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Finitized conformal spectrum of the Ising model on the cylinder and torus

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Abstract

The spectrum of the critical Ising model on a lattice with cylindrical and toroidal boundary conditions is calculated by commuting transfer matrix methods. Using a simple truncation procedure, we obtain the natural finitizations of the conformal spectra recently proposed by Melzer. These finitizations imply polynomial identities which in the large lattice limit give rise to the Rogers–Ramanujan identities for the $c = 1/2$ Virasoro characters.

1. Introduction

Though the square lattice Ising model [1] was solved by Onsager [2] over fifty years ago, a large number of papers devoted to this favourite toy model in statistical mechanics have appeared perennially ever since. Despite its simplicity, the Ising model has fascinating mathematical structures underlying its integrability, as well as simple but realistic applications to a wide variety of physical systems. Indeed, many well-known physicists have been motivated by this model, and such names as Onsager, Yang, Kasteleyn, Fisher, McCoy, Wu, Baxter and Capel, to name a few, are irretrievably associated with the Ising model [2–8].

Another much more modern chapter of statistical mechanics is the application of conformal invariance to critical systems [9,10]. For the particular case of the critical Ising model, Cardy [11] considered the finite-size effects on both toroidal and cylindrical

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geometries using the principles of conformal and modular invariance. He was thus able to make precise predictions about the form of the corrections to the eigenvalues of the row transfer matrix and to give the modular invariant (covariant) partition functions in terms of characters of the Virasoro algebra.

Cardy's predictions were indeed proven to be correct by analysis of the critical Ising model partition function in the scaling limit. In the case of toroidal boundary conditions, this was first carried out by Ferdinand and Fisher [12], who in fact derived the modular invariant partition function fifteen years prior to the advent of conformal invariance! In the case of cylindrical boundary conditions, Cardy's results were confirmed by Bugrij and Shadura [13].

It is the aim of this paper to extend the analysis of Refs. [12,13], following the approach of Baxter's commuting transfer matrices [7]. In particular, we solve functional equations for the row transfer matrix eigenvalues, which are satisfied as a consequence of the star-triangle relation. The reason for repeating the calculation of the conformal spectrum of the Ising model is the recent interest in fermionic character representations and the discovery of many "natural" polynomial finitizations of the $c < 1$ Virasoro characters; see for example Refs. [14–19]. In particular, we should mention Melzer's paper [15], in which finitizations were proposed for the unitary minimal Virasoro characters and modular invariant partition functions. Interestingly, the finitized characters had appeared previously in the corner transfer matrix calculations of a series of solvable lattice models by Andrews, Baxter and Forrester [20]. By computing the spectrum of the row transfer matrix for the Ising model, we show that also the row transfer matrix naturally acts as a generating function of finitized characters. This is similar to findings based on Bethe Ansatz studies of the row transfer matrix spectrum [16,19,21], but without the use of the so-called string-hypothesis.

The rest of this paper is organised as follows. In the next section we define the critical Ising model with cylindrical and toroidal boundary conditions, and review some of the predictions of conformal invariance concerning the spectrum of the model. Then in Section 3 we present functional equations satisfied by the transfer matrix eigenvalues and solve these exactly for arbitrary system size. In Section 4 we derive the finitized conformal spectra using a simple truncation procedure, and discuss the $c = 1/2$ character identities. We briefly discuss and summarize our results in Section 5.

2. The Ising model on the cylinder and torus

2.1. Definition of the model

Cylindrical geometry

To conveniently define the Ising model with a variety of boundary conditions, it is useful to define two different two-dimensional lattices. We define the lattice $\mathcal{L}$ as a square lattice rotated by 45 degrees, in which the rows have alternately $L-1$ and L faces. Similarly, we define the lattice $\mathcal{L}'$ as a rotated square lattice in which each row

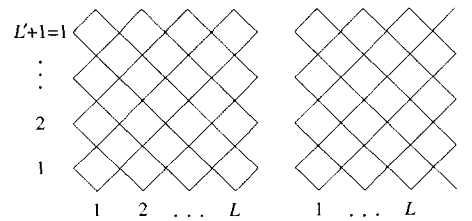

Fig. 1. The lattices $\mathcal{L}$ and $\mathcal{L}'$, respectively.

has L faces. Vertically both lattices have columns of L' faces, and we impose periodic boundary conditions in this direction by identifying the first and the $(L'+1)$th rows of faces. Both the lattices $\mathcal{L}$ and $\mathcal{L}'$ are depicted in Fig. 1. We note that $\mathcal{L}$ consists of $2L$ and $\mathcal{L}'$ of $2L+1$ (zigzagging) columns, and that both lattices have $2L'$ rows of edges. For later use we denote these numbers by N and M, so $M = 2L'$ and

$$N = 2L \text{ for } \mathcal{L}, \quad N = 2L+1 \text{ for } \mathcal{L}'. \tag{2.1}$$

Those sites of the lattice incident to only two edges define the boundary. Note that both $\mathcal{L}$ and $\mathcal{L}'$ have M left and right boundary sites.

We now define the usual Ising model on $\mathcal{L}$ and $\mathcal{L}'$ by putting Ising spins on each site of the lattice. We are concerned with the following four types of boundary conditions:

$++$ *(fixed) boundary* We choose the lattice $\mathcal{L}$ and fix the spins at the left and right boundaries to be $+1$.

$+-$ *(fixed) boundary* We choose the lattice $\mathcal{L}$ and fix the spins at the left and right boundaries to be $+1$ and -1, respectively.

mixed boundaries We choose the lattice $\mathcal{L}'$ and fix the spins at the left boundary to be $+1$, but place no restriction on the right boundary spins.

free boundaries We choose the lattice $\mathcal{L}$ and place no restriction on either boundary.

Denoting the set of spins not fixed by the boundary condition by $\{\sigma\}$, we define the partition function in each case to be

$$Z_{NM} = \sum_{\{\sigma\}} \exp\left(J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + K \sum_{\langle i,j \rangle} \sigma_i \sigma_j \right). \tag{2.2}$$

For fixed and mixed boundaries, the first sum within the parenthesis is over edges in odd columns and the second sum over edges in even columns of the lattice. Conversely, for free boundary conditions the first sum corresponds to even columns and the second sum to odd columns. Since we restrict ourselves to the critical Ising model, we have $\sinh(2J)\sinh(2K) = 1$. This condition can be conveniently parameterized by introducing a so-called spectral parameter u, so that

$$\sinh(2J) = \cot(2u), \quad \sinh(2K) = \tan(2u), \tag{2.3}$$

with $0 < u < \pi/4$.

Let us remark here that we could of course equally well have defined the $++$, $+-$ and the free boundary model on the lattice $\mathcal{L}'$ and the mixed boundary model on $\mathcal{L}$. The particular choices made above are purely for computational convenience and nothing else. In fact, we have performed the complete calculations on both lattices, but since the relevant results are independent of either choice, we have chosen to restrict ourselves to the above.

Toroidal geometry

To get the Ising model with the usual doubly-periodic boundary conditions, we take the lattice $\mathcal{L}$ and identify the first and the $(L+1)$th columns of faces. Again we define the partition function as in (2.2), but now the first sum within the parenthesis is over NW-SE edges, and the second sum over NE-SW edges.

2.2. Conformal properties

The critical properties of the Ising model are described by a conformal field theory characterised by a central charge $c = 1/2$ and conformal weights $\Delta = 0, 1/2$ and $1/16$. Here we review the relevant predictions which arise by applying the theory of conformal invariance to the model [11].

Cylindrical geometry

Consider the finite-size partition function Z_{NM} of the critical Ising model on a cylindrical lattice of N columns and M rows as described previously. The asymptotic behaviour of Z_{NM} in the limit of large N and M with the ratio M/N fixed is given by

$$Z_{NM}(u) = \mathrm{Tr}\Big(T(u)^M\Big) \sim \exp\Big(-NMf_\mathrm{b}(u) - Mf_\mathrm{s}(u)\Big)\, Z(q). \tag{2.4}$$

Here T is the transfer matrix to be defined in the next section, f_b is the bulk free energy, f_s is the surface free energy and $Z(q)$ is the universal conformal partition function with the modular parameter q given by

$$q = \exp(-2\pi \sin(4u) M/N). \tag{2.5}$$

To leading orders, each transfer matrix eigenvalue $\Lambda(u)$ reads [22]

$$\tfrac{1}{2}\log \Lambda(u) = -Nf_\mathrm{b} - f_\mathrm{s} + \frac{\pi}{N}\left(\frac{c}{24} - \Delta - n\right)\sin(4u) + o\left(\frac{1}{N}\right), \tag{2.6}$$

with Δ one of the conformal weights, and n a non-negative integer which is zero for the largest eigenvalue. Therefore, all eigenvalues naturally divide into three towers, and after taking the ratio of each eigenvalue with the largest, we can write

$$Z(q) = \sum_{\Delta} q^{-c/24+\Delta} \mathcal{N}(\Delta) \sum_{k} q^{n_k}, \tag{2.7}$$

with $\mathcal{N}(\Delta) = 0$ or 1. In fact, Cardy has predicted that [11]

$$Z(q) = \sum_{\Delta} \mathcal{N}(\Delta)\chi_\Delta(q), \tag{2.8}$$

with the operator content given by

$$\Big(\mathcal{N}(0),\mathcal{N}(1/2),\mathcal{N}(1/16)\Big)=\begin{cases}(1,0,0) & ++\ \text{boundary},\\ (0,1,0) & --\ \text{boundary},\\ (0,0,1) & \text{mixed boundaries},\\ (1,1,0) & \text{free boundaries}.\end{cases} \tag{2.9}$$

The $\chi_\Delta(q)$ in the above are the $c = 1/2$ Virasoro characters,

$$\begin{aligned} q^{1/48}\chi_0(q) &= \tfrac{1}{2}\left\{\prod_{k=1}^{\infty}\left(1+q^{k-1/2}\right)+\prod_{k=1}^{\infty}\left(1-q^{k-1/2}\right)\right\}\\ &= \sum_{\substack{m\ge 0\\ m\ \text{even}}}\frac{q^{m^2/2}}{(q)_m}=\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left\{q^{j(12j+1)}-q^{(3j+1)(4j+1)}\right\}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} q^{-23/48}\chi_{1/2} &= \tfrac{1}{2}q^{-1/2}\left\{\prod_{k=1}^{\infty}\left(1+q^{k-1/2}\right)-\prod_{k=1}^{\infty}\left(1-q^{k-1/2}\right)\right\}\\ &= \sum_{\substack{m\ge 0\\ m\ \text{odd}}}\frac{q^{(m^2-1)/2}}{(q)_m}=\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left\{q^{j(12j+5)}-q^{(3j+2)(4j+1)}\right\}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} q^{-1/24}\chi_{1/16}(q) &= \prod_{k=1}^{\infty}\left(1+q^{k}\right)\\ &= \sum_{m\ge 0}\frac{q^{m(m+1)/2}}{(q)_m}=\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left\{q^{j(12j-2)}-q^{(3j+1)(4j+2)}\right\}, \end{aligned} \tag{2.12}$$

where $(q)_m = \prod_{k=1}^{m}(1-q^k)$ for $m > 0$ and $(q)_0 = 1$. The three different forms of each of the characters constitute the $c = 1/2$ Rogers–Ramanujan identities, and will all appear in finitized (polynomial) form in our subsequent calculations.

Toroidal geometry

In the toroidal geometry, the partition function is still given by (2.4), but with

$$q = \exp\Big(-2\pi \mathrm{i} e^{-4\mathrm{i}u} M/N\Big). \tag{2.13}$$

Of course, since the lattice is periodic, there is no surface free energy. The finite-size corrections of the Ising model eigenvalues are given by [22,23]

$$\log \Lambda(u) = -Nf_b + \frac{2\pi}{N}\left[\left(\tfrac{1}{12}c - 2\Delta\right)\sin(4u) - \mathrm{i}\,n\,e^{-4\mathrm{i}u} + \mathrm{i}\,\bar{n}\,e^{4\mathrm{i}u}\right] + o\Big(\frac{1}{N}\Big), \tag{2.14}$$

with n and $\bar{n}$ non-negative integers, both of which are zero for the largest eigenvalue. So again the eigenvalues divide into three towers, and the modular invariant partition function can be written as

$$Z(q) = \sum_{\Delta} q^{-c/12+2\Delta} \mathcal{N}(\Delta) \sum_{k} q^{n_k} \sum_{\ell} (\bar{q})^{n_\ell}, \tag{2.15}$$

where $\mathcal{N}(\Delta)$ can be 0 or 1, and $\bar{q}$ is the complex conjugate of q. In fact, $\mathcal{N}(\Delta)$ is always 1, and the sums over k and ℓ can be expressed in terms of the $c = 1/2$ Virasoro characters to yield [11]

$$Z(q) = \chi_0(q)\chi_0(\bar{q}) + \chi_{1/2}(q)\chi_{1/2}(\bar{q}) + \chi_{1/16}(q)\chi_{1/16}(\bar{q}). \tag{2.16}$$

3. Calculation of the partition function

In this section we compute all eigenvalues of the transfer matrices for the Ising models defined in Section 2.1. For the case of toroidal boundary conditions, we omit the details altogether and simply state the results obtained by Baxter in Ref. [7].

3.1. Transfer matrices

Following Ref. [24], we define finite-width transfer matrices for each of the four different boundary conditions on the cylinder. First, for the fixed boundary conditions we set

$$T_{\sigma,\sigma'}(u) = \sum_{\{\sigma''\}} W\begin{pmatrix} +1 & \\ & \sigma_1'' \\ +1 & \end{pmatrix} W\begin{pmatrix} & \sigma_1' \\ \sigma_1'' & \\ & \sigma_1 \end{pmatrix} \cdots W\begin{pmatrix} \sigma_{L-1}' & \\ & \sigma_L'' \\ \sigma_{L-1} & \end{pmatrix} W\begin{pmatrix} & \tau \\ \sigma_L'' & \\ & \tau \end{pmatrix}, \tag{3.1}$$

with $\tau = +1$ for the $++$ case and $\tau = -1$ for the $+-$ case. The Boltzmann weights W are given by

$$\begin{aligned}
W\begin{pmatrix} \sigma & \\ & \sigma \\ \sigma & \end{pmatrix} &= \sqrt{2}\cos^2(u), & W\begin{pmatrix} & \sigma \\ \sigma & \\ & \sigma \end{pmatrix} &= \sqrt{2}\cos^2(\tfrac{1}{4}\pi - u), \\
W\begin{pmatrix} \sigma & \\ & -\sigma \\ \sigma & \end{pmatrix} &= \sqrt{2}\sin^2(u), & W\begin{pmatrix} & \sigma \\ -\sigma & \\ & \sigma \end{pmatrix} &= \sqrt{2}\sin^2(\tfrac{1}{4}\pi - u), \\
W\begin{pmatrix} \sigma & \\ & \sigma \\ -\sigma & \end{pmatrix} &= \sin(2u)/\sqrt{2}, & W\begin{pmatrix} & \sigma \\ \sigma & \\ & -\sigma \end{pmatrix} &= \cos(2u)/\sqrt{2}, \\
W\begin{pmatrix} -\sigma & \\ & \sigma \\ \sigma & \end{pmatrix} &= \sin(2u)/\sqrt{2}, & W\begin{pmatrix} & -\sigma \\ \sigma & \\ & \sigma \end{pmatrix} &= \cos(2u)/\sqrt{2},
\end{aligned} \tag{3.2}$$

with $\sigma = \pm 1$.

Similarly, for the mixed boundary conditions we define

$$T_{\sigma,\sigma'}(u) = \sum_{\{\sigma''\}} W\begin{pmatrix} +1 & \\ & \sigma_1'' \\ +1 & \end{pmatrix} W\begin{pmatrix} & \sigma_1' \\ \sigma_1'' & \\ & \sigma_1 \end{pmatrix} \cdots W\begin{pmatrix} \sigma_L' & \\ & \sigma_{L+1}'' \\ \sigma_L & \end{pmatrix}, \tag{3.3}$$

and for free boundaries

$$T_{\sigma,\sigma'}(u) = \sum_{\{\sigma''\}} W\begin{pmatrix} & \sigma_1' & \\ \sigma_1'' & & \\ & \sigma_1 & \end{pmatrix} W\begin{pmatrix} \sigma_1' & & \\ & & \sigma_2'' \\ \sigma_1 & & \end{pmatrix} \cdots W\begin{pmatrix} \sigma_L' & & \\ & & \sigma_{L+1}'' \\ \sigma_L & & \end{pmatrix}. \tag{3.4}$$

We remark here that for computational reasons we have normalized the weights in comparison with (2.2) so that in all cases $T(0)$ and $T(\pi/4)$ become the identity matrix. Specifically, the weights in the first column of (3.2) have been multiplied by the factor $\sin(2u)/\sqrt{2}$ and the weights in the second column by $\cos(2u)/\sqrt{2}$.

In Ref. [24] it was shown that each of the above-defined transfer matrices forms a commuting family, i.e., $T(u)T(v) = T(v)T(u)$. Furthermore, they all satisfy the same symmetry $T(u) = T(\pi/4 - u)$ and functional equation

$$T(u)T(u+\tfrac{1}{4}\pi) = \frac{\cos^{2(N+1)}(2u) - \sin^{2(N+1)}(2u)}{\cos(4u)} I, \tag{3.5}$$

with I the identity matrix.

In the following two sections we solve this functional equation to obtain all eigenvalues of the Ising model transfer matrices for arbitrary finite strip-width N.

3.2. Fixed and free boundary conditions

For these cases the functional equation (3.5) together with the commutativity of T implies that the eigenvalues of the transfer matrix satisfy

$$\Lambda(u)\Lambda(u+\tfrac{1}{4}\pi) = \left(\cos^{2(2L+1)}(2u) - \sin^{2(2L+1)}(2u)\right)/\cos(4u). \tag{3.6}$$

The right-hand side can simply be factorized to yield

$$\Lambda(u)\Lambda(u+\tfrac{1}{4}\pi) = \left(4e^{4iu}\right)^{-2L} \prod_{\substack{k=1 \\ k\neq L+1}}^{2L+1} \left(e^{8iu} + \tan^2\left(\frac{\pi(2k-1)}{4(2L+1)}\right)\right). \tag{3.7}$$

This in turn implies

$$\Lambda(u) = \epsilon \left(4ie^{4iu}\right)^{-L} \prod_{\substack{k=1 \\ k\neq L+1}}^{2L+1} \left(e^{4iu} + i\,\mu_k \tan\left(\frac{\pi(2k-1)}{4(2L+1)}\right)\right), \tag{3.8}$$

with $\epsilon^2 = \mu_k^2 = 1$ for all k.

From the symmetry $\Lambda(u) = \Lambda(\pi/4 - u)$, we have $\mu_k = \mu_{2L-k+2}$, and from $\Lambda(0) = 1$ we find $\prod_{k=1}^{L} \mu_k = \epsilon$. This allows the eigenvalue expression to be simplified to

$$\Lambda(u) = 2^{-L} \prod_{k=1}^{L} \left(\operatorname{cosec}\left(\frac{\pi(2k-1)}{2(2L+1)}\right) + \mu_k \sin(4u)\right). \tag{3.9}$$

Each μ_k in this expression can be chosen independently, and hence (3.9) gives rise to 2^L eigenvalues. This is indeed the dimension of the transfer matrix for free boundaries, but for either of the two fixed boundary cases the size of T is 2^{L-1}. To determine which choices of $\mu_1, \ldots, \mu_L$ correspond to eigenvalues for the $++$ transfer matrix and which to the $+-$ transfer matrix, we study the so-called braid limit $\mathrm{i}u \to \infty$. Appropriately renormalizing the transfer matrix and then taking this limit, we have

$$\Lambda^{\mathrm{b}} = \lim_{\mathrm{i}u \to \infty} \left(4\mathrm{i}\mathrm{e}^{-4\mathrm{i}u}\right)^L \Lambda(u) = \prod_{k=1}^{L} \mu_k. \tag{3.10}$$

From this we see that in the braid limit all eigenvalues of the transfer matrix are ± 1. This is in agreement with the braid limit of the functional equation (3.6), which reduces to $(\Lambda^{\mathrm{b}})^2 = 1$.

In fact, we can show that $\Lambda^{\mathrm{b}} = 1$ for all eigenvalues of the $++$ transfer matrix, and $\Lambda^{\mathrm{b}} = -1$ for all eigenvalues of the $+-$ transfer matrix. To see this we note that from the explicit form of the weights (3.2) it follows that

$$\lim_{\mathrm{i}u \to \infty} 4\mathrm{i}\mathrm{e}^{-4\mathrm{i}u} \sum_{\mu} W\begin{pmatrix} \sigma' & \\ \sigma & \mu \end{pmatrix} W\begin{pmatrix} & \tau \\ \mu & \tau \end{pmatrix} = \sigma\tau\,\delta_{\sigma,\sigma'}. \tag{3.11}$$

In the braid limit the fixed boundary transfer matrices are therefore diagonal, with diagonal elements

$$+\sigma_1\sigma_1\sigma_2\sigma_2 \ldots \sigma_{L-1}\sigma_{L-1}\tau = \tau, \tag{3.12}$$

where $\tau = 1$ for the $++$ boundary and -1 for the $+-$ boundary. Hence $\Lambda^{\mathrm{b}} = \tau$ for all eigenvalues, and (3.10) then implies that $\prod_{k=1}^{L} \mu_k = 1$ for the $++$ boundary and $\prod_{k=1}^{L} \mu_k = -1$ for the $+-$ boundary. This correctly yields 2^{L-1} eigenvalues for either of the fixed boundary cases, and 2^L eigenvalues for the free boundary model with unrestricted μ_k.

3.3. Mixed boundary conditions

This time we have to solve the functional equation

$$\Lambda(u)\Lambda(u + \tfrac{1}{4}\pi) = \left(\cos^{4(L+1)}(2u) - \sin^{4(L+1)}(2u)\right) / \cos(4u). \tag{3.13}$$

As before, we factorize the right-hand side, use the symmetry $\Lambda(u) = \Lambda(\pi/4 - u)$ and initial condition $\Lambda(0) = 1$, to find

$$\Lambda(u) = 2^{-L}\sqrt{L+1} \prod_{k=1}^{L} \left(\operatorname{cosec}\left(\frac{\pi k}{2(L+1)}\right) + \mu_k \sin(4u) \right), \tag{3.14}$$

with $\mu_k^2 = 1$ for all k. Since the size of the transfer matrix is 2^L, all independent choices of $\mu_1, \ldots, \mu_L$ are allowed.

3.4. Toroidal boundary conditions

Following Baxter [7], we solve the functional equation

$$\Lambda(u)\Lambda(u+\tfrac{1}{4}\pi) = \cos^{2L}(2u) + r\sin^{2L}(2u), \tag{3.15}$$

with $\Lambda(u) = \overline{\Lambda}(\pi/4 - u)$ and $\Lambda(0)$ the shift operator. Here r can take the values ± 1, and for $r = 1$ and L even one finds [7]

$$\Lambda(u) = \epsilon\sqrt{2}\left(2e^{2iu+i\pi/4}\right)^{-L}\prod_{k=1}^{L}\left(e^{4iu} + i\,\mu_k\tan\left(\frac{\pi(2k-1)}{4L}\right)\right), \tag{3.16}$$

with $\epsilon^2 = \mu_k^2 = 1$ for all k, and $\prod_{k=1}^{L}\mu_k = 1$. Similarly, for $r = -1$, the solution reads

$$\Lambda(u) = \epsilon\sqrt{L}\left(2e^{2iu+i\pi/4}\right)^{1-L}\prod_{k=1}^{L-1}\left(e^{4iu} + i\,\mu_k\tan\left(\frac{\pi k}{2L}\right)\right), \tag{3.17}$$

with $\epsilon^2 = \mu_k^2 = 1$ for all k.

We note that the total number of eigenvalues given by the above expressions is 2^{L+1} and that each eigenvalue appears with its negative.

4. Finitized conformal partition functions

In this section we use the solutions of the previous section to derive finitizations of the predicted conformal spectra.

4.1. Fixed boundary conditions

The ++ boundary

To compute the partition function (2.7) we have to take the ratio

$$\lim_{N\to\infty} N\log\left(\frac{\Lambda_j(u)}{\Lambda_0(u)}\right) \equiv -2\pi m_j\sin(4u) \tag{4.1}$$

for an infinite number of eigenvalues, where $\Lambda_0(u)$ is the largest eigenvalue and has all $\mu_k^{(0)} = 1$. Due to the form of the eigenvalue expression (3.9), only those terms in the product over k for which $\mu_k^{(j)}$ differs from 1 contribute to m_j. We first consider those $\Lambda_j(u)$ for which all $\mu_k^{(j)}$ are 1 for $k > L$, and define the finitized (and normalized) partition function [4]

$$X_0(L;q) = X_0(L) = \sum_{j=0}^{2^{L-1}-1} q^{m_j}. \tag{4.2}$$

Obviously, for $L \to \infty$ this sum becomes just (2.7).

[4] The notation $X_0(q)$ chosen here is in anticipation of the result; a finitized form of the $\chi_0(q)$ character.

In computing the ratio (4.1), we need

$$\lim_{N\to\infty} N\log\left(\frac{\operatorname{cosec}(\pi K/(N+1)) - \sin(4u)}{\operatorname{cosec}(\pi K/(N+1)) + \sin(4u)}\right) = -2\pi K\sin(4u), \tag{4.3}$$

with $K = k - 1/2$.

We then obtain to leading order

$$\left(\frac{\Lambda_j(u)}{\Lambda_0(u)}\right)^M \simeq \prod_{k=1}^{L} q^{\,(k-1/2)\,\delta_{\mu_k^{(j)},-1}}, \tag{4.4}$$

with q as in (2.5). This allows the finitized sum to be written as

$$\begin{aligned} X_0(L;q) &= \sum_{\{\mu\}_L^+} \prod_{k=1}^{L} q^{\,(k-1/2)\,\delta_{\mu_k,-1}} \\ &= \sum_{\{\mu\}_L^+} \prod_{k=1}^{L} \tfrac{1}{2}\left[\left(1+q^{k-1/2}\right) + \mu_k\left(1-q^{k-1/2}\right)\right], \end{aligned} \tag{4.5}$$

where here and in the following $\{\mu\}_L$ denotes the set of all sequences $\mu_1,\ldots,\mu_L$ with $\mu_k = \pm 1$, and $\{\mu\}_L^{\pm}$ denote the subsets of $\{\mu\}_L$ in which $\prod_{k=1}^{L}\mu_k = \pm 1$.

In the following we present two simple methods to evaluate the sum (4.5), each of which yields results of a manifestly different form. First, from the elementary considerations

$$\begin{aligned} \sum_{\{\mu\}_L^{\pm}} \prod_{k=1}^{L} \tfrac{1}{2}(a_k + \mu_k b_k) &= \tfrac{1}{2} \sum_{\{\mu\}_{L-1}} \left(a_L \pm b_L \prod_{k=1}^{L-1}\mu_k\right) \prod_{k=1}^{L-1} \tfrac{1}{2}(a_k+\mu_k b_k) \\ &= \tfrac{1}{2}\left[\prod_{k=1}^{L} a_k \pm \prod_{k=1}^{L} b_k\right], \end{aligned} \tag{4.6}$$

where in the last step we have used $\sum_\mu \mu = 0$, we get

$$X_0(L;q) = \tfrac{1}{2}\left\{\prod_{k=1}^{L}\left(1+q^{k-1/2}\right) + \prod_{k=1}^{L}\left(1-q^{k-1/2}\right)\right\}. \tag{4.7}$$

Alternatively, denoting the number of negative μ_k by m, and letting $1 \le k_1 < k_2 < \ldots < k_m \le L$ be their respective labels, we can write

$$\begin{aligned} X_0(L;q) &= \sum_{\substack{m=0 \\ m\ \text{even}}}^{L} \sum_{k_1=1}^{L} \sum_{k_2=k_1+1}^{L} \cdots \sum_{k_m=k_{m-1}+1}^{L} q^{k_1+k_2+\cdots+k_m-m/2} \\ &= \sum_{\substack{m=0 \\ m\ \text{even}}}^{L} q^{m^2/2} \sum_{k_1=0}^{L-m} \sum_{k_2=k_1}^{L-m} \cdots \sum_{k_m=k_{m-1}}^{L-m} q^{k_1+k_2+\cdots+k_m}. \end{aligned} \tag{4.8}$$

Clearly, the sum over $k_1, \dots, k_m$ is the generating function of partitions with largest part $\leq L - m$ and number of parts $\leq m$. As is well known [25], this can be expressed in terms of Gaussian polynomials, to result in

$$X_0(L;q) = \sum_{\substack{m=0 \\ m \text{ even}}}^{L} q^{m^2/2} \begin{bmatrix} L \\ m \end{bmatrix}, \tag{4.9}$$

where

$$\begin{bmatrix} L \\ m \end{bmatrix} = \begin{cases} \dfrac{(q)_L}{(q)_m (q)_{L-m}}, & 0 \leq m \leq L, \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Taking $L \to \infty$ in (4.7) and (4.9) gives two of the $\chi_0(q)$ character representations in (2.10), in accordance with the prediction of conformal field theory.

The $+-$ boundary

The only difference from the previous case is that in the eigenvalue expression (3.9) we have to take those eigenvalues for which $\prod_{k=1}^{L} \mu_k = -1$. Hence we compute a finitized partition function as the right-hand side of (4.5) with $\{\mu\}_L^+$ replaced by $\{\mu\}_L^-$, obtaining the (normalized) results

$$X_{1/2}(L;q) = \tfrac{1}{2} q^{-1/2} \left\{ \prod_{k=1}^{L} \left(1 + q^{k-1/2}\right) - \prod_{k=1}^{L} \left(1 - q^{k-1/2}\right) \right\}, \tag{4.11}$$

and

$$\begin{aligned} X_{1/2}(L;q) &= q^{-1/2} \sum_{\substack{m=1 \\ m \text{ odd}}}^{L} \sum_{k_1=1}^{L} \sum_{k_2=k_1+1}^{L} \cdots \sum_{k_m=k_{m-1}+1}^{L} q^{k_1+k_2+\cdots+k_m-m/2} \\ &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{L} q^{(m^2-1)/2} \begin{bmatrix} L \\ m \end{bmatrix}. \end{aligned} \tag{4.12}$$

Taking $L \to \infty$ in either (4.11) or (4.12) indeed gives the $\chi_{1/2}(q)$ character.

Free boundaries

Since the eigenvalues of the transfer matrix with free boundaries are those of the $++$ together with those of the $+-$ case, the operator content of (2.9) immediately follows from the previous two cases.

4.2. Mixed boundary conditions

In the calculation for mixed boundary conditions we employ the same procedure of deriving a finitized partition function. From (3.14) and (4.3) with $K = k$, we now have

$$\left(\frac{\Lambda_j(u)}{\Lambda_0(u)}\right)^M \simeq \prod_{k=1}^{L} q^{\,k\,\delta_{\mu_k^{(j)},-1}}. \tag{4.13}$$

This gives rise to

$$X_{1/16}(L;q) = \sum_{\{\mu\}_L} \prod_{k=1}^{L}\left(1 - \tfrac{1}{2}(1-\mu_k)(1-q^k)\right) = \prod_{k=1}^{L}\left(1+q^k\right), \tag{4.14}$$

and

$$\begin{aligned} X_{1/16}(L;q) &= \sum_{m=0}^{L}\sum_{k_1=1}^{L}\sum_{k_2=k_1+1}^{L}\cdots\sum_{k_m=k_{m-1}+1}^{L} q^{k_1+k_2+\cdots+k_m} \\ &= \sum_{m\geq 0} q^{m(m+1)/2}\begin{bmatrix} L \\ m \end{bmatrix}, \end{aligned} \tag{4.15}$$

correctly corresponding to finitizations of the $\chi_{1/16}(q)$ character.

4.3. *Toroidal boundary conditions*

To derive the modular invariant partition function from the eigenvalue expressions (3.16) and (3.17), we recall that, due to the factor ϵ, each eigenvalue occurs with its negative. To remove this trivial degeneracy we take

$$\prod_{k=\lfloor (L+3)/2\rfloor}^{L} \mu_k = \epsilon \quad \text{for } r=1, \qquad \prod_{k=\lfloor (L+2)/2\rfloor}^{L-1} \mu_k = \epsilon \quad \text{for } r=-1, \tag{4.16}$$

so that all eigenvalues have a non-negative real part. Using (4.16) allows (3.16) to be re-expressed as

$$\begin{aligned} \Lambda(u) = \sqrt{2}\left(2e^{2iu+i\pi/4}\right)^{-L} &\prod_{k=1}^{\lfloor (L+1)/2\rfloor}\left(e^{4iu} + i\,\mu_k \tan\left(\frac{\pi(2k-1)}{4L}\right)\right) \\ \times &\prod_{k=1}^{\lfloor L/2\rfloor}\left(\bar{\mu}_k e^{4iu} + i\cot\left(\frac{\pi(2k-1)}{4L}\right)\right), \end{aligned} \tag{4.17}$$

with $\bar{\mu}_k = \mu_{L-k+1}$, and (3.17) as

$$\begin{aligned} \Lambda(u) = \sqrt{L}\left(2e^{2iu+i\pi/4}\right)^{1-L} &\prod_{k=1}^{\lfloor L/2\rfloor}\left(e^{4iu} + i\,\mu_k \tan\left(\frac{\pi k}{2L}\right)\right) \\ \times &\prod_{k=1}^{\lfloor (L-1)/2\rfloor}\left(\bar{\mu}_k e^{4iu} + i\cot\left(\frac{\pi k}{2L}\right)\right), \end{aligned} \tag{4.18}$$

with $\bar{\mu}_k = \mu_{L-k}$. We remark that these rewritings of the eigenvalue expressions are by no means unique, and we could have given forms different to those above. However, the forms chosen here give rise to a natural finitization of the modular invariant partition function in terms of finitized Virasoro characters.

Taking the ratio of (4.17) and (4.18) with the largest eigenvalue (given by (4.17) with all $\mu_k = \bar{\mu}_k = 1$), and using the limits

$$\lim_{N\to\infty} N\log\left(\frac{e^{4iu} - i\tan(\pi K/N)}{e^{4iu} + i\tan(\pi K/N)}\right) = -2\pi i K e^{-4iu},$$
$$\lim_{N\to\infty} N\log\left(\frac{-e^{4iu} + i\cot(\pi K/N)}{e^{4iu} + i\cot(\pi K/N)}\right) = 2\pi i K e^{4iu}, \tag{4.19}$$

and the definition (2.13), we obtain a finitized and normalized modular invariant partition function

$$\begin{aligned} Z(L;q) = & \sum_{\{\mu\}^+_{\lfloor(L+1)/2\rfloor}} \prod_{k=1}^{\lfloor(L+1)/2\rfloor} \tfrac{1}{2}\left[\left(1+q^{k-1/2}\right) + \mu_k\left(1-q^{k-1/2}\right)\right] \\ & \times \sum_{\{\mu\}^+_{\lfloor L/2\rfloor}} \prod_{k=1}^{\lfloor L/2\rfloor} \tfrac{1}{2}\left[\left(1+(\bar{q})^{k-1/2}\right) + \mu_k\left(1-(\bar{q})^{k-1/2}\right)\right] \\ & + |q|^{1/8} \sum_{\{\mu\}_{\lfloor L/2\rfloor}} \prod_{k=1}^{\lfloor L/2\rfloor} \tfrac{1}{2}\left[\left(1+q^{k}\right) + \mu_k\left(1-q^{k}\right)\right] \\ & \times \sum_{\{\mu\}_{\lfloor(L-1)/2\rfloor}} \prod_{k=1}^{\lfloor(L-1)/2\rfloor} \tfrac{1}{2}\left[\left(1+(\bar{q})^{k}\right) + \mu_k\left(1-(\bar{q})^{k}\right)\right]. \end{aligned} \tag{4.20}$$

Here the factor $|q|^{1/8}$ arises from the scaling limit of the ratio of the largest eigenvalue in (4.18), which have all $\mu_k = \bar{\mu}_k = 1$, to the overall largest eigenvalue. Such a ratio can for example be computed using the Euler–Maclaurin formula.

The sums in (4.20) are precisely those encountered in Eqs. (4.5), (4.11) and (4.14), so we immediately can write

$$\begin{aligned} Z(L;q) = & X_0(\lfloor L/2\rfloor;\bar{q})\, X_0(\lfloor (L+1)/2\rfloor;q) \\ & + |q|\, X_{1/2}(\lfloor L/2\rfloor;\bar{q})\, X_{1/2}(\lfloor (L+1)/2\rfloor;q) \\ & + |q|^{1/8} X_{1/16}(\lfloor (L-1)/2\rfloor;\bar{q})\, X_{1/16}(\lfloor L/2\rfloor;q), \end{aligned} \tag{4.21}$$

which at the isotropic point ($q = \bar{q}$) is the finitized modular invariant partition function proposed by Melzer [15], in agreement with (2.16) in the limit $L \to \infty$.

4.4. Finitized characters

It is interesting to note that the various truncations of the Ising characters in the previous sections coincide with those recently proposed by Melzer [15]. The power of the finitized (polynomial) forms of the characters is that they give rise to non-trivial

q-series identities of the Rogers–Ramanujan type, some of which can be found in the famous list by Slater [26]. To see this we follow Melzer and remark that the following recurrences hold

$$\begin{aligned} X_0(L) &= X_0(L-1) + q^L X_{1/2}(L-1), \\ X_{1/2}(L) &= X_{1/2}(L-1) + q^{L-1} X_0(L-1), \\ X_{1/16}(L) &= (1+q^L) X_{1/16}(L-1). \end{aligned} \tag{4.22}$$

Supplemented with the initial conditions $X_0(0) = X_{1/16}(0) = 1$ and $X_{1/2}(0) = 0$, these can be solved to yield a third finitized $c = 1/2$ character representation, namely

$$\begin{aligned} X_0(L) &= \sum_{j=-\infty}^{\infty} \left\{ q^{j(12j+1)} \begin{bmatrix} 2L \\ L-4j \end{bmatrix} - q^{(3j+1)(4j+1)} \begin{bmatrix} 2L \\ L-4j-1 \end{bmatrix} \right\}, \\ X_{1/2}(L) &= \sum_{j=-\infty}^{\infty} \left\{ q^{j(12j+5)} \begin{bmatrix} 2L \\ L-4j-1 \end{bmatrix} - q^{(3j+2)(4j+1)} \begin{bmatrix} 2L \\ L-4j-2 \end{bmatrix} \right\}, \\ X_{1/16}(L) &= \sum_{j=-\infty}^{\infty} \left\{ q^{j(12j-2)} \begin{bmatrix} 2L \\ L-4j \end{bmatrix} - q^{(3j+1)(4j+2)} \begin{bmatrix} 2L \\ L-4j-2 \end{bmatrix} \right\}. \end{aligned} \tag{4.23}$$

To verify that these formulae indeed solve (4.22) we note the recurrences

$$\begin{bmatrix} L \\ m \end{bmatrix} = \begin{bmatrix} L-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} L-1 \\ m \end{bmatrix}, \quad \begin{bmatrix} L \\ m \end{bmatrix} = \begin{bmatrix} L-1 \\ m \end{bmatrix} + q^{L-m} \begin{bmatrix} L-1 \\ m-1 \end{bmatrix}. \tag{4.24}$$

By applying these twice the proof of (4.23) readily follows.

Collecting the three different finitizations for each character and letting the finitization parameter L go off to infinity, we thus obtain the $c = 1/2$ Rogers–Ramanujan identities of Eqs. (2.10)-(2.12).

5. Summary and discussion

The conformal spectrum of the Ising model with cylindrical and toroidal boundary conditions has been derived analytically. By a truncation of the scaling limit, polynomial finitizations of the Ising characters and modular invariant partition function were obtained. These finitizations agree with those proposed previously by Melzer. In the thermodynamic limit, in which the truncation reproduces the proper scaling limit, our results agree with those of Ferdinand and Fisher, and Bugrij and Shadura.

In conclusion, it is intriguing to speculate as to the meaning of the finitized modular invariant partition function. It is difficult to believe that it is no more special than any other finitization which can be constructed using the finitized $c = 1/2$ characters. However, whether it is possible to define a sensible finitization of modular invariance, which has the finitized partition function as an invariant, remains unclear.

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