The Mukhin–Varchenko conjecture for type A

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Abstract: We present a generalisation of the famous Selberg integral. This confirms the $\mathfrak{g} = \mathrm{A}_n$ case of a conjecture by Mukhin and Varchenko concerning the existence of a Selberg integral for each simple Lie algebra $\mathfrak{g}$.

Résumé: On présent une généralisation de la bien connue intégrale de Selberg. Cette généralisation vérifie le cas $\mathfrak{g} = \mathrm{A}_n$ de la conjecture de Mukhin et Varchenko concernant l'existence d'une intégrale de Selberg pour chaque algèbre simple de Lie $\mathfrak{g}$.

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0 Preliminaries

This paper is a shortened version, devoid of any proofs, of the paper *A Selberg integral for the Lie algebra* A_n [17].

1 $\mathfrak{g}$-Selberg integrals

In 1944 Selberg published the following remarkable multiple integral [12]. Let k be a positive integer, $t = (t_1, \dots, t_k)$ and $\mathrm{d}t = \mathrm{d}t_1 \cdots \mathrm{d}t_k$.

Theorem 1.1 (Selberg integral) *For $\alpha, \beta, \gamma \in \mathbb{C}$ such that*

$$\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\gamma) > -\min\{1/k, \mathrm{Re}(\alpha)/(k-1), \mathrm{Re}(\beta)/(k-1)\}$$

there holds

$$\int_{\Delta_k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} (t_i - t_j)^{2\gamma}\, \mathrm{d}t = \prod_{i=1}^{k} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha+\beta+(i+k-2)\gamma)\Gamma(\gamma)}, \quad (1.1)$$

where

$$\Delta_k = \{t \in \mathbb{R}^k :\ 0 \le t_k \le \cdots \le t_2 \le t_1 \le 1\}. \quad (1.2)$$

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When $k=1$ this simplifies to Euler's famous beta integral [3]

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\qquad \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0. \tag{1.3}$$

At the time of its publication the Selberg integral was largely overlooked, but now, more than 60 years later, it is widely regarded as one of the most fundamental and important hypergeometric integrals. It has connections and applications to orthogonal polynomials, random matrices, finite reflection groups, hyperplane arrangements, conformal field theory and more, see e.g., [4].

This paper is concerned with the intimate connection between Knizhnik–Zamolodchikov (KZ) equations and hypergeometric integrals of Selberg type [2, 11, 13, 15].

Let $\mathfrak{g}$ be a simple Lie algebra of rank n, with simple roots, fundamental weights, and Chevalley generators given by α_i, Λ_i and e_i, f_i, h_i for $1\le i\le n$. Let V_λ and V_μ be highest weight representations of $\mathfrak{g}$ with highest weights λ and μ, and let $u=u(z,w)$ be a function with values in $V_\lambda\otimes V_\mu$ solving the KZ equations

$$\kappa\,\frac{\partial u}{\partial z}=\frac{\Omega}{z-w}\,u,\qquad \kappa\,\frac{\partial u}{\partial w}=\frac{\Omega}{w-z}\,u,$$

where Ω is the Casimir element. Let $\mathrm{Sing}_{\lambda,\mu}[\nu]$ denote the space of singular vectors of weight ν in $V_\lambda\otimes V_\mu$

$$\mathrm{Sing}_{\lambda,\mu}[\nu]:=\{v\in V_\lambda\otimes V_\mu : h_iv=\nu(h_i)v,\ e_iv=0,\ 1\le i\le n\}.$$

Then, according to a theorem of Schechtman and Varchenko [13], solutions u with values in $\mathrm{Sing}_{\lambda,\mu}[\lambda+\mu-\sum_{i=1}^n k_i\alpha_i]$ are expressible in terms of $k:=k_1+\dots+k_n$ dimensional integrals as follows:

$$u(z,w)=\sum u_{IJ}(z,w)\,f^Iv_\lambda\otimes f^Jv_\mu$$

with coordinate functions u_{IJ} given by

$$u_{IJ}(z,w)=\int_\gamma \Psi(z,w;t)\omega_{IJ}(z,w;t)\mathrm{d}t.$$

In the above the sum is over all ordered multisets I and J with elements taken from $\{1,\dots,n\}$ such that their union contains the number i exactly k_i times, v_λ and v_μ are the highest weight vectors of V_λ and V_μ, $f^Iv=(\prod_{i\in I}f_i)v$, $t=(t_1,\dots,t_k)$, $\mathrm{d}t=\mathrm{d}t_1\cdots\mathrm{d}t_k$ and γ is a suitable integration cycle. The function ω_{IJ} is a rational function that will not concern us here and Ψ, known as the phase function, is defined as follows. The first k_1 integration variables are attached to the simple root α_1, the next k_2 integration variables are attached to the simple root α_2, and so on, such that $\alpha_{t_j}:=\alpha_i$ if $k_1+\dots+k_{i-1}<j\le k_1+\dots+k_i$. Then

$$\Psi(z,w;t)=(z-w)^{(\lambda,\mu)/\kappa}\prod_{i=1}^k(t_i-z)^{-(\lambda,\alpha_{t_i})/\kappa}(t_i-w)^{-(\mu,\alpha_{t_i})/\kappa}\prod_{1\le i<j\le k}(t_i-t_j)^{(\alpha_{t_i},\alpha_{t_j})/\kappa},$$

with $(\ ,\ )$ the bilinear symmetric form on $\mathfrak{h}^*$ (the space dual to the Cartan subalgebra $\mathfrak{h}$) normalised such that $(\theta,\theta)=2$ for the maximal root θ.

In [11] Mukhin and Varchenko formulated a remarkable conjecture regarding the normalised phase function

$$\Psi(t)=\prod_{i=1}^{k} t_i^{-(\lambda,\alpha_{t_i})/\kappa}(1-t_i)^{-(\mu,\alpha_{t_i})/\kappa}\prod_{1\le i<j\le n}|t_i-t_j|^{(\alpha_{t_i},\alpha_{t_j})/\kappa}.$$

They proposed that if the space $\mathrm{Sing}_{\lambda,\mu}[\lambda+\mu-\sum_{i=1}^{n}k_i\alpha_i]$ is one-dimensional, then the integral

$$\int_{\Delta}\Psi(t)\mathrm{d}t \tag{1.4}$$

(with $\Delta\subset[0,1]^k$ an appropriate integration domain not explicitly given) is expressible as a product of gamma functions.

For $\mathfrak{g}=\mathfrak{sl}_2=\mathrm{A}_1$ the integral (1.4) (with an appropriate choice for Δ) corresponds to the evaluation of the Selberg integral. Indeed, in this case there is just one fundamental weight and one simple root, with $(\Lambda_1,\alpha_1)=1$ and $(\alpha_1,\alpha_1)=2$. The most general choice for the weights λ and μ is $\lambda=\lambda_1\Lambda_1$ and $\mu=\mu_1\Lambda_1$, so that

$$\Psi(t)=\prod_{i=1}^{k} t_i^{-\lambda_1/\kappa}(1-t_i)^{-\mu_1/\kappa}\prod_{1\le i<j\le n}|t_i-t_j|^{2/\kappa}.$$

Identifying $(\kappa,\lambda_1,\mu_1)=(1/\gamma,(1-\alpha)/\gamma,(1-\beta)/\gamma)$ this becomes precisely the integrand of the Selberg integral.

For $\mathfrak{g}=\mathfrak{sl}_{n+1}=\mathrm{A}_n$, $k_1=\cdots=k_n=1$ and $\lambda=\lambda_n\Lambda_n$, $\mu=\sum_{i=1}^{n}\mu_i\Lambda_i$ the integral (1.4) was computed by Mukhin and Varchenko [11]. Using the standard ordering of the simple roots, so that $(\alpha_i,\alpha_j)=2\delta_{ij}-\delta_{|i-j|,1}$ one finds

$$\Psi(t)=t_n^{-\lambda_n/\kappa}\prod_{i=1}^{n}(1-t_i)^{-\mu_i/\kappa}\prod_{i=2}^{n}|t_i-t_{i+1}|^{-1/\kappa}.$$

Assuming $\Delta=\Delta_n$ (see (1.2)) the integral (1.4) is readily computed by iterating the beta integral (1.3). Identifying

$$(\kappa,\lambda_n,\mu_1,\dots,\mu_n)=(1/\gamma,(1-\alpha)/\gamma,(1-\beta_1)/\gamma,\dots,(1-\beta_n)/\gamma) \tag{1.5}$$

this yields

$$\Psi(t)=\Gamma(\alpha)\Gamma^{n-1}(1-\gamma)\prod_{j=1}^{n}\frac{\Gamma(\beta_1+\cdots+\beta_j+(1-j)\gamma)}{\Gamma(A_j+\beta_1+\cdots+\beta_j+(\delta_{j,n}-j)\gamma)}, \tag{1.6}$$

where $A_j=1+(1-\alpha)\delta_{j,n}$.

Essentially this same iterative method was applied by Mimachi and Takamuki [10] to deal with the tensor product of the vector representations of $\mathfrak{g}=\mathrm{B}_n,\ \mathrm{C}_n$ or D_n. Specifically, with $\lambda=\mu=\Lambda_1$ and

$$k_1=\cdots=k_{n-2}=1,\quad k_{n-1}=r\quad\text{and}\quad k_n=s$$

such that

$$(r,s)=\begin{cases}(2,2) & \mathrm{B}_n\\ (2,1) & \mathrm{C}_n\\ (1,1) & \mathrm{D}_n.\end{cases}$$

Mimachi and Takamuki computed (1.4) for a suitable choice of Δ by either iterating the $n=2$ case of the Selberg integral (in the case of B_n) or by iterating the beta integral (in the case of C_n and D_n).

In [14] Tarasov and Varchenko dealt with what is arguably the first nontrivial case of (1.4), proving an A_2 Selberg integral corresponding to (1.4) with $\lambda=\lambda_2\Lambda_2$, $\mu=\mu_1\Lambda_1+\mu_2\Lambda_2$ and arbitrary $k_1<k_2$. In this paper we present the generalisation of the Tarasov–Varchenko formula for all of the Lie algebras of type A. This result includes the Selberg integral (1.1) and the formula (1.6) as special cases. Specifically, we give the evaluation of (1.4) for $\mathfrak{g}=\mathfrak{sl}_{n+1}=\mathrm{A}_n$ in the case that $\lambda=\lambda_n\Lambda_n$, $\mu=\sum_{i=1}^n\mu_i\Lambda_i$ and $0\leq k_1\leq k_2\leq\cdots\leq k_n$. Again adopting the identification (1.5) our result is contained on the following theorem.

Theorem 1.2 (A_n Selberg integral) *For n a positive integer let $0\leq k_1\leq k_2\leq\cdots\leq k_n$ be integers and $k_0=k_{n+1}=0$. Assume (1.5) such that $\alpha,\beta_1,\ldots,\beta_n,\gamma\in\mathbb{C}$ with*

$$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta_1)>0,\ldots,\mathrm{Re}(\beta_n)>0,$$

$$-\min\{\mathrm{Re}(\alpha)/(k_n-1),1/k_n\}<\mathrm{Re}(\gamma)<1/k_n$$

and

$$-\mathrm{Re}(\beta_s)/(k_s-k_{s-1}-1)<\mathrm{Re}(\gamma)<\mathrm{Re}(\beta_s+\cdots+\beta_r)/(r-s)$$

for $1\leq s\leq r\leq n$. Then

$$\begin{aligned}\int_\Delta\Psi(t)\mathrm{d}t&=\prod_{s=1}^n\prod_{i=1}^{k_s}\frac{\Gamma(A_s+(i-k_{s+1}-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}\\&\quad\times\prod_{1\leq s\leq r\leq n}\prod_{i=1}^{k_s-k_{s-1}}\frac{\Gamma(\beta_s+\cdots+\beta_r+(i+s-r-1)\gamma)}{\Gamma(A_r+\beta_s+\cdots+\beta_r+(i+s-r+k_r-k_{r+1}-2)\gamma)},\end{aligned}$$

where $A_r=1+(\alpha-1)\delta_{r,n}$.

So far we have failed to specify the integration domain Δ, and we will do so in the next section. Let us remark now that for $n=1$ the domain is given by (1.2) so that we recover the Selberg integral (1.1). For general n and $k_1=\cdots=k_n=1$ the domain $\Delta=\Delta_n$ resulting in (1.6).

We also remark that Theorem 1.2 can be further generalised by inclusion of a Jack polynomial in the integrand. This corresponds to an A_n generalisation of the integrals of Aomoto (elementary symmetric function case) [1] and Kadell (the full Jack polynomial case) [6]. We refer the reader to [17] for details.

2 The domain Δ

For general choice of $k_1,\ldots,k_n$ the integration domain Δ is significantly more complicated than, for example, (1.2). It takes the form of a chain in the sense of algebraic topology and, in the case of $\mathfrak{g}=\mathrm{A}_2$, was first described in the work of Tarasov and Varchenko [14].

First we introduce a slight relabelling of the integration variables $t_1,\ldots,t_k$. Recall that the first k_1 variables are attached to the simple root α_1, the next k_2 variables are attached to α_2 and so on. For $1\leq s\leq n$ we now set

$$t^{(s)}=(t_{k_1+\cdots+k_{s-1}+1},\ldots,t_{k_1+\cdots+k_s})$$

so that $t^{(s)}$ contains those t_i attached to α_s. We also use the notation $t_i^{(s)}$ for $1 \le i \le k_s$ to denote the t_i contained in $t^{(s)}$.

Note that $\Psi(t)$ is symmetric under permutation of the components of $t^{(s)}$ so that without loss of generality we may assume the ordering

$$t_{k_s}^{(s)} \le \cdots \le t_2^{(s)} \le t_1^{(s)} \tag{2.1}$$

consistent with (1.2) when $n = 1$.

We also need an ordering between $t_i^{(s)}$ and $t_j^{(s+1)}$, i.e., between the variables attached to two adjacent simple roots (adjacent in the sense of the corresponding Dynkin diagram). To this end we introduce maps

$$M_s : \{1, \dots, k_s\} \to \{1, \dots, k_{s+1}\}$$

such that

$$M_s(i) \le M_s(i+1) \quad \text{and} \quad 1 \le M_s(i) \le k_{s+1} - k_s + i.$$

A standard counting argument shows that there are exactly c_{k_{s+1},k_s} admissible M_s, where $c_{n,k}$ is the row (n, k) entry in the Catalan triangle, or, equivalently, the number of standard Young tableaux of shape (n, k).

For example, if $k_1 = 2$ and $k_2 = 3$ then the conditions on $M := M_1 : \{1, 2\} \to \{1, 2, 3\}$ are

$$1 \le M(1) \le 2, \quad 1 \le M(2) \le 3, \quad \text{and} \quad M(1) \le M(2).$$

This permits exactly 5 distinct maps, mapping $(1, 2)$ to one of

$$(1, 1), (1, 2), (1, 3), (2, 2), (2, 3).$$

Given M_s we fix an ordering among the $t_i^{(s)}$ and $t_j^{(s+1)}$ by

$$t_{M_s(i)}^{(s+1)} \le t_i^{(s)} \le t_{M_s(i)-1}^{(s+1)} \quad \text{for } 1 \le i \le k_s, \tag{2.2}$$

where $t_0^{(s+1)} := \infty$. Given admissible maps $M_1, \dots, M_{n-1}$ define $D_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}$ as the set of points

$$(t_1, \dots, t_k) = (t_1^{(1)}, \dots, t_{k_1}^{(1)}, t_1^{(2)}, \dots, t_{k_2}^{(2)}, \dots, t_1^{(n)}, \dots, t_{k_n}^{(n)}) \in [0, 1]^k$$

such that (2.2) holds for all $1 \le s \le n-1$ and (2.1) holds for $1 \le s \le n$. Then the domain of integration, written as a chain, is given by

$$\Delta = \sum_{M_1,\dots,M_{n-1}} F_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}(\gamma) D_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}, \tag{2.3}$$

where

$$F_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}(\gamma) = \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \frac{\sin\bigl(\pi(i + k_{s+1} - k_s - M_s(i) + 1)\gamma\bigr)}{\sin\bigl(\pi(i + k_{s+1} - k_s)\gamma\bigr)}.$$

3 Remarks about the proof of Theorem 1.2

The proof of the A_n Selberg integral relies on a new q,t-binomial theorem for Macdonald polynomials [16, 17]. This q,t-binomial theorem can be reinterpreted as a q-integral, and taking the $q \to 1$ limit results in Theorem 1.2. This is analogous to the proof of the much simpler beta integral using the classical q-binomial theorem. In the latter case one makes the substitutions $(a,z) \mapsto (q^\beta, q^\alpha)$ in the q-binomial theorem

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k = \frac{(az;q)_\infty}{(z;q)_\infty} $$

(where $(a;q)_\infty = \prod_{0\geq 1}(1-aq^i)$ and $(a;q)_z = (a;q)_\infty/(aq^z;q)_\infty$) to obtain

$$ \int_0^1 t^{\alpha-1}(tq;q)_{\beta-1}\,\mathrm{d}_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}, $$

where Γ_q is the q-gamma function [5] and $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$, $-\beta \notin \{0,1,2,\dots\}$. Assuming that q is real and $\mathrm{Re}(\beta) > 0$ one can take the $q \to 1^-$ to obtain (1.3).

The Macdonald polynomials $P_\lambda(x;q,t)$ [8] satisfy a well-known generalisation of the q-binomial theorem [7, 9]

$$ {}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, t; x\right] := \sum_{\lambda} t^{n(\lambda)} \frac{(a;q,t)_\lambda}{c'_\lambda(q,t)}\, P_\lambda(x;q,t) = \prod_{i\geq 1} \frac{(ax_i;q)_\infty}{(x_i;q)_\infty}. \tag{3.1} $$

where $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$, $(a;q,t)_\lambda = \prod_{i\geq 1}(at^{1-i};q)_{\lambda_i}$ and $c'_\lambda(q,t)$ is a generalisation of $(q;q)_k$, standard in Macdonald polynomial theory. Writing (3.1) as a q-integral and taking the $q \to 1^-$ limit yields the Selberg integral (1.1).

To obtain the more general A_n Selberg integral the q,t-binomial theorem (3.1) needs further generalisation. Because this sum is significantly more difficult to describe than (3.1) we will not state it here in full generality. We remark however that the summand depends on n partitions, each attached to a different Macdonald polynomial. When all Macdonald polynomials are principally specialised one obtains

$$ \begin{aligned} {}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, t; x^{(1)},\dots,x^{(n)}\right] &= \prod_{s=1}^{n} \prod_{i=1}^{k_s-k_{s-1}} \frac{(az_s\cdots z_n t^{i+s+k_{s-1}+\cdots+k_{n-1}-n-1};q)_\infty}{(z_s\cdots z_n t^{i+s+k_{s-1}+\cdots+k_{n-1}-n-1};q)_\infty} \\ &\quad\times \prod_{1\leq s\leq r\leq n-1} \prod_{i=1}^{k_s-k_{s-1}} \frac{(qz_s\cdots z_r t^{i+s-r+k_{s-1}+\cdots+k_r-k_{r+1}-2};q)_\infty}{(z_s\cdots z_r t^{i+s-r+k_{s-1}+\cdots+k_{r-1}-1};q)_\infty}, \end{aligned} $$

where ${}_1\Phi_0$ is a suitable A_n generalisation of the series in (3.1), $k_0 = 0$ and $x^{(s)} = z_s(1,t,\dots,t^{k_s-1})$ for $1 \leq s \leq n$. It is this q,t-binomial theorem that gives rise to Theorem 1.2.

4 Two simple examples

To conclude we present the worked-out examples of the A_n Selberg integral for

1. $(k_1,\dots,k_{n-1},k_n) = (1,\dots,1,k)$

2. $\gamma = 0$.

Note that in (1) we no longer use k to denote the number of integration variables (i.e., the sum of the k_i). Also note that (1) for $k = 1$ corresponds to (1.6).

4.1 The case $(k_1, \dots, k_{n-1}, k_n) = (1, \dots, 1, k)$

In this case there is only one map M_s for $1 \le s \le n-2$, corresponding to the identity map $M_s(1) = 1$. For $s = n-1$, however, there are k different maps, given by $M_{n-1}(1) = a$ for $1 \le a \le k$.

If we relabel the integration variables $t_1^{(s)} \to u_s$ for $1 \le s \le n-1$ and $t_i^{(n)} \to t_i$ for $1 \le i \le k$, then the above implies the inequalities

$$O_\ell : \begin{cases} 0 \le t_k \le \cdots \le t_1 \le 1, \\ 0 \le u_{n-1} \le \cdots \le u_1 \le 1, \\ t_\ell \le u_{n-1} \le t_{\ell-1} \end{cases}$$

with $1 \le \ell \le k$ and $t_0 = 1$. As a result we obtain the following $(k+n-1)$-dimensional integral:

$$\begin{aligned} &\sum_{\ell=1}^{k} \frac{\sin(\pi(k-\ell+1)\gamma)}{\sin(\pi k\gamma)} \int_{O_\ell} \prod_{i=1}^{n-1}(1-u_i)^{\beta_i-1} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta_n-1} \prod_{1\le i<j\le k}(t_i-t_j)^{2\gamma} \\ &\qquad\qquad \times \prod_{i=1}^{n-2}(u_i-u_{i+1})^{-\gamma} \prod_{i=1}^{\ell-1}(t_i-u_{n-1})^{-\gamma} \prod_{i=\ell}^{k}(u_{n-1}-t_i)^{-\gamma}\, \mathrm{d}u\, \mathrm{d}t \\ &\quad = \Gamma(1-k\gamma)\Gamma^{n-2}(1-\gamma) \prod_{i=1}^{k} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)} \\ &\qquad\qquad \times \prod_{i=1}^{k-1} \frac{\Gamma(\beta_n+(i-1)\gamma)}{\Gamma(\alpha+\beta_n+(i+k-2)\gamma)} \prod_{i=1}^{n} \frac{\Gamma(\beta_1+\cdots+\beta_i+(1-i)\gamma)}{\Gamma(a_i+\beta_1+\cdots+\beta_i-i\gamma)}, \end{aligned}$$

where $a_1 = \cdots = a_{n-2} = 1$, $a_{n-1} = 1-(k-1)\gamma$, $a_n = \alpha + k\gamma$, $\mathrm{d}u = \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1}$ and $\mathrm{d}t = \mathrm{d}t_1 \cdots \mathrm{d}t_k$.

Let us denote the left-hand side of the above by $I_{n-1,k}(\alpha; \beta_1, \dots, \beta_n; \gamma)$. Next let $n \ge 3$ and replace the integration variable u_1 by v via

$$v = \frac{u_1 - u_2}{1 - u_2}.$$

Noting that $1 - u_1 = (1-v)(1-u_2)$ and $u_1 - u_2 = v(1-u_2)$ the integral over v may be identified as Euler's beta integral (1.3) with $\alpha = 1 - \gamma$. Therefore

$$I_{n-1,k}(\alpha; \beta_1, \dots, \beta_n; \gamma) = I_{n-2,k}(\alpha; \beta_1+\beta_2-\gamma, \beta_3, \dots, \beta_n; \gamma) \frac{\Gamma(1-\gamma)\Gamma(\beta_1)}{\Gamma(\beta_1-\gamma+1)}.$$

Iterating this recursion it follows that $I_{n-1,k}$ can be reduced to $I_{1,k}$, given by

$$\begin{aligned}\sum_{\ell=1}^{k}\frac{\sin(\pi(k-\ell+1)\gamma)}{\sin(\pi k\gamma)}\int_{O'_\ell}(1-u)^{\beta_1-1}\prod_{i=1}^{k}t_i^{\alpha-1}(1-t_i)^{\beta_2-1}\prod_{1\le i<j\le k}(t_i-t_j)^{2\gamma}&\\ \times\prod_{i=1}^{\ell-1}(t_i-u)^{-\gamma}\prod_{i=\ell}^{k}(u-t_i)^{-\gamma}\,\mathrm{d}u\,\mathrm{d}t&\\ =\frac{\Gamma(\beta_1)\Gamma(1-k\gamma)}{\Gamma(1+\beta_1-k\gamma)}\,\frac{\Gamma(\alpha+\beta_2+(2k-2)\gamma)}{\Gamma(\alpha+\beta_1+\beta_2+(k-2)\gamma)}\,\frac{\Gamma(\beta_1+\beta_2-\gamma)}{\Gamma(\beta_2+(k-1)\gamma)}&\\ \times\prod_{i=1}^{k}\frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta_2+(i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha+\beta_2+(i+k-2)\gamma)\Gamma(\gamma)}&,\end{aligned}$$

where

$$O'_\ell:\quad 0\le t_k\le\cdots\le t_\ell\le u\le t_{\ell-1}\le\cdots\le t_1\le 1.$$

4.2 The case $\gamma=0$

When $\gamma=0$ Theorem 1.2 collapses to the integral

$$\begin{aligned}\int_\Delta\prod_{s=1}^{n}\prod_{i=1}^{k_s}\big(1-t_i^{(s)}\big)^{\beta_s-1}\prod_{i=1}^{k_n}\big(t_i^{(n)}\big)^{\alpha-1}\,\mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)}&\\ =\prod_{s=1}^{n}\frac{1}{(k_s)!}\bigg(\frac{\Gamma(\alpha)\Gamma(\beta_s+\cdots+\beta_n)}{\Gamma(\alpha+\beta_s+\cdots+\beta_n)}\bigg)^{k_s-k_{s-1}}&,\end{aligned}$$

where Δ is given by the $\gamma\to 0$ limit of (2.3). Because the $t_i^{(s)}$ in the integrand are completely decoupled the problem of evaluating this integral is purely combinatorial. Introducing the partitions $\lambda^{(s)}$ for $1\le s\le n-1$ as $\lambda^{(s)}=(M_s(k_s),\dots,M_s(1))$ so that $\lambda^{(s)}$ has exactly k_s parts and $\lambda_i^{(s)}\le k_{s+1}-i+1$ the $\gamma=0$ integral may also be stated more explicitly as

$$\begin{aligned}\sum_{\substack{\lambda^{(1)},\dots,\lambda^{(n-1)}\\ 1\le\lambda_i^{(s)}\le k_{s+1}-i+1}}\prod_{s=1}^{n-1}\prod_{i=1}^{k_s}\frac{k_{s+1}-i-\lambda_i+2}{k_{s+1}-i+1}&\\ \times\int\prod_{s=1}^{n}\prod_{i=1}^{k_s}\big(1-t_i^{(s)}\big)^{\beta_s-1}\prod_{i=1}^{k_n}\big(t_i^{(n)}\big)^{\alpha-1}\,\mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)}&\\ =\prod_{s=1}^{n}\frac{1}{(k_s)!}\bigg(\frac{\Gamma(\alpha)\Gamma(\beta_s+\cdots+\beta_n)}{\Gamma(\alpha+\beta_s+\cdots+\beta_n)}\bigg)^{k_s-k_{s-1}}&,\end{aligned}$$

where the integration domain is given by

$$\max\Big\{t^{(s+1)}_{\lambda^{(s)}_{k_s-i+1}},t^{(s)}_{i+1}\Big\}\le t_i^{(s)}\le\min\Big\{t^{(s)}_{i-1},t^{(s+1)}_{\lambda^{(s)}_{k_s-i+1}-1}\Big\}\tag{4.1a}$$

for $1 \le s \le n-1$ and $1 \le i \le k_s$ (with $t_0^{(s)} = 1$ and $t_{k_s+1}^{(s)} = 0$), and

$$t_{i+1}^{(n)} \le t_i^{(n)} \le t_{i-1}^{(n)} \tag{4.1b}$$

for $1 \le i \le k_n$.

Thanks to the factor $\prod_{s=1}^{n-1}\prod_{i=1}^{k_s}(k_{s+1} - i - \lambda_i + 2)$ we may relax the condition $\lambda_i^{(s)} \le k_{s+1} - i + 1$ to $\lambda_1^{(s)} \le k_{s+1}$ so that the sum becomes

$$\sum_{\substack{\lambda^{(1)},\dots,\lambda^{(n-1)} \\ l(\lambda^{(s)})=k_s \\ \lambda_1^{(s)} \le k_{s+1}}} .$$

This result may be proved by elementary means using the following representation of the elementary symmetric function:

$$e_r(x) = \sum_{\substack{\lambda \\ l(\lambda)=r \\ \lambda_1 \le n}} \prod_{j=1}^{n} \frac{1}{m_j!} \prod_{i=1}^{r} (n - \lambda_i - i + 2)(x_{\lambda_i} - x_{\lambda_i - 1}),$$

with $x_0 = 0$, $x = (x_1, \dots, x_n)$, m_j the multiplicity of the part j in λ.

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