r+\frac12(\zeta-\sigma)\end{matrix}\Big]\Big\},\end{aligned}$$

where we have set $2L+\eta=\zeta\geq 0$ and $2j+\eta=\sigma$. To obtain this identity we take (4.3) and choose $a=q^{(\zeta-\sigma)/2}$, $b=q^{(\zeta+\sigma+2)/2}$ and perform a few trivial operations. □

## 5. Fermionic expressions for the one-dimensional configuration sums

From Definition 2.1 of the one-dimensional configuration sums we see that the sequence $\delta$ in Corollary 4.2 is not a sequence of manifestly positive polynomials (polynomials with positive integer coefficients). In applications of the corollary interesting $q$-series identities arise when there exist expressions that do have this property. Such constant-sign or fermionic representations for the configuration sums of the Andrews–Baxter–Forrester models have recently attracted a lot of attention [20, 21, 23, 34, 38, 39, 41, 54, 69, 70]. In this section we present some of the cited results for $X_{r,s}^{(p,p')}(L,b)$ in the simplest case when $\gcd(p,p')=1$ and $s$ and $b$ are so-called Takahashi lengths associated with the continued fraction expansion of $p/(p'-p)$. More complicated cases where $s$ and $b$ are not necessarily Takahashi lengths or where $(p,p')\neq 1$ can be found in [23] and [39], respectively.

Given $p,p'$ such that $\gcd(p,p')=1$ and $p<p'<2p$ define integers $n$ and $\nu_0,\dots,\nu_n$ by the continued fraction expansion

$$\frac{p}{p'-p}=[\nu_0,\nu_1,\dots,\nu_n].$$

Introduce partial sums of the $\nu_j$ as $t_m=\sum_{j=0}^{m-1}\nu_j$ for $1\le m\le n$ and set $t_0=-1$ and $d=t_{n+1}=-2+\sum_{j=0}^{n}\nu_j$. The $t_m$'s define a matrix $\mathcal{I}_B$ of size $d\times d$ with entries

$$(\mathcal{I}_B)_{i,j}=\begin{cases}\delta_{i,j+1}+\delta_{i,j-1} & \text{for } i\neq t_m\\ \delta_{i,j+1}+\delta_{i,j}-\delta_{i,j-1} & \text{for } i=t_m<d\\ \delta_{i,j+1}+\delta_{\nu_n,2}\delta_{i,j} & \text{for } i=d.\end{cases}$$

Viewing $\mathcal{I}_B$ as a generalized incidence matrix we define a corresponding fractional-level Cartan-type matrix $B=2I-\mathcal{I}_B$, where $I$ is the identity matrix. When $p'=p+1$ the matrix $B$ is a Cartan matrix of type A and when $p'=p+2$ it corresponds to a Cartan-type matrix of a tadpole graph.

For $1\le m\le n$ consider the recursion

$$x_{m+1}=x_{m-1}+\nu_m x_m.$$

We need two sets of integers $\{y_m\}_{m=0}^{n+1}$ and $\{\bar{y}_m\}_{m=0}^{n+1}$ approximating $p'$ and $p$, defined by the above recurrence and the initial conditions $y_{-1}=0$, $\bar{y}_{-1}=-1$, $y_0=\bar{y}_0=1$ $y_1=\nu_0+1$, $\bar{y}_1=\nu_0$. Hence $\bar{y}_m/(y_m-\bar{y}_m)=[\nu_0,\dots,\nu_{m-1}]$, $y_{n+1}=p'$ and $\bar{y}_{n+1}=p$. An important subset of $\mathbb{N}_{p'-1}$ is given by the "Takahashi lengths" $l_1,\dots,l_{d+2}$ defined as

$$l_{j+1}=y_{m-1}+(j-t_m)y_m,\qquad t_m<j\le t_{m+1}+\delta_{m,n}.$$

Clearly, for $p'=p+1$ the set of Takahashi lengths is just $\mathbb{N}_{p'-1}$. Similarly one may define the "truncated Takahashi lengths" $\bar{l}_1,\dots,\bar{l}_{d+2}$,

$$\bar{l}_{j+1}=\bar{y}_{m-1}+(j-t_m)\bar{y}_m,\qquad t_m<j\le t_{m+1}+\delta_{m,n},$$

which determine a subset of $\mathbb{Z}_p$. If $b=l_{j+1}$ is a Takahashi length then $\bar{b}$ denotes the truncated Takahashi length $\bar{l}_{j+1}$.

For vectors $\boldsymbol{u},\boldsymbol{v}\in\mathbb{Z}_+^{d+1}$ define

$$f(\boldsymbol{u},\boldsymbol{v})=\sum_{\boldsymbol{m}\in 2\mathbb{Z}^d+\boldsymbol{Q}_{\boldsymbol{u}+\boldsymbol{v}}} q^{\frac14\boldsymbol{m}B\boldsymbol{m}-\frac12\boldsymbol{A}_{\boldsymbol{u},\boldsymbol{v}}\boldsymbol{m}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}, \tag{5.1}$$

where

$$\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}=\prod_{j=1}^{d}\begin{bmatrix}m_j+n_j\\ m_j\end{bmatrix}$$

and where the following definitions are used. The variables $\boldsymbol{m}$ and $\boldsymbol{n}$ are related by the $(\boldsymbol{m},\boldsymbol{n})$-system

$$\boldsymbol{m}+\boldsymbol{n}=\frac12(\mathcal{I}_B\boldsymbol{m}+\boldsymbol{u}^*+\boldsymbol{v}^*)$$

where $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ denote the projections of $\boldsymbol{u}$ and $\boldsymbol{v}$ onto $\mathbb{Z}_+^d$. The linear term in the exponent of (5.1) is fixed by

$$(\boldsymbol{A}_{\boldsymbol{u},\boldsymbol{v}})_k = \begin{cases} u_k & \text{for } m \text{ odd} \\ v_k & \text{for } m \text{ even} \end{cases} \qquad t_m < k \le t_{m+1}.$$

Finally, $\boldsymbol{Q}_{\boldsymbol{u}} = \sum_{j=1}^{d+1} u_j \boldsymbol{Q}^{(j)}$ where $\boldsymbol{Q}^{(j)}$ is defined recursively as

$$Q_i^{(j)} = \begin{cases} \max\{j-i,0\} & \text{for } t_m \le i \le d \\ Q_{i+1}^{(j)} + Q_{t_{m'}+1}^{(j)} & \text{for } t_{m'-1} \le i < t_{m'}, 1 \le m' \le m \end{cases}$$

with $0 \le m \le n$ such that $t_m < j \le t_{m+1} + \delta_{m,n}$. When $\nu_n = 2$ we must take $Q_{t_n+1}^{(t_n+1)} = 0$.

When the conditions (2.5) are satisfied there exist fermionic expressions for the one-dimensional configuration sums (2.1) in terms of the function (5.1) [23]. Generally these are very complex and, as mentioned earlier, to keep formulas relatively simple we restrict our attention to $b$ and $s$ being Takahashi lengths (see [23, Eq. (10.3)]).

**Theorem 5.1.** *Let $1 \le p < p' < 2p$ such that $\gcd(p,p') = 1$ and let $b = l_{\beta+1}$, $s = l_{\sigma+1}$ be Takahashi lengths with $\beta \ge 1$ and $r = \bar{b} = \bar{l}_{\beta+1}$. Then*

$$X_{r,s}^{(p,p')}(L,b) = q^{\Delta_{b,s}} f(L\boldsymbol{e}_1 + \boldsymbol{u}_\beta, \boldsymbol{u}_\sigma), \tag{5.2}$$

*where $\boldsymbol{e}_i$ is the $i$th standard unit vector in $\mathbb{Z}^{d+1}$ ($\boldsymbol{e}_0 = 0$) and*

$$\boldsymbol{u}_i = \boldsymbol{e}_i - \sum_{k=m+1}^{n} \boldsymbol{e}_{t_k} \qquad \text{for } t_m < i \le t_{m+1} + \delta_{m,n}. \tag{5.3}$$

The explicit expression for $\Delta_{b,s}$ in the theorem is quite involved and is omitted here. Instead we fix it by requiring that

$$X_{r,s}^{(p,p')}(L,b;q=0) = 1,$$

for $L \ge |s-b|$. The relation between $b$ and $r$ given in the theorem corresponds to (2.7) with $c = b-1$. This explains why $\beta \ge 1$ (or $b = l_{\beta+1} \ge 2$). As a consequence $X_{0,s}^{(p,p')}(L,1)$, or, equivalently, $X_{1,s}^{(p,p')}(L,1)$, is not contained in (5.2). Using (2.9) these cases can however be obtained from [23, Eq. (10.2)] and [23, Eq. (8.68)] as follows.

**Theorem 5.2.** *For $1 \le p < p' < 2p$ such that $\gcd(p,p') = 1$ and $s = l_{\sigma+1}$ a Takahashi length,*

$$X_{0,s}^{(p,p')}(L,1) = q^{\frac{L}{2}+\Delta_s} f(L\boldsymbol{e}_1 + \boldsymbol{u}_0, \boldsymbol{u}_\sigma) \tag{5.4}$$

$$X_{0,p'-s}^{(p,p')}(L,1) = q^{\frac{L}{2}+\Delta'_s} f(L\boldsymbol{e}_1 + \boldsymbol{u}_0, \boldsymbol{u}_\sigma + \boldsymbol{u}_{d+1}). \tag{5.5}$$

As before, $\Delta_s$ and $\Delta'_s$ are determined by demanding that the left-hand side is 1 for $q = 0$, and $\boldsymbol{u}_i$ is as defined in equation (5.3).

Fermionic forms for $p' > 2p$ can be obtained from the previous two theorems by the duality transformation (2.3) (and equation (2.9) when $r = 0$, $b = 1$). Applying (2.4), this yields

$$X_{b-r,s}^{(p'-p,p')}(L,b) = q^{\frac{1}{4}(L^2-(b-s)^2)-\Delta_{b,s}} f(\boldsymbol{u}_\sigma, L\boldsymbol{e}_1 + \boldsymbol{u}_\beta)$$

and

$$X_{0,s}^{(p'-p,p')}(L,1) = q^{\frac{1}{4}(L^2-s^2+1)-\Delta_s} f(\boldsymbol{u}_\sigma, L\boldsymbol{e}_1+\boldsymbol{u}_0) \tag{5.6}$$

$$X_{0,p'-s}^{(p'-p,p')}(L,1) = q^{\frac{1}{4}(L^2-(p'-s)^2+1)-\Delta'_s} f(\boldsymbol{u}_\sigma+\boldsymbol{u}_{d+1}, L\boldsymbol{e}_1+\boldsymbol{u}_0). \tag{5.7}$$

## 6. Fermionic representations of $A_1^{(1)}$ string functions and parafermion characters

Our two main results obtained so far can be summarized as follows:

(1) The conjugate Bailey pairs $(\gamma,\delta)$ of Corollary 4.2 where $\gamma$ is a sequence of (generalized) $A_1^{(1)}$ string functions and $\delta$ a sequence of one-dimensional configuration sums.
(2) A fermionic representation for the sequences $\delta$ as formulated in Theorem 5.2 and equations (5.6) and (5.7).

As a consequence of these results we find fermionic or constant-sign expressions for the sequence $\gamma$ and thus for the $A_1^{(1)}$ string functions. Specifically, by Corollary 4.2 and equation (1.4) we have

$$\mathcal{C}_{m,\ell}^{(p,p')}(q) = \sum_{r=0}^{\infty} \frac{X_{0,\ell+1}^{(p,p')}(2r+m,1)}{(q)_r(q)_{r+m}}, \tag{6.1}$$

which for $p=1$ was found previously in Refs. [55, 56, 18]. Using (5.4) and (5.6) the following result arises.

**Corollary 6.1.** *For $1\le p<p'<2p$ with $\gcd(p,p')=1$ set $N=p'/p-2$, and let $m\in\mathbb{Z}_+$ and $\ell+1=l_{\sigma+1}$ a Takahashi length such that $\ell+m$ is even. Then*

$$\mathcal{C}_{m,\ell}^{N}(q) = q^{\Delta_{\ell+1}+\frac{1}{2}m} \sum_{r=0}^{\infty} \frac{q^r f((2r+m)\boldsymbol{e}_1+\boldsymbol{u}_0, \boldsymbol{u}_\sigma)}{(q)_r(q)_{r+m}} \tag{6.2}$$

*and*

$$\mathcal{C}_{m,\ell}^{-N/(N+1)}(q) = q^{\frac{1}{4}(m^2-\ell(\ell+2))-\Delta_{\ell+1}} \sum_{r=0}^{\infty} \frac{q^{r(r+m)} f(\boldsymbol{u}_\sigma, (2r+m)\boldsymbol{e}_1+\boldsymbol{u}_0)}{(q)_r(q)_{r+m}}.$$

Similarly, using (6.1), (5.5) and (5.7) we get

**Corollary 6.2.** *For $1\le p<p'<2p$ with $\gcd(p,p')=1$ set $N=p'/p-2$, and let $m\in\mathbb{Z}_+$ and $p'-\ell-1=l_{\sigma+1}$ a Takahashi length such that $\ell+m$ is even. Then*

$$\mathcal{C}_{m,\ell}^{N}(q) = q^{\Delta'_{p'-\ell-1}+\frac{1}{2}m} \sum_{r=0}^{\infty} \frac{q^r f((2r+m)\boldsymbol{e}_1+\boldsymbol{u}_0, \boldsymbol{u}_\sigma+\boldsymbol{u}_{d+1})}{(q)_r(q)_{r+m}}$$

*and*

$$\mathcal{C}_{m,\ell}^{-N/(N+1)}(q) = q^{\frac{1}{4}(m^2-\ell(\ell+2))-\Delta'_{p'-\ell-1}} \times \sum_{r=0}^{\infty} \frac{q^{r(r+m)} f(\boldsymbol{u}_\sigma+\boldsymbol{u}_{d+1}, (2r+m)\boldsymbol{e}_1+\boldsymbol{u}_0)}{(q)_r(q)_{r+m}}.$$

For most choices of $p$ and $p'$ we believe these results to be new. The simplest known summation formulas arise for $(p,p')=(1,3)$ and $(2,3)$ when we can employ Schur's [63] polynomial analogue of the Euler identity, $X^{(2,3)}_{1,\ell+1}(L)=1$, so that by (2.3) and (2.9)

$$X^{(1,3)}_{0,\ell+1}(L,1)=q^{\frac{1}{4}(L^2-\ell^2)}$$
$$X^{(2,3)}_{0,\ell+1}(L,1)=q^{\frac{1}{2}(L-\ell)}.$$

Considering $(p,p')=(1,3)$ we find from Corollary 4.2 that $\delta_L=a^Lq^{L^2+(\eta^2-\ell^2)/4}$, which we recognize as Bailey's original sequence $\delta$ of equation (1.6) up to an irrelevant factor $q^{(\eta^2-\ell^2)/4}$. Hence $\gamma_L=a^Lq^{L^2+(\eta^2-\ell^2)/4}/(aq)_\infty$ and

$$\mathcal{C}^1_{m,\ell}(q)=\frac{q^{\frac{1}{4}(m^2-\ell^2)}}{(q)_\infty}, \tag{6.3}$$

which is the well-known form of the level-1 string function [48, Sec. 4.6, Ex. 3]. Next let $(p,p')=(2,3)$. Then Schur's polynomial identity implies $\delta_L=q^{L+(\eta-\ell)/2}$ which corresponds to the specialization $r=q$ in the sequence $\delta$ of Bressoud and Singh given in equation (1.13). Accordingly, we find that the string function at level $-1/2$ can be represented as

$$\mathcal{C}^{-1/2}_{m,\ell}(q)=\frac{q^{\frac{1}{2}(m-\ell)}}{(q)^2_\infty}\sum_{i\in\mathbb{Z}_+}(-1)^iq^{\frac{1}{2}i(i+2m+1)}.$$

A constant-sign expression can be obtained from (6.2),

$$\mathcal{C}^{-1/2}_{m,\ell}(q)=q^{\frac{1}{2}(m-\ell)}\sum_{r=0}^{\infty}\frac{q^r}{(q)_r(q)_{r+m}}.$$

Using Heine's ${}_2\phi_1$ transformation formula [43, Eq. (III.3)]

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]=\frac{(abz/c)_\infty}{(z)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/a,c/b\\c\end{matrix};q,\frac{abz}{c}\right], \tag{6.4}$$

with $a=b=0$, $c=q^{m+1}$ and $z=q$, this can be transformed into

$$\mathcal{C}^{-1/2}_{m,\ell}(q)=\frac{q^{\frac{1}{2}(m-\ell)}}{(q)_\infty}\sum_{r=0}^{\infty}\frac{q^{r(r+m+1)}}{(q)_r(q)_{r+m}}$$

which has an explicit factor $1/(q)_\infty$ and hence also provides a fermionic expression for the parafermion characters $e^{-1/2}_{m,\ell}(q)$ of equation (3.10).

By far the most involved of the known cases is $(p,p')=(1,p')$ for arbitrary $p'\geq 3$. Then $N=p'-2\in\mathbb{N}$, $\ell\in\mathbb{Z}_{N+1}$, and from the fermionic representations (5.6) and (5.7) for the one-dimensional configuration sums we have

$$X^{(1,N+2)}_{0,\ell+1}(L,1)=q^{\frac{L^2-\ell^2}{4N}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}_+\\ \frac{L+\ell}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}}q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_\ell)}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix}, \tag{6.5}$$

with $\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(L\boldsymbol{e}_1+\boldsymbol{e}_\ell+\mathcal{I}\boldsymbol{m})$, and

$$X^{(1,N+2)}_{0,\ell+1}(L,1)=q^{\frac{L^2-\ell^2}{4N}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}_+\\ \frac{L-\ell}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}}q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_{N-\ell})}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix}, \tag{6.6}$$

with $\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(L\boldsymbol{e}_1+\boldsymbol{e}_{N-\ell}+\mathcal{I}\boldsymbol{m})$. Here $C$ is the $\mathrm{A}_{N-1}$ Cartan matrix, $\mathcal{I}$ the corresponding incidence matrix and $\boldsymbol{e}_i$ the $i$th standard unit vector in $\mathbb{Z}^{N-1}$ ($\boldsymbol{e}_0=\boldsymbol{e}_N=0$). Inserting (6.5) into (6.1) gives a fermionic formula for the integer-level string functions implied by [25, Eq. (4.7)].

Lepowsky and Primc [53] provide an alternative fermionic expression for the integer-level string functions as

$$\mathcal{C}^N_{m,\ell}(q)=\frac{q^{\frac{m^2-\ell^2}{4N}}}{(q)_\infty}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{N-1}\\ \frac{m+\ell}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}}\frac{q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_\ell)}}{(q)_{\boldsymbol{n}}}, \tag{6.7}$$

where $(q)_{\boldsymbol{n}}=\prod_{j=1}^{N-1}(q)_{n_j}$. Inserting (6.5)–(6.7) into (4.2) we obtain two sequences of conjugate Bailey pairs. Using the symmetry $\mathcal{C}^N_{m,\ell}(q)=q^{(m-\ell)/2}\mathcal{C}^N_{m-N,N-\ell}(q)$ these two sequences may be succinctly expressed as follows.

**Theorem 6.3.** *For $N\geq 1$, $\sigma\in\mathbb{Z}_2$, $\eta\in\mathbb{Z}_+$ and $\ell\in\mathbb{Z}_{N+1}$ such that $\ell+\eta+\sigma N$ is even, the following pair of sequences $(\gamma,\delta)$ forms a conjugate Bailey pair relative to $a=q^\eta$:*

$$\gamma_L=\frac{a^{L/N}q^{L^2/N}}{(aq)_\infty}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{N-1}\\ \frac{2L+\eta+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}}\frac{q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_\ell)}}{(q)_{\boldsymbol{n}}}$$

$$\delta_L=a^{L/N}q^{L^2/N}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{N-1}\\ \frac{2L+\eta+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}}q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_\ell)}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix},$$

*with $\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}((2L+\eta)\boldsymbol{e}_1+\boldsymbol{e}_\ell+\mathcal{I}\boldsymbol{m})$.*

These are the "higher-level" conjugate Bailey pairs of [61, Lemma 3] and [62, Cor. 2.1] (with the parameter $M$ therein sent to infinity and with the partition $\lambda$ therein having a single part).

To conclude this section we give some examples of (6.1) that are new. When we take $(p,p')=(2,5)$ we can express the string functions at level $1/2$ in terms of polynomials introduced by Schur [63] in his famous paper on the Rogers–Ramanujan identities. To be specific, from (5.6) we infer the following polynomial analogues of the Rogers–Ramanujan identities

$$\begin{aligned}
X^{(2,5)}_{0,1}(2L,1)&=q^L\Big(1+\sum_{n=1}^{L-1}q^{n(n+1)}{\textstyle\begin{bmatrix}2L-2-n\\ n\end{bmatrix}}\Big)\\
X^{(2,5)}_{0,2}(2L+1,1)&=q^L\sum_{n=0}^{L}q^{n^2}{\textstyle\begin{bmatrix}2L-n\\ n\end{bmatrix}}\\
X^{(2,5)}_{0,3}(2L,1)&=q^{L-1}\sum_{n=0}^{L-1}q^{n^2}{\textstyle\begin{bmatrix}2L-1-n\\ n\end{bmatrix}}\\
X^{(2,5)}_{0,4}(2L+1,1)&=q^{L-1}\sum_{n=0}^{L-1}q^{n(n+1)}{\textstyle\begin{bmatrix}2L-1-n\\ n\end{bmatrix}}.
\end{aligned}$$

We remark that the above results may also be derived using related polynomial identities for $X_{1,1}^{(2,5)}(2L,3)$, $X_{1,1}^{(2,5)}(2L+1,2)$, $X_{1,3}^{(2,5)}(2L,3)$ and $X_{1,3}^{(2,5)}(2L+1,2)$, due to Andrews [4]. Substituting the above four identities into (6.1) gives fermionic representation for the string functions at level 1/2. Fermionic forms for the corresponding parafermion characters $e_{m,\ell}^{1/2}$ can be obtained by pulling out an explicit factor $1/(q)_\infty$.

**Proposition 6.4.** *For $m\geq 0$ the level 1/2 string functions can be expressed as*

$$\mathcal{C}_{2m,0}^{1/2}(q)=\frac{q^m}{(q)_\infty}\sum_{r=0}^{\infty}\frac{q^r}{(q)_r}\Bigl(1+\sum_{n=1}^{m+\lfloor (r-2)/2\rfloor}q^{n(n+1)}\begin{bmatrix}r+2m-n-2\\ n\end{bmatrix}\Bigr)$$

$$\mathcal{C}_{2m+1,1}^{1/2}(q)=\frac{q^m}{(q)_\infty}\sum_{r=0}^{\infty}\frac{q^r}{(q)_r}\sum_{n=0}^{m+\lfloor r/2\rfloor}q^{n^2}\begin{bmatrix}r+2m-n\\ n\end{bmatrix}$$

$$\mathcal{C}_{2m,2}^{1/2}(q)=\frac{q^{m-1}}{(q)_\infty}\sum_{r=0}^{\infty}\frac{q^r}{(q)_r}\sum_{n=0}^{m+\lfloor (r-1)/2\rfloor}q^{n^2}\begin{bmatrix}r+2m-n-1\\ n\end{bmatrix}$$

$$\mathcal{C}_{2m+1,3}^{1/2}(q)=\frac{q^{m-1}}{(q)_\infty}\sum_{r=0}^{\infty}\frac{q^r}{(q)_r}\sum_{n=0}^{m+\lfloor (r-1)/2\rfloor}q^{n(n+1)}\begin{bmatrix}r+2m-n-1\\ n\end{bmatrix}.$$

*Proof.* We only present the proof of the second identity. The other three identities can be proven in a similar fashion. (The second rather than the first identity is chosen because all equations are more compact in this case.) We start with

$$\mathcal{C}_{2m+1,1}^{1/2}(q)=q^m\sum_{r=0}^{\infty}\sum_{n=0}^{r+m}\frac{q^{r+n^2}}{(q)_r(q)_{r+2m+1}}\begin{bmatrix}2r+2m-n\\ n\end{bmatrix}$$

and interchange the sums over $r$ and $n$ and shift $r\to r+n-m$. Then we again swap the order of summation yielding

$$\mathcal{C}_{2m+1,1}^{1/2}(q)=\Bigl(\sum_{r=m}^{\infty}\sum_{n=0}^{\infty}+\sum_{r=0}^{m-1}\sum_{n=m-r}^{\infty}\Bigr)\frac{q^{r+n(n+1)}}{(q)_{r+n-m}(q)_{r+n+m+1}}\begin{bmatrix}n+2r\\ n\end{bmatrix}. \tag{6.8}$$

Now consider the first double sum denoted by $S_1$ and write this as

$$S_1=\sum_{r=m}^{\infty}\frac{q^r}{(q)_{r-m}(q)_{r+m+1}}\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(q^{2r+1})_n}{(q)_n(q^{r-m+1})_n(q^{r+m+2})_n}.$$

Using the $q$-Kummer–Thomae–Whipple formula [43, (III.9)]

$${}_3\phi_2\Bigl[\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\Bigr]=\frac{(e/a,de/bc)_\infty}{(e,de/abc)_\infty}\,{}_3\phi_2\Bigl[\begin{matrix}a,d/b,d/c\\ d,de/bc\end{matrix};q,\frac{e}{a}\Bigr], \tag{6.9}$$

with $a,b\to\infty$, $c=q^{2r+1}$, $d=q^{r-m+1}$ and $e=q^{r+m+2}$ this can be put in the form

$$S_1=\frac{1}{(q)_\infty}\sum_{r=m}^{\infty}\sum_{n=0}^{r+m}\frac{q^{r+n(n+1)}}{(q)_{r+n-m}}\begin{bmatrix}r+m\\ n\end{bmatrix}.$$

Once more the order of summation is reversed, then $r$ is replaced by $r+m-n$ and the summation order is again changed. Thus,

$$S_1 = \frac{q^m}{(q)_\infty} \sum_{r=0}^{\infty} \frac{q^r}{(q)_r} \sum_{n=0}^{\min\{r,m+\lfloor r/2 \rfloor\}} q^{n^2} \begin{bmatrix} r+2m-n \\ n \end{bmatrix}.$$

Next we deal with $S_2$, given by the second double sum in (6.8). Shifting $n \to n+m-r$ gives

$$S_2 = \frac{q^m}{(q)_{2m+1}} \sum_{r=0}^{m-1} q^{(m-r)^2} \begin{bmatrix} r+m \\ 2r \end{bmatrix} \sum_{n=0}^{\infty} \frac{q^{n(n+2m-2r+1)}(q^{m+r+1})_n}{(q)_n (q^{m-r+1})_n (q^{2m+2})_n}.$$

By equation (6.9) with $a, b \to \infty$, $c = q^{m+r+1}$, $d = q^{m-r+1}$ and $e = q^{2m+2}$ this is equal to

$$S_2 = \frac{1}{(q)_\infty} \sum_{r=0}^{m-1} \sum_{n=0}^{2r} \frac{q^{r+(n+m-r)(n+m-r+1)}}{(q)_n} \begin{bmatrix} r+m \\ n+m-r \end{bmatrix}.$$

By an interchange of sums followed by the successive transformations $r \to n+m-r$ and $r \leftrightarrow n$ this becomes

$$S_2 = \frac{q^m}{(q)_\infty} \sum_{r=0}^{2m-2} \frac{q^r}{(q)_r} \sum_{n=r+1}^{m+\lfloor r/2 \rfloor} q^{n^2} \begin{bmatrix} r+2m-n \\ n \end{bmatrix}.$$

Computing $S_1 + S_2$ results in the claim of the proposition. $\square$

In our last example we take $(p, p') = (3, 4)$. The one-dimensional configuration sums for this case correspond to those of the celebrated Ising model of statistical mechanics, and the fermionic representations of the previous section can be simplified using the $q$-binomial theorem (4.5) or the $q$-Chu–Vandermonde sum (4.7). Specifically we have the polynomial identities

$$X_{0,1}^{(3,4)}(2L, 1) \pm q^{3/2} X_{0,3}^{(3,4)}(2L, 1) = q^L (\mp q^{1/2})_L \tag{6.10}$$

and

$$X_{0,2}^{(3,4)}(2L+1, 1) = q^L (-q)_L. \tag{6.11}$$

Substitution into (6.1) yields fermionic forms for the string functions at level $-2/3$. The next proposition states alternative expressions for these string functions which by (3.10) also imply fermionic forms for the corresponding parafermion characters.

**Proposition 6.5.** *For $m \geq 0$ the level $-2/3$ string functions satisfy the identities*

$$\mathcal{C}_{2m,0}^{-2/3}(q) = \frac{q^m}{2(q)_\infty} \sum_{r=0}^{\infty} \frac{q^{r^2/2+(m+1)r}}{(q)_r (q)_{r+2m}} \{(-q^{1/2})_{r+m} + (-1)^r (q^{1/2})_{r+m}\}$$

$$q^{3/2} \mathcal{C}_{2m,2}^{-2/3}(q) = \frac{q^m}{2(q)_\infty} \sum_{r=0}^{\infty} \frac{q^{r^2/2+(m+1)r}}{(q)_r (q)_{r+2m}} \{(-q^{1/2})_{r+m} - (-1)^r (q^{1/2})_{r+m}\}$$

$$\mathcal{C}_{2m+1,1}^{-2/3}(q) = \frac{q^m}{(q)_\infty} \sum_{r=0}^{\infty} \frac{q^{\binom{r}{2}+(m+2)r}(-q)_{r+m}}{(q)_r (q)_{r+2m+1}}.$$

*Proof.* Inserting the polynomial identities (6.10) and (6.11) into (6.1) one can apply the ${}_2\phi_1$ transformation (6.4) with $a=0$, $b=\mp q^{m+1/2}$, $c=q^{2m+1}$, $z=q$, and $a=0$, $b=-q^{m+1}$, $c=q^{2m+2}$, $z=q$, respectively. This yields identities for $\mathcal{C}^{-2/3}_{2m,0}(q)\pm q^{3/2}\mathcal{C}^{-2/3}_{2m,2}(q)$ and $\mathcal{C}^{-2/3}_{2m+1,1}(q)$ which immediately imply the expressions of the proposition. □

Note that one can apply (4.5) once again to rewrite

$$\frac{1}{2}\{(-q^{1/2})_{r+m}\pm(-1)^r(q^{1/2})_{r+m}\}=\sum_{n,\text{restriction}} q^{n^2/2}\begin{bmatrix} r+m\\ n\end{bmatrix},$$

where the restrictions are $n\equiv r \pmod 2$ and $n\not\equiv r \pmod 2$, respectively.

## 7. $\mathrm{A}_1^{(1)}$ branching functions

Let either $N_1$ or $N_2$ be a positive integer. Then the $\mathrm{A}_1^{(1)}$ branching functions are defined by [51]

$$\chi^{N_1}_{\ell_1}(z,q)\chi^{N_2}_{\ell_2}(z,q)=\sum_{\substack{\ell_3\in\mathbb{Z}_{p_3'-1}\\ \ell_1+\ell_2+\ell_3\in 2\mathbb{Z}}} B^{N_1,N_2}_{\ell_1,\ell_2,\ell_3}(q)\chi^{N_3}_{\ell_3}(z,q). \tag{7.1}$$

Here $N_1=p_1'/p_1-2$, $N_2=p_2'/p_2-2$ and $N_3=N_1+N_2=p_3'/p_3-2$, with $\gcd(p_i,p_i')=1$ for $i=1,2,3$. Note that $p_3=p_1p_2$ and $p_3'=p_1'p_2+p_2'p_1-2p_1p_2=p_2(p_1'+N_2p_1)$. Indeed $\gcd(p_3,p_3')=1$ since either $p_1=1$ or $p_2=1$.

In the following we are going to derive an explicit expression for the branching function following the method employed by Kac and Wakimoto in [51] (see also [35, 36]). The essence of this approach is to expand the character $\chi^{N_2}_{\ell_2}$ in terms of string functions and to then perform simple manipulations using the symmetries of the string functions to express the left-hand side of (7.1) as a linear combination of the $\chi^{N_3}_{\ell_3}$. The difference between our derivation below and that of Kac and Wakimoto is that we will not assume that $N_2$ is integer. Of course, since either $N_1$ or $N_2$ is (a positive) integer and $B^{N_1,N_2}_{\ell_1,\ell_2,\ell_3}=B^{N_2,N_1}_{\ell_2,\ell_1,\ell_3}$ one can without loss of generality assume that $N_2\in\mathbb{N}$. Nevertheless, dropping this assumption leads to a different representation of the branching functions. As will be shown in the next section, this has a natural interpretation in terms of the Bailey lemma. Before we commence our derivation we remark that because $N_2$ is no longer assumed to be integer we deal with string functions at (generally) non-integer level and hence we cannot rely on the symmetries employed in the Kac–Wakimoto derivation.

Insert (3.2) for $\chi^{N_1}_{\ell_1}(z,q)$ and (3.3) for $\chi^{N_2}_{\ell_2}(z,q)$ in the left-hand side of (7.1). Then, using the definition (3.1) of $\Theta_{n,m}(z,q)$, one obtains

$$\begin{aligned}P^{N_1,N_2}_{\ell_1,\ell_2}(q)&:=\chi^{N_1}_{\ell_1}(z,q)\chi^{N_2}_{\ell_2}(z,q)\sum_{\sigma=\pm1}\sigma\Theta_{\sigma,2}(z,q)\\ &=\sum_{\sigma=\pm1}\sum_{j\in\mathbb{Z}+\sigma\frac{\ell_1+1}{2p_1'}}\sum_{m\in 2\mathbb{Z}+\ell_2}\sigma z^{-\frac{1}{2}(m+2p_1'j)}q^{\frac{m^2}{4N_2}+p_1p_1'j^2}C^{N_2}_{m,\ell_2}(q).\end{aligned} \tag{7.2}$$

Now make the replacement $m \to m - 2p_1' j$ followed by $j \to \sigma(j + \frac{\ell_1+1}{2p_1'})$. Using $C^N_{m,\ell} = C^N_{-m,\ell}$ this gives

$$P^{N_1,N_2}_{\ell_1,\ell_2}(q) = q^{\frac{(\ell_1+1)^2}{4(N_1+2)}} \sum_{m\in 2\mathbb{Z}+\ell_1+\ell_2+1} z^{-\frac{1}{2}m} q^{\frac{1}{4N_2}(m-\ell_1-1)^2} b^{p_1',p_1'+N_2p_1,N_2}_{\ell_1+1,\ell_2,m}(q), \tag{7.3}$$

where we have introduced the function

$$\begin{aligned} b^{P,P',N}_{r,\ell,s}&(q) \\ &= \sum_{j\in\mathbb{Z}} \Big\{ q^{\frac{j}{N}(PP'j+P'r-Ps)} C^N_{2Pj+r-s,\ell}(q) - q^{\frac{1}{N}(Pj+r)(P'j+s)} C^N_{2Pj+r+s,\ell}(q) \Big\}. \end{aligned}$$

Note that the initial assumption that either $N_1$ or $N_2$ is a positive integer means that we are only concerned with $b^{P,P',N}_{r,\ell,s}(q)$ with either $(P'-P)/N = 1$ or $N \in \mathbb{N}$. This is crucial in the following lemma needed to rewrite the expression for $P^{N_1,N_2}_{\ell_1,\ell_2}(q)$.

**Lemma 7.1.** *Let $P \in \mathbb{N}$ and $N, P' \in \mathbb{Q}$ such that $N = p'/p - 2$ with $\gcd(p,p') = 1$ and $(P'-P)/N \in \mathbb{Z}_+$. When $(P'-P)/N = 1$ or $N \in \mathbb{N}$ the following periodicity holds:*

$$b^{P,P',N}_{r,\ell,s+2pP'}(q) = q^{-\frac{p}{N}(pPP'-P'r+Ps)} b^{P,P',N}_{r,\ell,s}(q). \tag{7.4}$$

*Proof.* After inserting the definition of $b^{P,P',N}_{r,\ell,s}$ in the above equation make the variable changes $j \to j+p$ in the first term and $j \to j-p$ in the second term of the left-hand side. Then, by the symmetry (3.4), equation (7.4) can be rewritten as

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \Big\{ q^{\frac{j}{N}(PP'j+P'r-Ps)} C^N_{2Pj+r-s-2pkN,\ell}(q) - q^{\frac{1}{N}(Pj-r)(P'j-s)} C^N_{2Pj-r-s-2pkN,\ell}(q) \Big\} \\ &\qquad = \sum_{j\in\mathbb{Z}} \Big\{ q^{\frac{j}{N}(PP'j+P'r-Ps)} C^N_{2Pj+r-s,\ell}(q) - q^{\frac{1}{N}(Pj-r)(P'j-s)} C^N_{2Pj-r-s,\ell}(q) \Big\} \end{aligned} \tag{7.5}$$

where $k = (P'-P)/N \in \mathbb{Z}_+$. When $N \in \mathbb{N}$ this follows directly from the symmetries (3.4) and (3.5), and in the remainder we assume that $N \in \mathbb{Q}$ and $k = 1$. The complication is now that we no longer have $C^N_{m,\ell} = C^N_{m-2N,\ell}$. In view of this let us first investigate the origin of this difficulty. Consider the expression (3.6) of the $\mathrm{A}^{(1)}_1$ string functions. The summand has two different terms corresponding to the two terms within the curly braces. In the first term make the variable change $j \to j-1$, $i \to i+2p$ and in the second term make the change $j \to j+1$, $j \to i+2p$. The result of these changes is exactly the same expression as before except that $m$ has been replaced by $m-2pN$ and that the sum over $i$ now runs over all integers greater than $-2p$. We may therefore conclude that

$$C^N_{m,\ell}(q) = C^N_{m-2pN,\ell}(q) + \bar{C}^N_{m-2pN,\ell}(q), \tag{7.6}$$

where

$$\begin{aligned} \bar{C}^N_{m,\ell}(q) &= \frac{q^{\frac{(\ell+1)^2}{4(N+2)}-\frac{m^2}{4N}}}{\eta^3(\tau)} \\ &\quad \times \sum_{i=1}^{2p-1} \sum_{j\in\mathbb{Z}} (-1)^i q^{\frac{1}{2}i(i-m)+pj(p'j+\ell+1)} \Big\{ q^{-\frac{1}{2}i(2p'j+\ell+1)} - q^{\frac{1}{2}i(2p'j+\ell+1)} \Big\}. \end{aligned}$$

By a shift $j \to j-1$ in the second term of the summand this becomes

$$\begin{aligned} \bar{C}^N_{m,\ell}(q) &= \frac{q^{\frac{(\ell+1)^2}{4(N+2)}-\frac{m^2}{4N}}}{\eta^3(\tau)} \\ &\quad \times \sum_{i=1}^{2p-1}\sum_{j\in\mathbb{Z}}(-1)^i q^{\frac{1}{2}i(i-m)+pj(p'j+\ell+1)-\frac{1}{2}i(2p'j+\ell+1)}\Bigl\{1-q^{\frac{1}{2}(i-p)(2p'j-p'+\ell+1)}\Bigr\}, \end{aligned}$$

which shows that the $i=p$ term in the summand vanishes and hence that $\bar{C}^N_{m,\ell}(q)=0$ for $N$ integer.

Inserting (7.6) into equation (7.5) with $k=1$ we are done with the lemma if we prove that

$$\begin{aligned} \sum_{j\in\mathbb{Z}}\Bigl\{ & q^{\frac{j}{N}(PP'j+P'r-Ps)}\bar{C}^N_{2Pj+r-s-2pN,\ell}(q) \\ & - q^{\frac{1}{N}(Pj-r)(P'j-s)}\bar{C}^N_{2Pj-r-s-2pN,\ell}(q)\Bigr\}=0. \end{aligned}$$

Using the explicit form for $\bar{C}^N_{m,\ell}(q)$, this is equivalent to showing that

$$\begin{aligned} \sum_{i=1}^{2p-1}\sum_{j\in\mathbb{Z}} & (-1)^i q^{\frac{1}{2}i(i-r+s)+jp(p'j+\ell+1)-\frac{1}{2}i(2p'j+\ell+1)}\Bigl\{1-q^{\frac{1}{2}(i-p)(2p'j-p'+\ell+1)}\Bigr\} \\ & \times q^{p((i-p)N+r-s)}\sum_{\mu\in\mathbb{Z}}\Bigl\{q^{\mu(\mu P+r-P(i-2p))}-q^{(\mu-i+2p)(\mu P-r)}\Bigr\}=0. \end{aligned}$$

After the shift $\mu\to i-2p-\mu$ in the second term in the sum over $\mu$ we are done. $\square$

From (3.4) it follows that $b^{P,P,N}_{r,\ell,s}(q) = -q^{\frac{rs}{N}}b^{P,P',N}_{r,\ell,-s}(q)$ so that in combination with Lemma 7.1

$$b^{P,P',N}_{r,\ell,2pP'-s}(q) = -q^{-\frac{1}{N}(pP-r)(pP'-s)}b^{P,P',N}_{r,\ell,s}(q). \tag{7.7}$$

In view of (7.4) and (7.7), it becomes natural to dissect the sum over $m$ in (7.3) using

$$\sum_{m\in2\mathbb{Z}+\ell_1+\ell_2+1} f_m = \sum_{k\in\mathbb{N}}\Bigl\{\sum_{\substack{\ell_3\in\mathbb{Z}_{p_3'}\\ \ell_1+\ell_2+\ell_3\in2\mathbb{Z}}} f_{2p_3'k+\ell_3+1} + \sum_{\substack{\ell_3+1\in\mathbb{Z}_{p_3'}\\ \ell_1+\ell_2+\ell_3\in2\mathbb{Z}}} f_{2p_3'k-\ell_3-1}\Bigr\}.$$

Observing that $b^{P,P',N}_{r,\ell,0}(q)=0$ and, by (7.7), also $b^{P,P',N}_{r,\ell,pP'}(q)=0$, equation (7.3) can then be written as

$$\begin{aligned} P^{N_1,N_2}_{\ell_1,\ell_2}(q) &= q^{\frac{(\ell_1+1)^2}{4(N_1+2)}+\frac{(\ell_3-\ell_1)^2}{4N_2}} \sum_{\substack{\ell_3\in\mathbb{Z}_{p_3'-1}\\ \ell_1+\ell_2+\ell_3\in2\mathbb{Z}}} b^{p_1',p_1'+N_2p_1,N_2}_{\ell_1+1,\ell_2,\ell_3+1}(q) \\ &\quad\times\sum_{k\in\mathbb{Z}}\Bigl\{z^{-\frac{1}{2}(2p_3'k+\ell_3+1)}q^{p_3k(p_3'k+\ell_3+1)} - z^{-\frac{1}{2}(2p_3'k-\ell_3-1)}q^{p_3k(p_3'k-\ell_3-1)}\Bigr\} \\ &= q^{\frac{((p_1'+N_2p_1)(\ell_1+1)-p_1'(\ell_3+1))^2}{4N_2p_1'(p_1'+N_2p_1)}} \sum_{\substack{\ell_3\in\mathbb{Z}_{p_3'-1}\\ \ell_1+\ell_2+\ell_3\in2\mathbb{Z}}} b^{p_1',p_1'+N_2p_1,N_2}_{\ell_1+1,\ell_2,\ell_3+1}(q)\sum_{\sigma=\pm1}\sigma\Theta_{\sigma(\ell_3+1),p_3'}(z,q^{p_3}). \end{aligned}$$

Comparing with (7.1) and (7.2) we can read off the branching functions.

**Theorem 7.2.** *For $N_1 = p_1'/p_1 - 2$ and $N_2 = p_2'/p_2 - 2$ with $\gcd(p_1, p_1') = \gcd(p_2, p_2') = 1$, such that $p_1 = 1$ or $p_2 = 1$ we have*

$$B^{N_1,N_2}_{r-1,\ell,s-1}(q) = B^{N_2,N_1}_{\ell,r-1,s-1}(q) = q^{\frac{(P'r-Ps)^2}{4N_2PP'}} \times \sum_{j\in\mathbb{Z}} \Bigl\{ q^{\frac{j}{N_2}(PP'j+P'r-Ps)} C^{N_2}_{2Pj+r-s,\ell}(q) - q^{\frac{1}{N_2}(Pj+r)(P'j+s)} C^{N_2}_{2Pj+r+s,\ell}(q) \Bigr\}.$$

*Here $P = p_1'$, $P' = p_1' + N_2 p_1$, $r \in \mathbb{N}_{P-1}$, $\ell + 1 \in \mathbb{N}_{p_2'-1}$ and $s \in \mathbb{N}_{p_2P'-1}$.*

When $N_2 \in \mathbb{N}$ this is Theorem 3.1 of [51] for $\mathrm{X}^{(r)}_N = \mathrm{A}^{(1)}_1$.

For comparison with later expressions it will be convenient to normalize the branching functions and to express them in terms of the modified string functions. Hence we introduce

$$\mathcal{B}^{N_1,N_2}_{r-1,\ell,s-1}(q) = \sum_{j\in\mathbb{Z}} q^{p_1 j(p_1'j+r)} \Bigl\{ \mathcal{C}^{N_2}_{2p_1'j+r-s,\ell}(q) - \mathcal{C}^{N_2}_{2p_1'j+r+s,\ell}(q) \Bigr\}, \tag{7.8}$$

where

$$B^{N_1,N_2}_{r-1,\ell,s-1}(q) = q^{\frac{(P'r-Ps)^2}{4N_2PP'} + \frac{(\ell+1)^2}{4(N_2+2)} - \frac{(r-s)^2}{4N_2} - \frac{1}{8}} \mathcal{B}^{N_1,N_2}_{r-1,\ell,s-1}(q).$$

## 8. Bose-Fermi identities

In Section 6 we have applied Corollary 4.2 to derive fermionic representations for the $\mathrm{A}^{(1)}_1$ string functions, but so far we have not yet employed the result of Corollary 4.2 in the context of the Bailey lemma. This is what we will do next. To simplify the notation we abbreviate the polynomial identities (5.2)–(5.4) as

$$X^{(p,p')}_{r,s}(L,b) = F^{(p,p')}_{r,s}(L,b). \tag{8.1}$$

From these, Bailey pairs relative to $q^{|b-s|}$ can be extracted [10, 40]. Together with the conjugate Bailey pairs of Corollary 4.2 these Bailey pairs (given by [24, Eq. (3.6)]) may be substituted into Bailey's equation (1.2). Omitting the details we find the following theorem.

**Theorem 8.1.** *For $i = 1, 2$, let $1 \le p_i < p_i' < 2p_i$ such that $\gcd(p_i, p_i') = 1$ and set $N_i = p_i'/p_i - 2$. Let $b$ and $s$ be Takahashi lengths with respect to the continued fraction decomposition of $p_1/(p_1' - p_1)$ and let $r = \bar{b}$. Let $\ell + 1$ be a Takahashi length with respect to the continued fraction decomposition of $p_2/(p_2' - p_2)$. Then, for $\eta = |b - s|$ with $\eta + \ell$ even,*

$$\sum_{j\in\mathbb{Z}} \Bigl\{ q^{j(p_1p_1'j+rp_1'-sp_1)} \mathcal{C}^{N_2}_{2p_1'j+b-s,\ell}(q) - q^{(p_1j+r)(p_1'j+s)} \mathcal{C}^{N_2}_{2p_1'j+b+s,\ell}(q) \Bigr\} = \sum_{L=0}^{\infty} F^{(p_1,p_1')}_{r,s}(2L+\eta, b) F^{(p_2,p_2')}_{0,\ell+1}(2L+\eta, 1)/(q)_{2L+\eta}. \tag{8.2}$$

Many similar theorems can be derived. For example, we could have iterated the Bailey pair implied by (8.1) (see [24, Eq. (3.8)]) before substituting it into (1.2). Alternatively one can derive identities for $N_1 > 0$, $N_2 < 0$, or $N_1 < 0$, $N_2 > 0$ or $N_1, N_2 > 0$.

In general we have not been able to identify the left-hand side of (8.2), but when either $N_1$ or $N_2$ is a positive integer one can recognize the left-hand side of the above identities as $\mathrm{A}_1^{(1)}$ branching function. First assume $N_2$ is integer and $r$ is even. Using the symmetries $\mathcal{C}^{N_2}_{m-2N_2,\ell}(q)=q^{N_2-m}\mathcal{C}^{N_2}_{m,\ell}(q)$ and $\mathcal{C}^{N_2}_{m,\ell}(q)=\mathcal{C}^{N_2}_{-m,\ell}(q)$, the left-hand side of (8.2) becomes

$$q^{\frac{1}{4}r(2s-2b-N_2r)}\mathcal{B}^{N_1,N_2}_{s-1,\ell,b+N_2r-1}(q).$$

For $N_2$ integer and $r$ odd we can use $\mathcal{C}^{N_2}_{m-N_2,\ell}(q)=q^{(N_2-m-\ell)/2}\mathcal{C}^{N_2}_{m,N_2-\ell}(q)$ and $\mathcal{C}^{N_2}_{m,\ell}(q)=\mathcal{C}^{N_2}_{-m,\ell}(q)$ to rewrite the left-hand side of (8.2) as

$$q^{\frac{1}{4}r(2s-2b-N_2r)+\frac{1}{4}(N_2-2\ell)}\mathcal{B}^{N_1,N_2}_{s-1,N_2-\ell,b+N_2r-1}(q).$$

Finally, for $N_1$ integer we must have $r=0$, $b=1$ or $r=1$, $b-1\in\mathbb{N}_{p_1'-2}$ and the left-hand side of (8.2) can be simplified to

$$\mathcal{B}^{N_1,N_2}_{s-1,\ell,0}(q)\qquad\text{and}\qquad\mathcal{B}^{N_1,N_2}_{p_1'-s-1,\ell,p_1'-b-1}(q),$$

respectively.

Given the above results let us connect to the discussion in Sections 1 and 7 on the duality between Bailey and conjugate Bailey pairs and on the symmetry of the branching functions. If $r=0$ and $b=1$ the right-hand side of (8.2) is symmetric under the simultaneous interchange $N_1\leftrightarrow N_2$ and $s\leftrightarrow\ell+1$. In terms of Bailey and conjugate Bailey pairs this corresponds to the transformation

$$(\beta^{(N_1)},\delta^{(N_2)})\leftrightarrow(\bar{\beta}^{(N_2)},\bar{\delta}^{(N_1)})$$

with

$$\begin{aligned}\bar{\beta}^{(N_2)}_L&=\delta^{(N_2)}_L/(q)_{2L+\eta}\\ \bar{\delta}^{(N_1)}_L&=\beta^{(N_1)}_L(q)_{2L+\eta},\end{aligned}$$

where $\beta^{(N_1)}_L=F^{(p_1,p_1')}_{0,s}(2L+\eta,1)/(q)_{2L+\eta}$ and $\delta^{(N_2)}_L=F^{(p_2,p_2')}_{0,\ell+1}(2L+\eta,1)$. This result is to be compared with (1.16).

Similarly, using (2.9), the right-hand side of (8.2) is symmetric under the interchange $N_1\leftrightarrow N_2$ and $s\leftrightarrow\ell+1$ if $r=1$ and $b=1$, which corresponds to the transformation

$$\begin{aligned}\bar{\beta}^{(N_2)}_L&=q^{-L-(\eta-\ell)/2}\delta^{(N_2)}_L/(q)_{2L+\eta}\\ \bar{\delta}^{(N_1)}_L&=q^{L+(\eta-\ell)/2}\beta^{(N_1)}_L(q)_{2L+\eta},\end{aligned}$$

where $\beta^{(N_1)}_L=q^{-L-(\eta-\ell)/2}F^{(p_1,p_1')}_{0,s}(2L+\eta,1)/(q)_{2L+\eta}$ and $\delta^{(N_2)}_L=F^{(p_2,p_2')}_{0,\ell+1}(2L+\eta,1)$.

Carrying out the corresponding transformations on $\alpha$ and $\gamma$ yields another expression for the left-hand side of (8.2) which involves the modified string functions at level $N_1$. When either $N_1$ or $N_2$ is a positive integer we recognize the resulting identities as the special cases $\ell_3=0$ or $\ell_3=N_1$ of the symmetry $B^{N_1,N_2}_{\ell_1,\ell_2,\ell_3}=B^{N_2,N_1}_{\ell_2,\ell_1,\ell_3}$, as expected.

Finally we present some explicit identities that follow by application of Bailey's lemma and the conjugate Bailey pairs of Corollary 4.2. In Refs. [10, Eqs. (2.12), (2.13)] and [11, Eqs. (3.47), (3.48)] one can find the following generalization of (1.12),

$$\alpha_L=\frac{(1-aq^{2L})(a)_L(-1)^Lq^{\binom{L}{2}}}{(1-a)(q)_L}\qquad\text{and}\qquad\beta_L=\delta_{L,0}. \tag{8.3}$$

Inserting this and (4.2) into equation (1.2) and performing some series manipulations gives a generalized Euler identity for the modified string functions.

**Proposition 8.2.** *For $1 \le p < p'$, $\ell \in \mathbb{Z}_{p'-1}$, $\eta \in \mathbb{Z}_{p'}$ such that $\ell + \eta$ is even,*

$$\sum_{L=-\infty}^{\infty} (-1)^L q^{\binom{L}{2}} \mathcal{C}^{(p,p')}_{2L+\eta,\ell}(q) = \delta_{\ell,\eta}.$$

Recalling (6.3), this is the classical Euler identity for $(p,p') = (1,3)$. For $p = 1$ and arbitrary $p'$ this is the $\mathrm{A}_1^{(1)}$ case of equation (2.1.17) of Ref. [51].

Before we can proof the proposition we need a technical lemma.

**Lemma 8.3.** *If $f_m = f_{-m}$ then*

$$\begin{aligned}(8.4)\qquad \sum_{L=0}^{\infty} (1-q^{2L+\eta})(q^{L+1})_{\eta-1} (-1)^L q^{\binom{L}{2}} f_{2L+\eta} \\ = \sum_{k=0}^{\lfloor \eta/2 \rfloor} \left\{ \begin{bmatrix} \eta \\ k \end{bmatrix} - \begin{bmatrix} \eta \\ k-1 \end{bmatrix} \right\} \sum_{L=-\infty}^{\infty} (-1)^L q^{\binom{L}{2}} f_{2L+\eta-2k}.\end{aligned}$$

*Proof.* First observe that

$$\sum_{k=0}^{\lfloor \eta/2 \rfloor} \left\{ \begin{bmatrix} \eta \\ k \end{bmatrix} - \begin{bmatrix} \eta \\ k-1 \end{bmatrix} \right\} \sum_{L=k-\eta+1}^{k-1} (-1)^L q^{\binom{L}{2}} f_{2L+\eta-2k} = 0.$$

To prove this shift $L \to L + k$ in the first term in the curly braces and successively $k \to \eta - k + 1$ and $L \to \eta - L + 1$ in the second term in the curly braces. Using the symmetry of $f_m$ the resulting terms can be combined to

$$\sum_{L=1}^{\eta-1} f_{2L-\eta} \sum_{k=0}^{\eta} (-1)^{k-L} q^{\binom{k-L}{2}} \begin{bmatrix} \eta \\ k \end{bmatrix} = \sum_{L=1}^{\eta-1} f_{2L-\eta} (-1)^L q^{\binom{L+1}{2}} (q^{-L})_\eta = 0,$$

where the middle term follows by application of the $q$-binomial theorem (4.5) and the last term by $(q^{-a})_b = 0$ for $0 \le a < b$. With this result we can write the sum over $L$ in the right-hand side of equation (8.4) as a sum over $L \le k - \eta$ and $L \ge k$. Then using the symmetry of $f_m$ the right-hand side becomes

$$\begin{aligned}\sum_{L=0}^{\infty} f_{2L+\eta} \sum_{k=0}^{\lfloor \eta/2 \rfloor} \left\{ \begin{bmatrix} \eta \\ k \end{bmatrix} - \begin{bmatrix} \eta \\ k-1 \end{bmatrix} \right\} \left\{ (-1)^{L+k} q^{\binom{L+k}{2}} + (-1)^{k-\eta-L} q^{\binom{k-\eta-L}{2}} \right\} \\ = \sum_{L=0}^{\infty} f_{2L+\eta} \sum_{k=0}^{\eta} (-1)^{L+k} q^{\binom{L+k}{2}} (1 + q^{L+k}) \begin{bmatrix} \eta \\ k \end{bmatrix} \\ = \sum_{L=0}^{\infty} f_{2L+\eta} (-1)^L q^{\binom{L}{2}} \left\{ (q^L)_\eta + q^L (q^{L+1})_\eta \right\}.\end{aligned}$$

Comparing with the left-hand side of (8.4) we are done since $(a)_n + a(aq)_n = (1 - a^2 q^n)(aq)_{n-1}$. $\square$

*Proof of Proposition 8.2.* Inserting (8.3) and (4.2) into (1.2) gives the identity

$$X^{(p,p')}_{0,\ell+1}(\eta, 1) = \sum_{L=0}^{\infty} (1 - q^{2L+\eta})(q^{L+1})_{\eta-1} (-1)^L q^{\binom{L}{2}} \mathcal{C}^{(p,p')}_{2L+\eta,\ell}(q),$$

for $\eta+\ell$ even and $\ell+1\in\mathbb{N}_{p'-1}$. Applying Lemma 8.3 this can be simplified to

$$X^{(p,p')}_{0,\ell+1}(\eta,1)=\sum_{k=0}^{\lfloor \eta/2\rfloor}\left\{\begin{bmatrix}\eta\\k\end{bmatrix}-\begin{bmatrix}\eta\\k-1\end{bmatrix}\right\}\sum_{L=-\infty}^{\infty}(-1)^L q^{\binom{L}{2}}\mathcal{C}^{(p,p')}_{2L+\eta-2k,\ell}(q).$$

Now observe that for $\eta\le p'-1$ the only contribution to $X^{(p,p')}_{0,\ell+1}(\eta,1)$ comes from the $j=0$ term in the summand of (2.1). Therefore,

$$X^{(p,p')}_{0,\ell+1}(\eta,1)=\begin{bmatrix}\eta\\(\eta+\ell)/2\end{bmatrix}-\begin{bmatrix}\eta\\(\eta-\ell-2)/2\end{bmatrix}=\sum_{k=0}^{\lfloor \eta/2\rfloor}\left\{\begin{bmatrix}\eta\\k\end{bmatrix}-\begin{bmatrix}\eta\\k-1\end{bmatrix}\right\}\delta_{\eta-2k,\ell}.$$

By induction on $\eta$ this implies Proposition 8.2. □

Our last identity follows by a straightforward generalization of the proof of Theorem 4.1 of Ref. [62], which corresponds to $p=1$ in the result given below.

**Theorem 8.4.** *For $1\le p<p'$, $\ell\in\mathbb{Z}_{p'-1}$ and integers $\delta,k,i$ such that $\delta\in\mathbb{Z}_2$, $k\ge 2$ and $i\in\mathbb{N}_k$,*

$$\sum_{L=-\infty}^{\infty}(-1)^L q^{((2k+\delta-2)L+2k-2i+\delta)L/2}\mathcal{C}^{(p,p')}_{2L,\ell}(q)$$
$$=\sum_{n_1,\dots,n_{k-1}\ge 0}\frac{q^{N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}X^{(p,p')}_{0,\ell+1}(2N_1,1)}{(q)_{n_1}\cdots(q)_{n_{k-2}}(q^{2-\delta};q^{2-\delta})_{n_{k-1}}},$$

*where $N_j=n_j+\cdots+n_{k-1}$.*

By Jacobi's triple product identity (1.9) and the fermionic expressions for the string function and configuration sums given earlier in the paper, the above identities can be recognized as (i) Andrews' analytic counterpart of Gordon's partition theorem when $(p,p')=(1,3)$ and $\delta=1$ [6], (ii) Bressoud's generalization thereof to even moduli when $(p,p')=(1,3)$ and $\delta=0$ [27], (iii) generalizations of the Göllnitz–Gordon partition identities due to Andrews and Bressoud when $(p,p')=(1,4)$ and $\delta=1$ [8, 28], (iv) Rogers–Ramanujan type identities by Bressoud when $(p,p')=(1,4)$ and $\delta=0$ [28].

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Department of Mathematics, M.I.T., Cambridge, MA 02139, U.S.A

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands