INHOMOGENEOUS LATTICE PATHS, GENERALIZED KOSTKA POLYNOMIALS AND A_{n-1} SUPERNOMIALS

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Abstract. Inhomogeneous lattice paths are introduced as ordered sequences of rectangular Young tableaux thereby generalizing recent work on the Kostka polynomials by Nakayashiki and Yamada and by Lascoux, Leclerc and Thibon. Motivated by these works and by Kashiwara's theory of crystal bases we define a statistic on paths yielding two novel classes of polynomials. One of these provides a generalization of the Kostka polynomials while the other, which we name the A_{n-1} supernomial, is a q-deformation of the expansion coefficients of products of Schur polynomials. Many well-known results for Kostka polynomials are extended leading to representations of our polynomials in terms of a charge statistic on Littlewood–Richardson tableaux and in terms of fermionic configuration sums. Several identities for the generalized Kostka polynomials and the A_{n-1} supernomials are proven or conjectured. Finally, a connection between the supernomials and Bailey's lemma is made.

1. Introduction

Lattice paths play an important rôle in combinatorics and exactly solvable lattice models of statistical mechanics. In particular, the one-dimensional configuration sums necessary for the calculation of order parameters of lattice models are generating functions of lattice paths (see for example [4, 12, 16]). A classic example of lattice paths is given by sequences of upward and downward steps. The number of such paths consisting of λ_1 steps up and λ_2 steps down is given by the binomial coefficient $\binom{\lambda_1+\lambda_2}{\lambda_1} = (\lambda_1+\lambda_2)!/\lambda_1!\lambda_2!$ which is the expansion coefficient of

$$(x_1+x_2)^L = \sum_{\substack{\lambda_1,\lambda_2 \\ \lambda_1+\lambda_2=L}} x_1^{\lambda_1} x_2^{\lambda_2} \binom{L}{\lambda_1}.$$

An important q-deformation of the binomial is the q-binomial

$$\begin{bmatrix} \lambda_1+\lambda_2 \\ \lambda_1 \end{bmatrix} = \begin{cases} \frac{(q^{\lambda_2+1})_{\lambda_1}}{(q)_{\lambda_1}} & \text{for } \lambda_1,\lambda_2 \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where $(x)_n = (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1})$. The q-binomial can be interpreted as the generating function of all paths with λ_1 steps up and λ_2 steps down where each path is weighted as follows. Let $p_1,\ldots,p_{\lambda_1+\lambda_2}$ denote the steps of the path where we label a step up by 1 and a step down by 2. Then the weight of the path is given by $\sum_{i=1}^{\lambda_1+\lambda_2-1} i\chi(p_i < p_{i+1})$ where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

Other q-functions have occurred, such as a q-deformation of the trinomial coefficients [2, 3] in the expansion of $(x_1^2+x_1x_2+x_2^2)^L$ or, more generally, of the $(N+1)$-nomial coefficients [24, 44, 53] in the expansion of h_N^L where h_N is the complete symmetric polynomial in the variables x_1 and

x_2 of degree N. In a study of Rogers–Ramanujan-type identities the following generalizations of the multinomial coefficients were introduced [47]

$$h_1^{L_1}\dots h_N^{L_N} = \sum_{\substack{\lambda_1,\lambda_2\\ \lambda_1+\lambda_2=\ell_N}} x_1^{\lambda_1}x_2^{\lambda_2}\binom{\boldsymbol{L}}{\lambda_1-\frac{1}{2}\ell_N}, \tag{1.2}$$

where $\boldsymbol{L}=\sum_{i=1}^N L_i\boldsymbol{e}_i\in\mathbb{Z}_{\geq 0}^N$ with $\boldsymbol{e}_i$ the ith unit vector in $\mathbb{Z}^N$ and $\ell_i=\sum_{j=1}^N \min\{i,j\}L_j$. Since Equation (1.2) reduces to the definition of the $(i+1)$-nomial coefficient when $\boldsymbol{L}=L\boldsymbol{e}_i$ (up to a shift in the lower index), the expansion coefficient in (1.2) was coined (A_1) supernomial. In [47] it was shown that many Rogers–Ramanujan-type identities admit bounded analogues involving the following q-deformation of the supernomial

$$\begin{bmatrix}\boldsymbol{L}\\ a\end{bmatrix} = \sum_{j_1+\cdots+j_N=a+\frac{\ell_N}{2}} q^{\sum_{k=1}^{N-1}(\ell_{k+1}-\ell_k-j_{k+1})j_k}\begin{bmatrix}L_N\\ j_N\end{bmatrix}\begin{bmatrix}L_{N-1}+j_N\\ j_{N-1}\end{bmatrix}\dots\begin{bmatrix}L_1+j_2\\ j_1\end{bmatrix} \tag{1.3}$$

for $\boldsymbol{L}\in\mathbb{Z}_{\geq 0}^N$ and $a+\frac{1}{2}\ell_N=0,1,\dots,\ell_N$. However, the question whether (1.3) or the Rogers–Ramanujan-type identities involving (1.3) can be interpreted as generating functions of weighted lattice paths remained unanswered. Incidentally, the polynomials in Equation (1.3) have occurred in Butler's study [7, 8, 9] of finite abelian groups.

In a seemingly unrelated development, Nakayashiki and Yamada [40] introduced the notion of "inhomogeneous" lattice paths by considering paths in which each of the elementary steps p_i can be chosen from a different set $\mathcal{B}_i$. The main result of their work is a new combinatorial representation of the Kostka polynomial as the generating function of inhomogeneous paths where either all $\mathcal{B}_i$ are sets of fully symmetric (one-row) Young tableaux or all $\mathcal{B}_i$ are sets of fully antisymmetric (one-column) Young tableaux. An equivalent description of the Kostka polynomials, formulated in terms of the plactic monoid, was found by Lascoux, Leclerc and Thibon [33].

The purpose of this paper is to elucidate the connection of the work of Nakayashiki and Yamada and of Lascoux, Leclerc and Thibon on the Kostka polynomials with that of [47] on supernomials and to extend all of them. In particular, we introduce inhomogeneous lattice paths based on Young tableaux with mixed symmetries, or more precisely, on Young tableaux of rectangular shape. Motivated by the theory of crystal bases [19] we assign weights to these paths and relate their generating functions to q-deformations of the (A_{n-1}) supernomials defined through products of Schur polynomials, in the spirit of Equation (1.2). By imposing suitable restrictions on the inhomogeneous lattice paths, we obtain new polynomials that include the Kostka polynomials as a special case. For these generalized Kostka polynomials we derive several extensions of classical results such as a Lascoux–Schützenberger-type representation [35] in terms of a charge statistic, and a representation akin to that of Kirillov and Reshetikhin [27] based on rigged configurations. We furthermore prove and conjecture several identities involving the A_{n-1} supernomials and generalized Kostka polynomials and briefly comment on a Bailey-type lemma [6] for "antisymmetric" supernomials. The A_{n-1} supernomials include polynomials previously studied in [7, 8, 9, 15].

The rest of this paper is organized as follows. Section 2 serves to set the notation used throughout the paper and to review some basic definitions and properties of Young tableaux, words and Kostka polynomials. In Section 3 inhomogeneous lattice paths based on rectangular Young

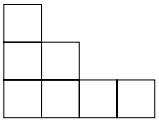
tableaux are introduced (Definition 3.1). A statistic on such paths originating from crystal-base theory is used to define A_{n-1} supernomials (Definition 3.5) and generalized Kostka polynomials (Definition 3.9) as generating functions of inhomogeneous paths. We furthermore map the paths underlying the generalized Kostka polynomials onto Littlewood–Richardson (LR) tableaux (Definition 3.6). In Section 4 an initial cyclage and charge statistic on LR tableaux is defined (Definition 4.3) which enables us in the subsequent section to prove a Lascoux–Schützenberger-type representation for generalized Kostka polynomials (Corollary 5.2). Section 6 deals with more general λ-(co)cyclages on LR tableaux, showing that these cyclages impose a ranked poset structure on the set of LR tableaux (Theorem 6.3). These results are used in Section 7 to prove a duality formula for the generalized Kostka polynomials (Theorem 7.1) and recurrence relations for the A_{n-1} supernomials and the generalized Kostka polynomials (Theorem 7.4). In Section 8 the recurrences are employed to obtain a Kirillov–Reshetikhin-type expression for the generalized Kostka polynomials (Theorem 8.2). We finally conclude in Section 9 with some conjectured polynomial identities and with a Bailey-like lemma involving the A_{n-1} supernomials.

2. Young tableaux, words and Kostka polynomials

This section reviews some definitions and properties of Young tableaux, words and Kostka polynomials and sets out the notation and terminology used throughout the paper. For more details the reader may consult [9, 13, 37].

Throughout, we denote by $|A|$ the cardinality of a set A and we define $|\mu| = \sum_i \mu_i$ for an array of numbers $\mu = (\mu_1, \mu_2, \ldots)$.

2.1. **Young tableaux and words.** We begin by recalling some definitions regarding partitions. A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a weakly decreasing sequence of non-negative integers such that only finitely many $\lambda_i \neq 0$. We write $\lambda \vdash n$ if $|\lambda| = n$. Partitions which differ only by a string of zeros are identified. Each partition can be depicted by a Young diagram, which (adopting the "French" convention) is a collection of boxes with left-adjusted rows of decreasing length from bottom to top. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition then the corresponding Young diagram has λ_i boxes in the ith row from the bottom. For example



is the Young diagram corresponding to the partition $(4, 2, 1)$. The nonzero elements λ_i are called the parts of λ. The height of λ is the number of parts and its width equals the largest part. At times it is convenient to denote a partition λ with L_i parts equal to i by $\lambda = (1^{L_1} 2^{L_2} \cdots)$. The partition $\lambda^\top$ is the partition corresponding to the transposed diagram of λ obtained by reflecting along the diagonal, i.e., if $\lambda = (1^{L_1} 2^{L_2} \cdots N^{L_N})$ then $(\lambda^\top)_i = L_i + \cdots + L_N$. The addition $\lambda + \mu$ of the partitions λ and μ is defined by the addition of their parts $(\lambda + \mu)_i = \lambda_i + \mu_i$. The parts of the partition $\lambda \cap \mu$ are given by $(\lambda \cap \mu)_i = \min\{\lambda_i, \mu_i\}$, and λ/μ denotes the skew shape obtained by removing the boxes of μ from λ. By $\lambda \geq \mu$ in dominance order we mean $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i. The set of "rectangular" partitions (i.e., partitions with rectangular diagram) is denoted by $\mathcal{R}$.

In this paper we will often encounter arrays of rectangular partitions. For such an array $\mu = (\mu_1, \ldots, \mu_L) \in \mathcal{R}^L$, define $\mu^\star = (\mu_1^\top, \ldots, \mu_L^\top)$ and $|\mu| = |\mu_1| + \cdots + |\mu_L|$. When all

components of μ have height 1 ordered according to decreasing width, i.e., $\mu_i = (k_i)$ with $k_1 \geq \cdots \geq k_L$, one can identify μ with the partition $(k_1, \ldots, k_L)$. Notice, however, that $\mu^\star \neq \mu^\top$ in this case. There is the following partial order on $\mathcal{R}^L$ modulo reordering. Define $\lambda^{(a)}$ as the partition obtained from $\lambda \in \mathcal{R}^L$ by putting the widths of all components of λ of height a in decreasing order. Then $\lambda \geq \mu$ for $\lambda, \mu \in \mathcal{R}^L$ if $\lambda^{(a)} \geq \mu^{(a)}$ for all a by the dominance order on partitions.

Next we consider Young tableaux. Let $X = \{x_1 < x_2 < \cdots < x_n\}$ be a totally ordered alphabet of non-commutative indeterminates. A Young tableau over X is a filling of a Young diagram such that each row is weakly increasing from left to right and each column is strictly increasing from bottom to top. The Young diagram (or, equivalently, partition) underlying a Young tableau T is called the shape of T and the height of T is the height of its shape. The content of a Young tableau T is an array $\mu = (\mu_1, \ldots, \mu_n)$ where μ_i is the number of boxes filled with x_i. The set of all Young tableaux of shape λ and content μ is denoted by $\mathrm{Tab}(\lambda, \mu)$. It is clear that $\mathrm{Tab}(\lambda, \mu) = \emptyset$ unless $|\lambda| = |\mu|$.

Young tableaux can also be represented by words over the alphabet X. Let $\mathcal{X}$ be the free monoid generated by X. By the Schensted bumping algorithm [43] one can associate a Young tableau to each word $w \in \mathcal{X}$ denoted by $[w]$. Knuth [30] introduced equivalence relations on words generated by

$$\begin{aligned} zxy &\equiv xzy \qquad (x \leq y < z), \\ yxz &\equiv yzx \qquad (x < y \leq z), \end{aligned} \tag{2.1}$$

for $x, y, z \in X$ and showed that $[w] = [w']$ if and only if $w \equiv w'$. The word w_T obtained from a Young tableau T by reading its entries successively from left to right down the page is called a word in row-representation or, for short, a row-word. Since $T = [w_T]$, the Schensted and row-reading algorithms provide an one-to-one correspondence between the plactic monoid $\mathcal{X}/\equiv$ and the set of Young tableaux over X. Using the above correspondence, we say that a word has shape λ and content μ if the corresponding Young tableau $[w]$ is in $\mathrm{Tab}(\lambda, \mu)$. Furthermore, the product of two Young tableaux S and T is defined as $S \cdot T = [w_S w_T]$ where $w_S w_T$ is the word formed by juxtaposing the row-words w_S and w_T.

Finally, the following definitions for words are needed. A word $w = w_1 w_2 \cdots w_k$ with $w_i \in X$ is called a Yamanouchi word if, for all $1 \leq i \leq k$, the sequence $w_k \cdots w_i$ contains at least as many x_1 as x_2, at least as many x_2 as x_3 and so on. We call a word balanced if all of the letters in X occur an equal number of times. Let w be a word on the two-letter alphabet $\{x < y\}$ and recursively connect all pairs yx in w as in the following example:

$$w = \underbrace{y\underbrace{y\underbrace{y\underbrace{x\underbrace{y\underbrace{x\underbrace{x\underbrace{x}y}}}}}}}.$$

A pair $\underbrace{y\cdots x}$ is called an inverted pair. All letters of w which do not belong to an inverted pair are called non-inverted, and the subword of w containing its non-inverted letters is of the form $x^r y^s$.

2.2. **Kostka polynomials.** Throughout this paper $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ where $x_1, \ldots, x_n$ are commutative variables (not to be confused with the *non*commutative letters in the alphabet X) and

$\lambda = (\lambda_1, \ldots, \lambda_n)$. The Schur polynomial s_λ in the variables $x_1, \ldots, x_n$ is defined as

$$s_\lambda(x) = \sum_{T \in \mathrm{Tab}(\lambda, \cdot)} x^T, \tag{2.2}$$

where $x^T := x^{\mathrm{content}(T)}$. The Kostka polynomials $K_{\lambda\mu}(q)$ arise as the connection coefficients between the Schur and Hall–Littlewood polynomials [37],

$$s_\lambda(x) = \sum_{\mu \vdash |\lambda|} K_{\lambda\mu}(q) P_\mu(x; q). \tag{2.3}$$

Here λ and μ are partitions and $K_{\lambda\mu}(q) \neq 0$ if and only if $|\lambda| = |\mu|$ and $\lambda \geq \mu$.

A combinatorial interpretation of the Kostka polynomials was obtained by Lascoux and Schützenberger [35], who showed that

$$K_{\lambda\mu}(q) = \sum_{T \in \mathrm{Tab}(\lambda, \mu)} q^{c(T)}, \tag{2.4}$$

where $c(T)$ is the charge of a Young tableau defined below.

Let $T \in \mathrm{Tab}(\cdot, \mu)$ be a Young tableau of content μ over $X = \{x_1 < x_2 < \cdots < x_n\}$ and let $T_{\min} = T_{\min}(\mu) := [x_1^{\mu_1} \cdots x_n^{\mu_n}]$ be the one-row tableau of content μ. When $T \neq T_{\min}$ and $w_T = x_i u$ the initial cyclage $\mathcal{C}$ on T is defined as $\mathcal{C}(T) = [u x_i]$. The cocharge $\mathrm{co}(T)$ of $T \in \mathrm{Tab}(\cdot, \mu)$ is the number of times one has to apply $\mathcal{C}$ to obtain $T_{\min}$. The charge of T is defined as $c(T) = \|\mu\| - \mathrm{co}(T)$ where $\|\mu\|$ is the cocharge of the Young tableau $T_{\max} := [x_n^{\mu_n} \cdots x_1^{\mu_1}]$, given by $\|\mu\| = \sum_{i<j} \min\{\mu_i, \mu_j\}$.

To illustrate the above definitions take, for example, $T = [x_3 x_2 x_1^2 x_2]$. Then

$$\begin{aligned} \mathcal{C}(T) &= [x_2 x_1^2 x_2 x_3], \\ \mathcal{C}^2(T) &= [x_1^2 x_2 x_3 x_2] = [x_3 x_1^2 x_2^2], \\ \mathcal{C}^3(T) &= [x_1^2 x_2^2 x_3], \end{aligned}$$

so that $\mathrm{co}(T) = 3$ and $c(T) = 1$ in this example.

Another combinatorial description of the Kostka polynomials, in terms of rigged configurations, is due to Kirillov and Reshetikhin [27] and provides an explicit formula for calculating the Kostka polynomials as

$$K_{\lambda\mu}(q) = \sum_{\alpha} q^{C(\alpha)} \prod_{a,i \geq 1} \begin{bmatrix} P_i^{(a)}(\alpha) + \alpha_i^{(a)} - \alpha_{i+1}^{(a)} \\ \alpha_i^{(a)} - \alpha_{i+1}^{(a)} \end{bmatrix}. \tag{2.5}$$

The summation is over sequences $\alpha = (\alpha^{(0)}, \alpha^{(1)}, \ldots)$ of partitions such that $\alpha^{(0)} = \mu^\top$ and $|\alpha^{(a)}| = \lambda_{a+1} + \lambda_{a+2} + \cdots$. Furthermore

$$P_i^{(a)}(\alpha) = \sum_{k=1}^{i} (\alpha_k^{(a-1)} - 2\alpha_k^{(a)} + \alpha_k^{(a+1)}) \tag{2.6}$$

and

$$C(\alpha)=\sum_{a,i\geq 1}\binom{\alpha_i^{(a-1)}-\alpha_i^{(a)}}{2}, \tag{2.7}$$

where $\binom{a}{2}=a(a-1)/2$ for $a\in\mathbb{Z}$. Expressions of the type (2.5) are often called fermionic as they can be interpreted as the partition function for a system of quasi-particles with fractional statistics obeying Pauli's exclusion principle [22, 23].

In Section 3.2 a third combinatorial representation of the Kostka polynomials as the generating function of paths will be discussed. This representation is due to Lascoux, Leclerc and Thibon [33] and Nakayashiki and Yamada [40] and is the starting point for our generalized Kostka polynomials. As we will see in subsequent sections, these generalized Kostka polynomials also admit representations stemming from Equations (2.4) and (2.5).

3. A_{n-1} supernomials and generalized Kostka polynomials

This section deals with paths defined as ordered sequences of rectangular Young tableaux. Assigning weights to the paths, we consider the generating functions over two different sets of paths called unrestricted and classically restricted. These are treated in Sections 3.1 and 3.2, respectively. As will be shown in Section 7, the generating functions over the set of unrestricted paths are A_{n-1} generalizations of the A_1 supernomials (1.3). The generating functions over the set of classically restricted paths lead to generalizations of the Kostka polynomials.

3.1. Unrestricted paths and A_{n-1} supernomials.

Denote by $\mathcal{B}_\lambda$ the set $\mathrm{Tab}(\lambda,\cdot)$ of Young tableaux of shape λ over the alphabet $\{1,2,\cdots,n\}$. An element of $\mathcal{B}_\lambda$ is called a step and an ordered sequence of L steps is a path of length L denoted by $p_L\otimes\cdots\otimes p_1$. We treat here only paths with rectangular steps p_i, i.e., $p_i\in\mathcal{B}_{\mu_i}$ for $\mu_i\in\mathcal{R}$. Let us however emphasize that the steps in a path can have different shapes indicated by the subscript i on μ_i. Paths with this property are called inhomogeneous [40].

The reason for the tensor product notation for paths (treated here as ordered sequences of steps only) is for notational convenience, but is motivated by the relation to the theory of crystal bases [19]. In this setting $\mathcal{B}_{(i^a)}$ is usually labelled by $\mathcal{B}_{i\Lambda_a}$ where Λ_a are the fundamental weights of A_{n-1}. The set $\mathcal{B}_{i\Lambda_a}$ is called a perfect crystal and parametrizes a basis of the irreducible highest weight module of A_{n-1} with highest weight $i\Lambda_a$ [21]. There exist crystal bases for all integrable highest weight modules and they are compatible with the tensor product structure.

Definition 3.1 (Unrestricted paths)**.** *For fixed integers $n\geq 2$ and $L\geq 0$ let $\lambda\in\mathbb{Z}^n_{\geq 0}$ and $\mu=(\mu_1,\ldots,\mu_L)\in\mathcal{R}^L$. The set of paths $\mathcal{P}_{\lambda\mu}$ is defined as*

$$\mathcal{P}_{\lambda\mu}=\{p_L\otimes\cdots\otimes p_1|\ p_i\in\mathcal{B}_{\mu_i}\ \textit{and}\ \sum_{i=1}^{L}\mathrm{content}(p_i)=\lambda\}. \tag{3.1}$$

To each path $P\in\mathcal{P}_{\lambda\mu}$ we assign an energy $h(P)\in\mathbb{Z}_{\geq 0}$ as

$$h(P)=\sum_{i=1}^{L-1}ih(p_{i+1}\otimes p_i), \tag{3.2}$$

where $h(p\otimes p')$ for the steps $p \in \mathcal{B}_\nu$ and $p' \in \mathcal{B}_{\nu'}$ is defined as the number of boxes in the product $p \cdot p'$ that lie outside the Young diagram $\nu + \nu'$ or, more formally, as

$$h(p \otimes p') = |\nu + \nu'| - |\text{shape}(p \cdot p') \cap (\nu + \nu')|. \tag{3.3}$$

Example 3.1.

$$\text{Let} \quad P = p_2 \otimes p_1 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}. \quad \text{Then} \quad p_2 \cdot p_1 = \begin{array}{|c|c|c|}\hline 3 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 2 \\ \hline \end{array}$$

and $\text{shape}(p_2 \cdot p_1) = (3, 2, 1)$. Hence $h(P) = |(4, 2)| - |(3, 2, 1) \cap (4, 2)| = 6 - 5 = 1$.

The cardinality $S_{\lambda\mu}$ of $\mathcal{P}_{\lambda\mu}$ does not depend on the ordering of μ, i.e.,

$$S_{\lambda\mu} = S_{\lambda\tilde{\mu}}, \tag{3.4}$$

where $\tilde{\mu}$ is a permutation of μ. In general the generating function of $\mathcal{P}_{\lambda\mu}$ with paths weighted by the energy function h does not have this symmetry. To obtain a weight function such that the resulting generating function does respect this symmetry we introduce an isomorphism $\sigma : \mathcal{B}_\alpha \otimes \mathcal{B}_{\alpha'} \to \mathcal{B}_{\alpha'} \otimes \mathcal{B}_\alpha$ for $\alpha, \alpha' \in \mathcal{R}$ between two successive steps. Let $p \otimes p' \in \mathcal{B}_\alpha \otimes \mathcal{B}_{\alpha'}$. Then $\sigma(p \otimes p') = \tilde{p}' \otimes \tilde{p}$, where $\tilde{p}'$ and $\tilde{p}$ are the unique Young tableaux of shape α' and α, respectively, which satisfy

$$p \cdot p' = \tilde{p}' \cdot \tilde{p}. \tag{3.5}$$

The uniqueness of the Young tableaux $\tilde{p}'$ and $\tilde{p}$ is ensured since the Littlewood–Richardson coefficients have the symmetry $c^{\beta}_{\alpha\alpha'} = c^{\beta}_{\alpha'\alpha}$ and for rectangular shapes α and α' obey $c^{\beta}_{\alpha\alpha'} \leq 1$. Notice that σ is the identity if p and p' have the same shape.

Definition 3.2 (Isomorphism)**.** *For a path $P = p_L \otimes \cdots \otimes p_1 \in \mathcal{P}_{\lambda\mu}$ we define the isomorphism σ_i as*

$$\sigma_i(P) = p_L \otimes \cdots \otimes \sigma(p_{i+1} \otimes p_i) \otimes \cdots \otimes p_1. \tag{3.6}$$

The group generated by the isomorphisms σ_i is the symmetric group, i.e., $\sigma_i^2 = \text{Id}$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geq 2$. The proof of the braiding relation is non-trivial (see [49, 51]).

Definition 3.3 (Orbit)**.** *The set $\mathcal{O}_P$ is the orbit of the path $P \in \mathcal{P}_{\lambda\mu}$ under the group generated by the isomorphisms σ_i.*

The weight of a path P is now given by the mean of the energy function h over the orbit of P.

Definition 3.4 (Weight)**.** *For $P \in \mathcal{P}_{\lambda\mu}$, the weight function $H : \mathcal{P}_{\lambda\mu} \to \mathbb{Z}_{\geq 0}$ is defined as*

$$H(P) = \frac{1}{|\mathcal{O}_P|} \sum_{P' \in \mathcal{O}_P} h(P'). \tag{3.7}$$

It is not obvious from (3.7) that the weight $H(P)$ of a path P is indeed integer. This will follow from Theorem 5.1. Before we continue to define the generating functions over the set of paths $\mathcal{P}_{\lambda\mu}$, some remarks on the relation of our definitions to lattice paths of exactly solvable lattice models and the theory of crystal bases are in order.

Remark 3.1. For homogeneous paths, i.e., $P \in \mathcal{P}_{\lambda\mu}$ with $\mu_1 = \cdots = \mu_L$, the weight simplifies to $H(P) = h(P)$ which is the weight function of configuration sums of $A^{(1)}_{n-1}$ solvable lattice models. For example, for $p, p' \in \mathcal{B}_{(N)}$, the energy function $h(p \otimes p')$ coincides with the one of [12, 16] (and references therein) given by

$$h(p \otimes p') = \max_{\tau \in S_N} \Big\{ \sum_{i=1}^{N} \chi(p_i > p'_{\tau_i}) \Big\}. \tag{3.8}$$

Here $p_i, p'_i \in \{1, 2, \ldots, n\}$ are the letters in $p = [p_1 \cdots p_N]$ and $p' = [p'_1 \cdots p'_N]$, S_N is the permutation group on $1, 2, \ldots, N$, $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. An alternative combinatorial expression of (3.8) in terms of so-called nonmovable tableaux is given in [26]. When $p, p' \in \mathcal{B}_{(1^N)}$, our energy function reduces to $h(p \otimes p') = \min_{\tau \in S_N} \big\{ \sum_{i=1}^{N} \chi(p_i > p'_{\tau_i}) \big\}$ of [41].

Nakayashiki and Yamada [40] defined weight functions on inhomogeneous paths when either μ or $\mu^\star$ is a partition, i.e., when $|\mu_1| \geq \cdots \geq |\mu_L|$ and either $\text{height}(\mu_i) = 1$ for all i or $\text{width}(\mu_i) = 1$ for all i. Their isomorphism, defined in terms of graphical rules (Rule 3.10 and 3.11 of [40]), is a special case of the isomorphism σ. The expression for $H(P)$ that they give is quite different to that of Equation (3.7) even though it is the same function for the subset of paths they consider. For example when $\text{height}(\mu_i) = 1$ for all i, H of [40] is, in our normalization, given by

$$H(P) = \sum_{i=2}^{L} \sum_{j=1}^{i-1} h(p_i \otimes p_j^{(i-1)}),$$

where $P = p_L \otimes \cdots \otimes p_1 \in \mathcal{P}_{\lambda\mu}$, $p_i^{(i)} = p_i$ and $p_j^{(i)} = p'_i$ with $P' = \sigma_{i-1} \circ \sigma_{i-2} \circ \cdots \circ \sigma_j(P)$ for $j < i$.

Lascoux, Leclerc and Thibon [33] defined a weight function $b(T)$ for Young tableaux T as the mean over certain orbits very similar in spirit to Equation (3.7) (see [33, Theorem 5.1]). In fact, when $\text{height}(\mu_i) = 1$ for all i, each path $P \in \mathcal{P}_{\cdot\mu}$ can be mapped to a Young tableau $T \in \text{Tab}(\cdot, \mu)$ (by the virtue of the map ω of Equation (3.16) below, i.e., $T = [\omega(P)]$), and in this case one finds that $H(P) = \|\mu\| - b(T)$ where we recall that $\|\mu\| = \sum_{i<j} \min\{\mu_i, \mu_j\}$.

Remark 3.2. Kashiwara [19] defined raising and lowering operators f_i and e_i $(0 \leq i \leq n-1)$ acting on elements of a crystal $\mathcal{B}_{i\Lambda_a}$. Set $\mathcal{B}_k = \mathcal{B}_{i_k \Lambda_{a_k}}$ $(k = 1, 2)$. Then for $p_1 \in \mathcal{B}_1$ and $p_2 \in \mathcal{B}_2$ the lowering operators act on the tensor product $p_2 \otimes p_1$ as follows

$$e_i(p_2 \otimes p_1) = \begin{cases} p_2 \otimes e_i p_1 & \text{if } \varphi_i(p_1) \geq \varepsilon_i(p_2), \\ e_i p_2 \otimes p_1 & \text{if } \varphi_i(p_1) < \varepsilon_i(p_2), \end{cases}$$

where $\varepsilon_i(b) = \max\{k | e_i^k(b) \neq 0\}$ and $\varphi_i(b) = \max\{k | f_i^k(b) \neq 0\}$. The action of f_i on a tensor product is defined in a similar way. (Note that to conform with the rest of this paper the order of the tensor product is inverted in comparison to the usual definitions). Let $e_i(p_2 \otimes p_1) \neq 0$. Up to an additive constant the energy function used in crystal theory is recursively defined as

$$E(e_i(p_2 \otimes p_1)) = \begin{cases} E(p_2 \otimes p_1) + 1 & \text{if } i = 0 \text{ and } \varphi_0(p_1) \geq \varepsilon_0(p_2),\ \varphi_0(\tilde{p}_2) \geq \varepsilon_0(\tilde{p}_1), \\ E(p_2 \otimes p_1) - 1 & \text{if } i = 0 \text{ and } \varphi_0(p_1) < \varepsilon_0(p_2),\ \varphi_0(\tilde{p}_2) < \varepsilon_0(\tilde{p}_1), \\ E(p_2 \otimes p_1) & \text{otherwise.} \end{cases} \tag{3.9}$$

Here $p_2 \otimes p_1 \mapsto \tilde{p}_1 \otimes \tilde{p}_2$ with $p_1, \tilde{p}_1 \in \mathcal{B}_1$ and $p_2, \tilde{p}_2 \in \mathcal{B}_2$ is an isomorphism obeying certain conditions [17, 18, 20]. The isomorphism σ defined through (3.5) yields the isomorphism of crystal theory. Up to a sign the energy $h(p_2 \otimes p_1)$ as defined in (3.3) provides an explicit expression for the recursively defined energy of (3.9), i.e., $E(p_2 \otimes p_1) = -h(p_2 \otimes p_1)$. We do not prove these statements in this paper.

Let us now define the A_{n-1} supernomial as the generating function of the set of paths $\mathcal{P}_{\lambda\mu}$ weighted by H of Definition 3.4.

Definition 3.5 (Supernomials)**.** *Let $\lambda \in \mathbb{Z}^n_{\geq 0}$ and $\mu \in \mathcal{R}^L$. Then the supernomial $S_{\lambda\mu}(q)$ is defined as*

$$S_{\lambda\mu}(q) = \sum_{P \in \mathcal{P}_{\lambda\mu}} q^{H(P)}. \tag{3.10}$$

Since $H(P) = H(P')$ for $P' \in \mathcal{O}_P$ it is clear that

$$S_{\lambda\mu}(q) = S_{\lambda\tilde{\mu}}(q), \tag{3.11}$$

where $\tilde{\mu}$ is a permutation of μ.

To conclude this section we comment on the origin of the terminology supernomial as first introduced in [47]. Recalling Definition 3.1 of the set of paths $\mathcal{P}_{\lambda\mu}$ and Equation (2.2) for the Schur polynomial, one finds that

$$s_{\mu_1}(x) \cdots s_{\mu_L}(x) = \sum_{\lambda \vdash |\mu|} S_{\lambda\mu} x^\lambda, \tag{3.12}$$

where $S_{\lambda\mu} := S_{\lambda\mu}(1) = |\mathcal{P}_{\lambda\mu}|$. For homogeneous paths, i.e., $\mu_1 = \cdots = \mu_L$, this is the usual definition for various kinds of multinomial coefficients. The supernomials (or, more precisely, q-supernomial coefficients) can thus be viewed as q-deformations of generalized multinomial coefficients.

3.2. **Classically restricted paths and generalized Kostka polynomials.** Analogous to the previous section we now introduce classically restricted paths and their generating function. To describe the set of classically restricted paths we first map paths onto words and then specify the restriction on these words.

For our purposes it will be most convenient to label the alphabet underlying the words associated to paths as

$$X^a = \{x_1^{(1)} < x_1^{(2)} < \cdots < x_1^{(a_1)} < x_2^{(1)} < \cdots < x_2^{(a_2)} < \cdots < x_L^{(1)} < \cdots < x_L^{(a_L)}\} \tag{3.13}$$

for some fixed integers $1 \leq a_i \leq n$. As before, $\mathcal{X}^a$ denotes the free monoid generated by X^a. The i-subword of a word $w \in \mathcal{X}^a$ is the subword of w consisting of the letters $x_i^{(j)}$ $(1 \leq j \leq a_i)$ only. More generally, the $(i_1, \ldots, i_\ell)$-subword of w is the subword consisting of the letters with subscripts $i_1, \ldots, i_\ell$ only.

We are interested in the following subset of $\mathcal{X}^a$

$$\mathcal{W} = \{w \in \mathcal{X}^a | \text{ each } i\text{-subword of } w \text{ is a balanced Yamanouchi word}\}. \tag{3.14}$$

By the Schensted bumping algorithm each word $w \in \mathcal{W}$ corresponds to a Young tableau $[w]$ over the alphabet X^a. It is an easy matter to show that if $w \in \mathcal{W}$ is Knuth equivalent to $w' \in \mathcal{X}^a$, then $w' \in \mathcal{W}$. Hence it makes sense to consider the set of Young tableaux over X^a corresponding

to $\mathcal{W}/\equiv$. Instead of labelling the content of such a tableau by $\mu=(\mu_1^{(1)},\dots,\mu_L^{(a_L)})$ (where $\mu_i^{(j)}$ is the number of $x_i^{(j)}$'s), we set $\mu=(\mu_1,\dots,\mu_L)$ where $\mu_i=(\mu_i^{(1)},\dots,\mu_i^{(a_i)})$ so that $\mu\in\mathcal{R}^L$ with height$(\mu_i)=a_i$. The set of words $w\in\mathcal{W}$ with content$([w])=\mu$ is denoted $\mathcal{W}_\mu$.

Definition 3.6 (Littlewood–Richardson tableaux)**.** *Let $L\geq 0$ and $n\geq 2$ be integers, λ a partition and $\mu\in\mathcal{R}^L$. Then the set of LR tableaux of shape λ and content μ is defined as*

$$\mathrm{LRT}(\lambda,\mu)=\{T|w_T\in\mathcal{W}_\mu \text{ and } \mathrm{shape}(T)=\lambda\}. \tag{3.15}$$

The set of LR tableaux $\mathrm{LRT}(\lambda,\mu)$ reduces to the set $\mathrm{Tab}(\lambda,\mu)$ of Young tableaux over $X=\{x_1<\cdots<x_L\}$ when $a_1=a_2=\cdots=a_L=1$ in (3.13).

We now define a map

$$\omega:\mathcal{P}_{\lambda\mu}\to\mathcal{W}_\mu \tag{3.16}$$

in the following way. Let $P=p_L\otimes\cdots\otimes p_1\in\mathcal{P}_{\lambda\mu}$ and let (j,k) denote the kth row of p_j. Set $P'=P$, w to the empty word and carry out the following procedure $|\mu|$ times:

> If (j,k) labels the position of the rightmost, maximal entry in P', obtain a new P' by removing this maximal entry from P' and append $x_j^{(k)}$ to w.

The resulting word w defines $\omega(P)$. Equivalently, $[\omega(P)]$ is the column insertion recording tableaux of $\mathrm{col}(p_L)\dots\mathrm{col}(p_1)$ where $\mathrm{col}(T)$ is the column word of the tableau T.

The word $\omega(P)$ obtained via the above procedure is indeed in $\mathcal{W}_\mu$. The Yamanouchi condition is guaranteed in each intermediate w in the construction thanks to the fact that all steps p_j are Young tableaux. Since all p_j have rectangular shape $\omega(P)$ is a balanced Yamanouchi word. Note that the integer a_j in the alphabet (3.13) used in Definition (3.14) of $\mathcal{W}$ is exactly the height of step p_j.

Example 3.2. To illustrate the map ω take for example

$$P=\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\otimes\begin{array}{|c|}\hline 4\\\hline 2\\\hline 1\\\hline\end{array}.$$

Then $\omega(P)=x_1^{(3)}x_3^{(2)}x_1^{(2)}x_2^{(1)}x_3^{(1)}x_3^{(2)}x_1^{(1)}x_2^{(1)}x_3^{(1)}$ and the corresponding LR tableau (identifying $x_j^{(k)}$ with $j^{(k)}$) is

$$\begin{array}{|c|c|c|c|}\hline 1^{(3)}&3^{(2)}&&\\\hline 1^{(2)}&3^{(1)}&3^{(2)}&\\\hline 1^{(1)}&2^{(1)}&2^{(1)}&3^{(1)}\\\hline\end{array}. \tag{3.17}$$

The map ω has the following properties which are needed later.

Lemma 3.1. *Let $P=p_L\otimes\cdots\otimes p_1\in\mathcal{P}_{\lambda\mu}$. Then:*

(*i*) *The shape of $p_{i+1}\cdot p_i$ is the same as the shape of the $(i,i+1)$-subword of $\omega(P)$.*

(*ii*) *Let $w=w_1\dots w_{|\mu|}:=\omega(P)$ and $w'=w'_1\dots w'_{|\mu|}:=\omega(\sigma_i(P))$. Then* $\mathrm{shape}(w_\ell\dots w_{\ell'})=\mathrm{shape}(w'_\ell\dots w'_{\ell'})$ *for all $1\leq\ell\leq\ell'\leq|\mu|$. In particular, $\omega(P)$ and $\omega(\sigma_i(P))$ have the same shape, and if $\omega(P)$ is in row-representation then so is $\omega(\sigma_i(P))$.*

Proof. (i) Let $L(w,k)$ be the largest possible sum of the lengths of k disjoint increasing sequences extracted from the word w. If w has shape ν then $L(w,k) = \nu_1 + \cdots + \nu_k$ (see for example Lemma 1 on page 32 of [13]). Since $\mathrm{shape}(p_{i+1} \cdot p_i) = \mathrm{shape}(w_{p_{i+1}} w_{p_i})$ (i) is proven if we can show that $L(w_{p_{i+1}} w_{p_i}, k) = L(\omega(p_{i+1} \otimes p_i), k)$ for all k. Let $s_1, \ldots, s_k$ be disjoint increasing sequences of $w_{p_{i+1}} w_{p_i}$ such that the sum of their lengths is $L(w_{p_{i+1}} w_{p_i}, k)$. Now ω successively maps the rightmost largest element of P to $x_\ell^{(m)}$ if its position is in step ℓ and row m. Interpreting $s_1, \ldots, s_k$ as decreasing sequences from right to left we see that the image of s_j under ω is an increasing sequence from left to right in $\omega(p_{i+1} \otimes p_i)$. Conversely, each increasing sequence of $\omega(p_{i+1} \otimes p_i)$ is a decreasing sequence from right to left of $w_{p_{i+1}} w_{p_i}$ which proves $L(w_{p_{i+1}} w_{p_i}, k) = L(\omega(p_{i+1} \otimes p_i), k)$.

(ii) Since σ_i only changes the $(i, i+1)$-subword of $\omega(P)$ it suffices to prove (ii) for paths $P = p \otimes p'$ of length two. Set $\tilde{p}' \otimes \tilde{p} := \sigma(p \otimes p')$ so that $p \cdot p' = \tilde{p}' \cdot \tilde{p}$. In particular $p \cdot p'$ and $\tilde{p}' \cdot \tilde{p}$ have the same shape. Hence from (i) we conclude that $\omega(p \otimes p')$ and $\omega(\sigma(p \otimes p'))$ have the same shape as well. Denoted by $w_p^{(\ell,\ell')} w_{p'}^{(\ell,\ell')}$ and $w_{\tilde{p}'}^{(\ell,\ell')} w_{\tilde{p}}^{(\ell,\ell')}$ the words obtained from $w_p w_{p'}$ and $w_{\tilde{p}'} w_{\tilde{p}}$ after successively removing the $\ell - 1$ rightmost biggest letters and the $|\mu| - \ell'$ leftmost smallest letters, respectively. Then the arguments of point (i) still go through, that is, $L(w_p^{(\ell,\ell')} w_{p'}^{(\ell,\ell')}, k) = L(w_\ell \ldots w_{\ell'}, k)$ and $L(w_{\tilde{p}'}^{(\ell,\ell')} w_{\tilde{p}}^{(\ell,\ell')}, k) = L(w'_\ell \ldots w'_{\ell'}, k)$ for all k. Since $w_p^{(\ell,\ell')} w_{p'}^{(\ell,\ell')} \equiv w_{\tilde{p}'}^{(\ell,\ell')} w_{\tilde{p}}^{(\ell,\ell')}$, it follows that $\mathrm{shape}(w_\ell \ldots w_{\ell'}) = \mathrm{shape}(w'_\ell \ldots w'_{\ell'})$. From this one can immediately deduce that $\omega(P)$ is in row-representation if and only if $\omega(\sigma_i(P))$ is in row-representation. □

Definition 3.7 (Classically restricted paths)**.** *Let λ be a partition such that* $\mathrm{height}(\lambda) \leq n$ *and let $\mu \in \mathcal{R}^L$. The set of classically restricted paths $\overline{\mathcal{P}}_{\lambda\mu}$ is defined as*

$$\overline{\mathcal{P}}_{\lambda\mu} = \{P \in \mathcal{P}_{\lambda\mu} | \, \mathrm{shape}(\omega(P)) = \lambda\}. \tag{3.18}$$

Since each path $P \in \mathcal{P}_{\lambda\mu}$ contains λ_i boxes filled with i, the condition that $\omega(P)$ has shape λ implies that $\omega(P)$ is in row representation. Hence $\overline{\mathcal{P}}_{\lambda\mu}$ is isomorphic to $\mathrm{LRT}(\lambda, \mu)$.

Let us now introduce the restricted analogue of the supernomials of Definition 3.5.

Definition 3.8. *For λ a partition with* $\mathrm{height}(\lambda) \leq n$ *and $\mu \in \mathcal{R}^L$ define*

$$\tilde{K}_{\lambda\mu}(q) = \sum_{P \in \overline{\mathcal{P}}_{\lambda\mu}} q^{H(P)}. \tag{3.19}$$

When μ is a partition, i.e., $\mu \in \mathcal{R}^L$ such that its components μ_i are one-row partitions of decreasing size, $\tilde{K}_{\lambda\mu}(q)$ reduces to the cocharge Kostka polynomial. This follows from the work of Nakayashiki and Yamada [40] and Lascoux, Leclerc and Thibon [33] and the relation between the weight (3.7) and their statistics as explained in Remark 3.1, and is our motivation for the definition of generalized Kostka polynomials for all $\mu \in \mathcal{R}^L$.

Definition 3.9 (Generalized Kostka polynomials)**.** *For λ a partition with* $\mathrm{height}(\lambda) \leq n$ *and $\mu \in \mathcal{R}^L$, the generalized Kostka polynomial $K_{\lambda\mu}(q)$ is defined as*

$$K_{\lambda\mu}(q) = q^{\|\mu\|} \tilde{K}_{\lambda\mu}(1/q), \tag{3.20}$$

where $\|\mu\| = \sum_{i<j} |\mu_i \cap \mu_j|$.

Lemma 3.1 ensures that if $P' \in \mathcal{O}_P$ with $P \in \overline{\mathcal{P}}_{\lambda\mu}$ then $\mathrm{shape}(\omega(P')) = \lambda$. Therefore $P' \in \overline{\mathcal{P}}_{\lambda\tilde{\mu}}$ for some permutation $\tilde{\mu}$ of μ, and hence also $\tilde{K}_{\lambda\tilde{\mu}}(q) = \tilde{K}_{\lambda\mu}(q)$ in analogy with Equation (3.11). We also find that

$$K_{\lambda\mu}(q) = K_{\lambda\tilde{\mu}}(q) \tag{3.21}$$

since $\|\mu\| = \|\tilde{\mu}\|$.

We remark that the lattice path representation of the Kostka polynomials was used in [31] to express the Demazure characters as a sum over Kostka polynomials. It also yields expressions for the $\mathrm{A}^{(1)}_{n-1}/\mathrm{A}_{n-1}$ branching functions in terms of the Kostka polynomials [26, 40] and has been employed in [15] to obtain various further $\mathrm{A}^{(1)}_{n-1}$ branching- and string function identities.

4. Initial cyclage and cocharge for LR tableaux

In this section we define the notions of initial cyclage, cocharge and charge for LR tableaux which "paves the path" for Section 5 where an expression of the Lascoux–Schützenberger-type (2.4) for the generalized Kostka polynomials is derived. In the case when $\mathrm{LRT}(\lambda,\mu)$ coincides with $\mathrm{Tab}(\lambda,\mu)$ our definitions reduce to the usual definitions of the initial cyclage etc. as mentioned in Section 2.2.

The definition of the initial cyclage for $T \in \mathrm{LRT}(\cdot,\mu)$ has to be altered when $a \neq (1,\dots,1)$ for the alphabet X^a as given in (3.13). Namely, if $T = \left[x_i^{(a_i)}u\right] \in \mathrm{LRT}(\cdot,\mu)$ with $x_i^{(a_i)}u$ in row-representation (by the Yamanouchi condition on each of the i-subwords the first letter has to be $x_i^{(a_i)}$ for some i), then $T' := \left[ux_i^{(a_i)}\right]$ obtained by cycling the first letter is not in $\mathrm{LRT}(\cdot,\mu)$ since the Yamanouchi condition on the i-subword of $ux_i^{(a_i)}$ is violated if $a_i > 1$. To repair this fault we define the initial cyclage for $T = [w]$ with $w = x_i^{(a_i)}u$ in row-representation by considering the following chain of transformations

$$w \to w^{(a_i)} \to w^{(a_i-1)} \to \cdots \to w^{(1)} \in \mathcal{W}_\mu. \tag{4.1}$$

Here $w^{(a_i)} := ux_i^{(a_i)}$ and $w^{(j)}$ for $1 < j \leq a_i$ is a word such that (*i*) the k-subword for each $k \neq i$ is a balanced Yamanouchi word, (*ii*) the i-subword is balanced, (*iii*) the subword consisting of the letters $x_i^{(j)}$ and $x_i^{(j-1)}$ has one non-inverted $x_i^{(j-1)}$ and one non-inverted $x_i^{(j)}$ and (*iv*) the subword consisting of the letters $x_i^{(k)}$ and $x_i^{(k-1)}$ with $k \neq j$ has no non-inverted letters. The transformation $w^{(j+1)} \to w^{(j)}$ is defined as follows. Consider the subword of $w^{(j+1)}$ consisting of the letters $x_i^{(j)}$ and $x_i^{(j+1)}$ only. Determine all inverted pairs for this subword and exchange the two non-inverted letters $x_i^{(j)}$ and $x_i^{(j+1)}$ (making them an inverted pair). The resulting word is $w^{(j)}$. Clearly, this means that $w^{(1)} \in \mathcal{W}_\mu$.

Definition 4.1 (Initial cyclage)**.** *The initial cyclage $\mathcal{C}$ on $T \in \mathrm{LRT}(\cdot,\mu)$ is defined as*

$$\mathcal{C}(T) = \left[w^{(1)}\right], \tag{4.2}$$

where $w^{(1)} \in \mathcal{W}_\mu$ is the last word in the chain of transformations (4.1)*.*

Example 4.1. If

$$T = \begin{array}{|c|c|c|}\hline 2^{(2)} & 3^{(1)} & \multicolumn{1}{c}{} \\ \hline 1^{(2)} & 2^{(2)} & \multicolumn{1}{c}{} \\ \hline 1^{(1)} & 2^{(1)} & 2^{(1)} \\ \hline \end{array} \quad \text{then} \quad \mathcal{C}(T) = \begin{array}{|c|c|c|}\hline 3^{(1)} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1^{(2)} & 2^{(2)} & 2^{(2)} \\ \hline 1^{(1)} & 2^{(1)} & 2^{(1)} \\ \hline \end{array},$$

where the words $w^{(2)}$ and $w^{(1)}$ in the chain (4.1) are given by $w^{(2)} = x_3^{(1)}x_1^{(2)}x_2^{(2)}x_1^{(1)}x_2^{(1)}x_2^{(1)}x_2^{(2)}$ and $w^{(1)} = x_3^{(1)}x_1^{(2)}x_2^{(2)}x_1^{(1)}x_2^{(1)}x_2^{(2)}x_2^{(1)}$.

For $\mathrm{Tab}(\cdot,\mu)$ the initial cyclage defines a partial order ranked by the cocharge, that is, $\mathrm{co}(T) := \mathrm{rank}(T)$. In particular, $\mathrm{co}(T) = \mathrm{co}(T') + 1$ if $T' = \mathcal{C}(T)$ and the minimal element in this poset is $T_{\min} = [x_1^{\mu_1}\cdots x_n^{\mu_n}]$. For $\mathrm{LRT}(\cdot,\mu)$ we would like to mimic this structure, that is, we wish to turn $\mathrm{LRT}(\cdot,\mu)$ into a ranked poset with minimal element defined as the LR tableau $T_{\min} = \left[x_1^{\mu_1}x_2^{\mu_2}\cdots x_L^{\mu_L}\right]$, where $x_i^{\mu_i}$ abbreviates $(x_i^{(a_i)}x_i^{(a_i-1)}\cdots x_i^{(1)})^j$ for $\mu_i = (j^{a_i}) \in \mathcal{R}$. Note that for $a \neq (1,\ldots,1)$, $T_{\min}$ is no longer a one-row tableau but a tableau of shape $\mu_1+\mu_2+\cdots+\mu_L$ where the jth row is filled with the letters $x_i^{(j)}$ (all possible i) only.

Having fixed $T_{\min}$ we observe two important differences between the sets $\mathrm{LRT}(\cdot,\mu)$ (for $a \neq (1,\ldots,1)$) and $\mathrm{Tab}(\cdot,\mu)$.

Remark 4.1.

(1) If for $T, T_{\min} \in \mathrm{Tab}(\cdot,\mu)$, w_T and $w_{T_{\min}}$ both start with the same letter then $T = T_{\min}$. Generally this is not true for $T, T_{\min} \in \mathrm{LRT}(\cdot,\mu)$. Indeed, the LR tableau (3.17) of Example 3.2 starts with $x_1^{(3)}$, but is not the minimal LR tableaux, which reads

$$T_{\min} = \begin{array}{|c|c|c|c|c|}\hline 1^{(3)} & & & & \\ \hline 1^{(2)} & 3^{(2)} & 3^{(2)} & & \\ \hline 1^{(1)} & 2^{(1)} & 2^{(1)} & 3^{(1)} & 3^{(1)} \\ \hline \end{array}\,.$$

(2) For $a = (1,\ldots,1)$ the initial cyclage $\mathcal{C}$ has no fixed points. In other words there is no $T \in \mathrm{Tab}(\cdot,\mu)$ with $T \neq T_{\min}$ such that $\mathcal{C}(T) = T$. For $\mathrm{LRT}(\cdot,\mu)$, however, fixed points may occur. The following LR tableau, for example, is not minimal but obeys $\mathcal{C}(T) = T$:

$$T = \begin{array}{|c|c|c|}\hline 4^{(3)} & & \\ \hline 1^{(2)} & 3^{(1)} & 4^{(2)} \\ \hline 1^{(1)} & 2^{(1)} & 4^{(1)} \\ \hline \end{array}\,. \tag{4.3}$$

The second remark shows that the initial cyclage $\mathcal{C}$ does not induce a ranked poset structure on $\mathrm{LRT}(\cdot,\mu)$ and hence needs further modification. For this purpose we define the "dropping" and "insertion" operators $\mathcal{D}$ and $\mathcal{U}$, respectively. Let $T \in \mathrm{LRT}(\cdot,\mu)$ with $\mu \in \mathcal{R}^L$. If, for some fixed $i \in \{1,\ldots,L\}$ and all $j = 1,\ldots,\mathrm{height}(T)$ the jth row of T contains $x_i^{(j)}$ (there may be more than one $x_i^{(j)}$ in row j), then drop all the boxes containing the letters $x_i^{(j)}$. Repeat this operation on the reduced tableau until no more letters can be dropped. The final tableau defines $\mathcal{D}(T)$. Obviously the condition for dropping occurs if and only if $\mathrm{height}(T) = a_i$ for some i. The operator $\mathcal{U}$ is somewhat intricate in that we only define $\mathcal{U}\circ\mathcal{O}\circ\mathcal{D}$, where $\mathcal{O}$ can be any content preserving operator acting on LR tableaux. So assume $T' = (\mathcal{O}\circ\mathcal{D})(T)$. Then $\mathcal{U}$ acts on T' by reinserting all boxes that have been dropped by $\mathcal{D}$, inserting a box with filling $x_i^{(j)}$ in the jth row such that the conditions for a Young tableau are satisfied (i.e., each row remains non-decreasing and each column strictly increasing on X^a). The insertion operator $\mathcal{U}$ is not to be confused with the insertion of boxes defined by the Schensted algorithm. $\mathcal{U}$ never bumps any boxes.

Remark 4.2. Note that $\mathcal{U}\circ\mathcal{D} = \mathrm{Id}$, but $\mathcal{D}\circ\mathcal{U}\circ\mathcal{O}\circ\mathcal{D} = \mathcal{D}\circ\mathcal{O}\circ\mathcal{D}$ and not $\mathcal{O}\circ\mathcal{D}$.

Definition 4.2 (Modified initial cyclage)**.**
The modified initial cyclage $\overline{\mathcal{C}} : \mathrm{LRT}(\cdot,\mu) \to \mathrm{LRT}(\cdot,\mu)$ *is defined as*

$$\overline{\mathcal{C}} = \mathcal{U} \circ \mathcal{C} \circ \mathcal{D}. \tag{4.4}$$

Note that $\mathcal{D}(T) = T$ when $\mathrm{height}(T) > \max\{a_1, \ldots, a_L\}$, in which case $\overline{\mathcal{C}} = \mathcal{C}$.

Finally we are in the position to define the cocharge and charge of an LR tableau.

Definition 4.3 (Cocharge and charge)**.** *Let* $T \in \mathrm{LRT}(\cdot,\mu)$.

(*i*) *The cocharge* $\mathrm{co}(T)$ *of* T *is the number of times one has to apply* $\overline{\mathcal{C}}$ *to obtain the minimal LR tableau* $T_{\min}$.

(*ii*) *The charge is* $c(T) = \|\mu\| - \mathrm{co}(T)$ *where* $\|\mu\| = \sum_{i<j} |\mu_i \cap \mu_j|$.

Example 4.2. For T in (4.3) we have

$$\mathcal{D}(T) = \begin{array}{|c|}\hline 3^{(1)} \\ \hline 2^{(1)} \\ \hline \end{array}\,; \qquad (\mathcal{C}\circ\mathcal{D})(T) = \begin{array}{|c|c|}\hline 2^{(1)} & 3^{(1)} \\ \hline \end{array}\,; \qquad \overline{\mathcal{C}}(T) = (\mathcal{U}\circ\mathcal{C}\circ\mathcal{D})(T) = \begin{array}{|c|c|c|c|}\hline & & & 4^{(3)} \\ \hline 1^{(2)} & 4^{(2)} & & \\ \hline 1^{(1)} & 2^{(1)} & 3^{(1)} & 4^{(1)} \\ \hline \end{array}\,.$$

Since $\overline{\mathcal{C}}(T) = T_{\min}$ and $\|\mu\| = 7$ we see that $\mathrm{co}(T) = 1$ and $c(T) = 6$.

Further examples of the action of the modified initial cyclage can be found in Appendix B.

As will be shown in Section 6, the modified initial cyclage $\overline{\mathcal{C}}$ indeed turns $\mathrm{LRT}(\cdot,\mu)$ into a ranked poset with co := rank. This implies in particular that $\mathrm{co}(T)$ is a bounded non-negative integer. In fact, we will do more and define more general λ-cyclages which induce a ranked poset structure on $\mathrm{LRT}(\cdot,\mu)$. We also show in Section 6 that the charge is non-negative and that $\|\mu\| = \mathrm{co}(T_{\max})$ where $T_{\max} = \left[x_L^{\mu_L} \cdots x_1^{\mu_1}\right]$ has maximal cocharge.

For convenience we write $\mathrm{co}(w)$ instead of $\mathrm{co}([w])$ for $w \in \mathcal{W}_\mu$.

5. Charge statistic representation for the generalized Kostka polynomials

There is a relation between the weight-function of Definition 3.4 and the cocharge of Definition 4.3 which is stated in Theorem 5.1. This relation enables us to derive an expression for the generalized Kostka polynomials stemming from the Lascoux and Schützenberger representation (2.4) given in Corollary 5.2. Section 5.2 is devoted to the proof of Theorem 5.1.

5.1. Relation between cocharge and weight.

To state the precise relation between cocharge and weight, we need to introduce the anti-automorphism Ω on words in $\mathcal{W}$. Recall that for every alphabet $X = \{x_1 < x_2 < \cdots < x_L\}$ there exists the dual alphabet $X^* = \{x_L^* < x_{L-1}^* < \cdots < x_1^*\}$. Setting $(x_i^*)^* = x_i$, $(X^*)^* = X$. The letter x_i^* which is often identified with x_{L+1-i} is called dual to x_i. Under Ω a word $w = x_{i_1} x_{i_2} \ldots x_{i_k}$ in the monoid $\mathcal{X}$ is mapped to $w^* = x_{i_k}^* x_{i_{k-1}}^* \ldots x_{i_1}^*$ in $\mathcal{X}^*$. Obviously, Ω is an involution. For the alphabet X^a of Equation (3.13) one may identify $(x_i^{(j)})^*$ with $x_{L+1-i}^{(a_i+1-j)}$ so that $(X^a)^*$ becomes X^{a^*} with $a^* = (a_{L+1}, \ldots, a_1)$. One can easily show that if $w \in \mathcal{W}$ over X^a, then $\Omega(w) \in \mathcal{W}^*$ with $\mathcal{W}^* = \{w \in \mathcal{X}^{a^*} |$ each i-subword of w is a balanced Yamanouchi word$\}$. Since Ω respects the Knuth equivalence relations, Ω is also well-defined on LR tableaux by setting $\Omega(T) = \Omega(w_T)$. On paths $P = p_L \otimes \cdots \otimes p_1 \in \mathcal{P}_{\lambda\mu}$ we define $\Omega_{\mathrm{p}}(P) = \Omega(p_1) \otimes \cdots \otimes \Omega(p_L)$.

Recall the map $w : \mathcal{P}_{\lambda\mu} \to \mathcal{W}_\mu$ defined in (3.16) from paths to words. With this we can now state the following theorem.

Theorem 5.1 (Weight-cocharge relation)**.** *For $n \geq 2$ and $L \geq 0$ integers let $\lambda \in \mathbb{Z}^n_{\geq 0}$ and $\mu \in \mathcal{R}^L$. Then for $P \in \mathcal{P}_{\lambda\mu}$ the weight $H(P)$ is a non-negative integer and*

$$H(P) = \mathrm{co}(\Omega \circ \omega(P)). \tag{5.1}$$

In the special case when $n = 2$ and $\mu = (1^{|\lambda|})$ a similar relation was noticed in [11]. Theorem 5.1 generalizes [33, Theorem 5.1] valid when μ is a partition. It also implies that the generalized Kostka polynomials can be expressed as the generating function of LR tableaux with the charge statistic. This is summarized in the following corollary.

Corollary 5.2. *The generalized Kostka polynomial $K_{\lambda\mu}(q)$ can be expressed as*

$$K_{\lambda\mu}(q) = \sum_{T \in \mathrm{LRT}(\lambda,\mu)} q^{c(T)}. \tag{5.2}$$

Proof. We start by rewriting the generalized cocharge Kostka polynomial $\tilde{K}_{\lambda\mu}(q)$. Recalling that $\overline{\mathcal{P}}_{\lambda\mu}$ is isomorphic to $\mathrm{LRT}(\lambda, \mu)$ it follows from (3.19) and (5.1) that

$$\tilde{K}_{\lambda\mu}(q) = \sum_{T \in \mathrm{LRT}(\lambda,\mu)} q^{\mathrm{co}(\Omega(T))}. \tag{5.3}$$

Since Ω does not change the shape of a tableau (see for example [13]), but changes its content $\mu = (\mu_1, \ldots, \mu_L)$ to $(\mu_L, \ldots, \mu_1)$, and since $\tilde{K}_{\lambda\mu}(q) = \tilde{K}_{\lambda\tilde{\mu}}(q)$ for a permutation $\tilde{\mu}$ of μ, we can drop Ω in (5.3). Recalling Equation (3.20) and $c(T) = \|\mu\| - \mathrm{co}(T)$ completes the proof of (5.2). □

5.2. **Proof of Theorem 5.1.** The proof of Theorem 5.1 requires several steps. First we use the fact that all paths with Knuth equivalent words have the same weight.

Lemma 5.3. *Let $P, P' \in \mathcal{P}_{\cdot\mu}$ such that $\omega(P) \equiv \omega(P')$. Then $H(P) = H(P')$.*

Proof. Since $\omega(P) \equiv \omega(P')$ also $\omega(p_{i+1} \otimes p_i) \equiv \omega(p'_{i+1} \otimes p'_i)$ (see Lemma 3 on page 33 of [13]). In particular, they have the same shape and hence, by (i) of Lemma 3.1, $p_{i+1} \cdot p_i$ and $p'_{i+1} \cdot p'_i$ have the same shape. This implies $h(p_{i+1} \otimes p_i) = h(p'_{i+1} \otimes p'_i)$ for all i and therefore $h(P) = h(P')$. Since σ_i only changes steps p_i and p_{i+1} and since, by (ii) of Lemma 3.1, $\omega(P)$ and $\omega(\sigma_i(P))$ have the same shape, it follows that $\omega(\sigma_i(P)) \equiv \omega(\sigma_i(P'))$. Hence, repeating the argument, $h(\sigma_i(P)) = h(\sigma_i(P'))$. This implies $H(P) = H(P')$. □

Thanks to the above lemma it suffices to prove (5.1) for just one representative path P for each $T \in \mathrm{LRT}(\cdot, \mu)$ such that $[\omega(P)] = T$. Let us now find a suitable set of such paths.

Define $\mathcal{P}_\mu := \mathcal{P}_{\lambda\mu}$ where $\lambda = (1^{|\mu|})$ and $|\mu| = \sum_i |\mu_i|$ so that for $P \in \mathcal{P}_\mu$ each letter $1, \ldots, |\mu|$ occurs exactly once. There is a bijection between $\mathcal{P}_\mu$ and $\mathcal{W}_\mu$. The map from $\mathcal{P}_\mu$ to $\mathcal{W}_\mu$ is just given by ω of Equation (3.16). The inverse map

$$\omega^{-1} : \mathcal{W}_\mu \to \mathcal{P}_\mu \tag{5.4}$$

is given as follows. Let $w = w_{|\mu|} w_{|\mu|-1} \cdots w_1$ be a word in $\mathcal{W}_\mu$. Then reading w from left to right, place i in the rightmost empty box in row k of step j if $w_i = x_j^{(k)}$. Obviously, since $w \in \mathcal{W}_\mu$ the steps of the resulting path P are Young tableaux of rectangular shapes μ_i.

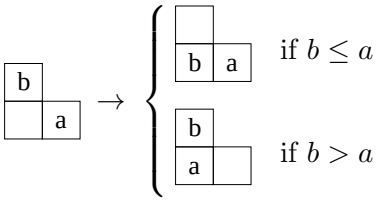


FIGURE 1. Sliding mechanism

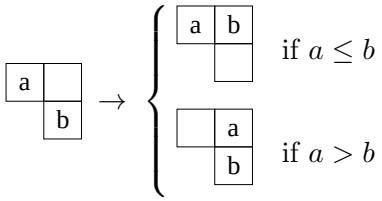


FIGURE 2. Inverse sliding mechanism

Denote the set of paths $P \in \mathcal{P}_\mu$ with $\omega(P)$ in row-representation by $\overline{\mathcal{P}}_\mu$. Since $\mathrm{LRT}(\cdot,\mu)$ and $\mathcal{W}_\mu/\equiv$ are isomorphic, the bijection between $\mathcal{P}_\mu$ and $\mathcal{W}_\mu$ also implies a bijection between $\overline{\mathcal{P}}_\mu$ and $\mathrm{LRT}(\cdot,\mu)$ still denoted by ω. By Lemma 5.3 we are thus left to prove (5.1) for all $P \in \overline{\mathcal{P}}_\mu$.

The bijection between $\mathrm{LRT}(\cdot,\mu)$ and $\overline{\mathcal{P}}_\mu$ induces a modified initial cyclage $\overline{\mathcal{C}}_{\mathrm{p}} : \overline{\mathcal{P}}_\mu \to \overline{\mathcal{P}}_\mu$ defined as $\overline{\mathcal{C}}_{\mathrm{p}} := \omega^{-1} \circ \overline{\mathcal{C}} \circ \omega$. Before setting out for the proof of (5.1) let us study some of the properties of this induced function.

First consider the induced map $\mathcal{C}_{\mathrm{p}} := \mathcal{S} \circ \omega^{-1} \circ \mathcal{C} \circ \omega$ of the initial cyclage $\mathcal{C}$ of Definition 4.1. Here $\mathcal{S}$ is a shift operator which decreases the letters in each step of $P \in \mathcal{P}_\mu$ by one and hence makes $\mathcal{C}_{\mathrm{p}}(P)$ a path over $\{0,1,\ldots,|\mu|-1\}$. The reason for including $\mathcal{S}$ in the definition of $\mathcal{C}_{\mathrm{p}}$ is merely for convenience so that $\mathcal{C}_{\mathrm{p}}$ acts only on one step of paths in $\overline{\mathcal{P}}_\mu$ as will be shown in Lemma 5.4. One may always undo the effect of $\mathcal{S}$ by acting with $\mathcal{S}^{-1}$ which adds one to each entry of a path. To state the precise action of $\mathcal{C}_{\mathrm{p}}$ on a path, let us briefly review Schützenberger's (inverse) sliding mechanism [48]. Suppose there is an empty box with neighbours to the right and above. Then slide the smaller of the two neighbours in the hole; if both neighbours are equal choose the one above. Similarly for the inverse sliding mechanism consider an empty box with neighbours to the left and below. Slide the bigger of the two neighbours in the hole; if both are equal choose the one below. If there is only one neighbour in either case slide this one into the empty box. The sliding and inverse sliding mechanisms are illustrated in Figures 1 and 2, respectively.

Lemma 5.4. *Let $P = p_L \otimes \cdots \otimes p_1 \in \overline{\mathcal{P}}_\mu$ be a path over $\{1,2,\ldots,|\mu|\}$ and let the letter $|\mu|$ be contained in step p_i. Then*

(*i*) *$\mathcal{C}_{\mathrm{p}}$ acts only on step p_i of P, i.e., $\mathcal{C}_{\mathrm{p}}(P) = p_L \otimes \cdots \otimes \mathcal{C}_{\mathrm{p}}(p_i) \otimes \cdots \otimes p_1$, and*

(*ii*) *$\mathcal{C}_{\mathrm{p}}(p_i)$ is obtained by first removing $|\mu|$ from the top-right box of p_i, then using the inverse sliding mechanism to move the empty box to the bottom-left corner and finally inserting 0 into the empty box.*

Proof. Since $P \in \overline{\mathcal{P}}_\mu$ and since the largest letter $|\mu|$ occurs in step i, the word $\omega(P)$ is in row-representation and of the form $\omega(P) = x_i^{(a_i)} u$. Let $T = [\omega(P)]$. In the chain of transformations (4.1) with $w = \omega(P)$ only the i-subword of w gets changed and all letters in $w^{(1)}$ not in the i-subword are shifted one position to the left. Hence $\mathcal{S} \circ \omega^{-1} \circ \mathcal{C}(T)$ leaves all but the ith step in P invariant which implies (*i*). To prove (*ii*) observe that in row $j+1$ the empty box moves to the left up to the point where the left neighbour is smaller than the neighbour below. Under the map ω these two neighbours correspond to the two non-inverted letters $x_i^{(j)}$ and $x_i^{(j+1)}$ in $w^{(j+1)}$ of (4.1) used for the definition of $\mathcal{C}$. □

Some properties of the initial cyclage $\mathcal{C}_{\rm p}$, the map ω, the involution $\Omega_{\rm p}$ and the isomorphism σ_i are summarized in the following lemma. For $P \in \mathcal{P}_{\lambda\mu}$ we set $h_i(P) = h(p_{i+1} \otimes p_i)$.

Lemma 5.5. *For $\lambda \in \mathbb{Z}^n_{\geq 0}$ and $\mu \in \mathcal{R}^L$ we have on $\mathcal{P}_{\lambda\mu}$*

$$h_{L-i} = h_i \circ \Omega_{\rm p}, \tag{5.5}$$

$$\Omega_{\rm p} \circ \sigma_i = \sigma_{L-i} \circ \Omega_{\rm p}, \tag{5.6}$$

and on $\overline{\mathcal{P}}_\mu$

$$\Omega_{\rm p} = \omega^{-1} \circ \Omega \circ \omega, \tag{5.7}$$

$$[\sigma_i, \mathcal{C}_{\rm p}] = 0, \tag{5.8}$$

$$\mathcal{C}_{\rm p} = \Omega_{\rm p} \circ \mathcal{C}_{\rm p}^{-1} \circ \Omega_{\rm p}, \tag{5.9}$$

where $\mathcal{C}_{\rm p}^{-1}$ is defined as follows. It acts on the step with the smallest entry in $P \in \overline{\mathcal{P}}_\mu$ by removing the 1, moving the empty box by the sliding mechanism to the top right corner and inserting $|\mu| + 1$.

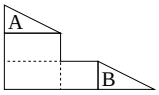
Proof. Let $P \in \mathcal{P}_{\lambda\mu}$. The energy $h_i(P)$ is determined by the shape of $p_{i+1} \cdot p_i$. Hence $h_i(\Omega_{\rm p}(P))$ is determined by the shape of $\Omega(p_{L-i}) \cdot \Omega(p_{L+1-i}) = \Omega(p_{L+1-i} \cdot p_{L-i})$. But Ω leaves the shape of a Young tableau invariant (see for example [13]), yielding (5.5).

Since the isomorphism σ_i acts only locally on $p_{i+1} \otimes p_i$ and $\Omega_{\rm p}$ reverses the order of the steps, it suffices to prove (5.6) for a path of length two. Define $\tilde{p}_1 \otimes \tilde{p}_2 = \sigma(p_2 \otimes p_1)$ so that $\tilde{p}_1 \cdot \tilde{p}_2 = p_2 \cdot p_1$. Acting on the last equation with Ω yields $\Omega(\tilde{p}_2) \cdot \Omega(\tilde{p}_1) = \Omega(p_1) \cdot \Omega(p_2)$. Since Ω does not change the shape of a Young tableau and because of the uniqueness of the decomposition into the product of two rectangular Young tableaux we conclude that $\sigma(\Omega(p_1) \otimes \Omega(p_2)) = \Omega(\tilde{p}_2) \otimes \Omega(\tilde{p}_1)$ which proves (5.6).

Equation (5.7) follows in a straightforward manner from the definitions of ω and ω^{-1}.

Let $P \in \overline{\mathcal{P}}_\mu$ and let the letter $|\mu|$ be contained in step p_j of P. By (i) of Lemma 5.4, $\mathcal{C}_{\rm p}$ acts only on step p_j, and σ_i acts only on $p_{i+1} \otimes p_i$. Hence the proof of (5.8) reduces to showing that $[\sigma, Z_p] = 0$ on $\mathcal{P}_{(\mu_1,\mu_2)}$. Here $Z_p = \mathcal{S} \circ \omega^{-1} \circ Z \circ \omega$ and $Z : \mathcal{W}_\mu \to \mathcal{W}_\mu$ is defined as $Z(w) = w^{(1)}$ where $w^{(1)}$ as given in (4.1) (note that w need not be in row-representation). Let $P = p_2 \otimes p_1 \in \mathcal{P}_{(\mu_1,\mu_2)}$ and set $w = w_1 \dots w_{|\mu|} := \omega(P)$ and $\tilde{w} = \tilde{w}_1 \dots \tilde{w}_{|\mu|} := \omega(\sigma(P))$. The map $\omega : \mathcal{P}_\mu \to \mathcal{W}_\mu$ is a bijection. Since for a given shape λ the set $\mathrm{LRT}(\lambda, (\mu_1, \mu_2))$ can have at most one element, a word $w \in \mathcal{W}_{(\mu_1,\mu_2)}$ is uniquely specified by $\mathrm{shape}(w_1 \dots w_k)$ for all $1 \leq k \leq |\mu|$. Hence (5.8) amounts to showing that, for $w' = w'_1 \dots w'_{|\mu|} := Z(w)$ and $\tilde{w}' = \tilde{w}'_1 \dots \tilde{w}'_{|\mu|} := Z(\tilde{w})$, $\mathrm{shape}(w'_1 \dots w'_k) = \mathrm{shape}(\tilde{w}'_1 \dots \tilde{w}'_k)$ for all $1 \leq k \leq |\mu|$. By construction, $\mathrm{shape}(w'_1 \dots w'_k) = \mathrm{shape}(w_2 \dots w_{k+1})$ and $\mathrm{shape}(\tilde{w}'_1 \dots \tilde{w}'_k) = \mathrm{shape}(\tilde{w}_2 \dots \tilde{w}_{k+1})$ for all $1 \leq k < |\mu|$ and by Lemma 3.1 (ii) $\mathrm{shape}(w_2 \dots w_k) = \mathrm{shape}(\tilde{w}_2 \dots \tilde{w}_k)$. Hence we are left to show that $\mathrm{shape}(w') = \mathrm{shape}(\tilde{w}')$. This is can be done explicitly. In particular, one may use that the shape of the product of two rectangular Young tableaux has the following form

$$\mathrm{shape}(p_2 \cdot p_1) = \tag{5.10}$$



where A and B are partitions and the two overlapping rectangles are the shapes of p_1 and p_2; one may be contained in the other. Note that A is the complement of B, so that knowing

A (B) fixes the shape. By Lemma 3.1 (i) also $\omega(p_2 \otimes p_1)$ has the shape (5.10). Let $b = \mathrm{shape}(w_1 \dots w_{|\mu|})/\mathrm{shape}(w_2 \dots w_{|\mu|})$ and $b' = \mathrm{shape}(w'_1 \dots w'_{|\mu|})/\mathrm{shape}(w_2 \dots w_{|\mu|})$. One may show that (i) if $b \in A$ then $b' \in B'$, (ii) if $b \in B$ then $b' \in A'$ and (iii) if $b \notin A \cup B$ then $b' \notin A' \cup B'$. Since b is the same for both w and $\tilde{w}$ this implies that $\mathrm{shape}(w') = \mathrm{shape}(\tilde{w}')$.

For the proof of (5.9) one can consider a path consisting of just a single step thanks to (i) of Lemma 5.4. Suppose p has M boxes filled with the numbers $1, \dots, M$. From (ii) of Lemma 5.4 we know that $\mathcal{C}_{\mathrm{p}}$ acts by the inverse sliding mechanism and by definition $\mathcal{C}_{\mathrm{p}}^{-1}$ acts by the sliding mechanism. Ω acts on rectangular Young tableaux by rotation of 180° and dualizing all letters. But since the inverse sliding mechanism is the same as the sliding mechanism after rotation of 180° and dualizing, as is easily seen from Figures 1 and 2, Equation (5.9) follows. □

After these preliminaries we come to the heart of the proof of Theorem 5.1. By Lemma 5.3 we are left to prove Equation (5.1) for all $P \in \overline{\mathcal{P}}_\mu$ and by (5.7) this is equivalent to

$$H'(P) := H(\Omega_{\mathrm{p}}(P)) = \mathrm{co}(\omega(P)). \tag{5.11}$$

We will show that $\overline{\mathcal{C}}_{\mathrm{p}} = \omega^{-1} \circ \overline{\mathcal{C}} \circ \omega$ decreases the weight H' of paths in $\overline{\mathcal{P}}_\mu$ by one, i.e.,

$$H'(P) - H'(\overline{\mathcal{C}}_{\mathrm{p}}(P)) = 1 \quad \text{for } P \in \overline{\mathcal{P}}_\mu \text{ and } P \neq P_{\min}, \tag{5.12}$$

where $P_{\min} := \omega^{-1}(w_{\min})$ with $w_{\min} = w_{\min}(\mu) = x_1^{\mu_1} \cdots x_L^{\mu_L}$ the word corresponding to the minimal LR tableau $T_{\min}$. By definition $\mathrm{co}(T_{\min}) = 0$ and one finds by direct computation that also $H'(P_{\min}) = 0$. (This can be deduced from the fact that in $P_{\min}$ the number i cannot be contained in a step to the left of the step containing $i-1$; this is also true for any $P \in \mathcal{O}_{P_{\min}}$ as $P = \omega^{-1}(w_{\min}(\tilde{\mu}))$ for some permutation $\tilde{\mu}$ of μ). The equation $H'(P_{\min}) = \mathrm{co}(T_{\min}) = 0$ together with (5.12) implies that $H'(P)$ and thus $H(P)$ are integers. By definition $H'(P)$ is finite and non-negative. Suppose there exists a $P \in \overline{\mathcal{P}}_\mu$ such that $m-1 < H'(P) < m$ for some integer m. Then we conclude from (5.12) that $H'(\overline{\mathcal{C}}_{\mathrm{p}}^m(P)) < 0$ which contradicts the non-negativity of H'. Since $\mathrm{co}(T) - \mathrm{co}(\overline{\mathcal{C}}(T)) = 1$ Equation (5.12) implies (5.11) for all $P \in \overline{\mathcal{P}}_\mu$.

Using (3.2), (3.7), (5.5), (5.6) and $\Omega_{\mathrm{p}}^2 = \mathrm{Id}$ one finds that

$$H'(P) = H(\Omega_{\mathrm{p}}(P)) = \frac{1}{|\mathcal{O}_P|} \sum_{P' \in \mathcal{O}_P} \sum_{i=1}^{L-1} (L-i) h_i(P'). \tag{5.13}$$

Hence to show (5.12) one needs to relate the energies $h_i(P)$ and $h_i(\overline{\mathcal{C}}_{\mathrm{p}}(P))$. Let us first focus on the relation between the energies of P and $\mathcal{C}_{\mathrm{p}}(P)$. Following [33] we decompose the orbit $\mathcal{O}_P$ of P into chains. Let $U, V \in \mathcal{O}_P$ with largest entries in step i and $i-1$, respectively. Then write $U \rightsquigarrow V$ if $\sigma_{i-1}(U) = V$ ($i = 2, 3, \dots, L$). Connected components of the resulting graph are called chains. With this notation we have the following lemma which is proven in Appendix A.

Lemma 5.6. *For $P \in \overline{\mathcal{P}}_\mu$ with $\mu \in \mathcal{R}^L$ define the vector $\boldsymbol{h}(P) = (h_1(P), h_2(P), \dots, h_{L-1}(P))$. For a chain $\gamma = \{P_m \rightsquigarrow P_{m-1} \rightsquigarrow \cdots \rightsquigarrow P_\ell\}$ such that $\sigma_{k-1}(P_k) = P_{k-1}$ and $Q_j = \mathcal{C}_{\mathrm{p}}(P_j)$ the following relations hold,*

$$\begin{aligned} &\boldsymbol{h}(Q_m) - \boldsymbol{h}(P_m) = \boldsymbol{e}_m \\ &\boldsymbol{h}(Q_k) - \boldsymbol{h}(P_k) = 0 \qquad\qquad \text{for } \ell < k < m \\ &\boldsymbol{h}(Q_\ell) - \boldsymbol{h}(P_\ell) = -\boldsymbol{e}_{\ell-1} \end{aligned} \tag{5.14}$$

and if $m = \ell$,

$$\boldsymbol{h}(Q_m) - \boldsymbol{h}(P_m) = \boldsymbol{e}_m - \boldsymbol{e}_{m-1}. \tag{5.15}$$

Here $\boldsymbol{e}_m$ $(1 \leq m \leq L-1)$ *are the canonical basis vectors of* $\mathbb{Z}^{L-1}$ *and* $\boldsymbol{e}_0 = \boldsymbol{e}_L = 0$.

Thanks to Equation (5.8) $\{Q_m, Q_{m-1}, \ldots, Q_\ell\}$ is a subset of $\mathcal{O}_{\mathcal{C}_{\mathrm{p}}(P)}$. Defining $H'_\gamma(P) := \frac{1}{|\gamma|}\sum_{P'\in\gamma}\sum_{i=1}^{L-1}(L-i)h_i(P')$ for a subset $\gamma \subset \mathcal{O}_P$, Lemma 5.6 ensures that

$$H'_\gamma(P) - H'_{\mathcal{C}_{\mathrm{p}}(\gamma)}(\mathcal{C}_{\mathrm{p}}(P)) = 1 \tag{5.16}$$

for $\gamma = \{P_m \rightsquigarrow P_{m-1} \rightsquigarrow \cdots \rightsquigarrow P_\ell\}$ as long as $\ell > 1$. For the case treated in [33], where $[\omega(P)] \in \mathrm{Tab}(\cdot, \mu)$ is an ordinary Young tableau, ℓ is always bigger than one when $P \neq P_{\min}$ and hence the proof of Theorem 5.1 is complete in this case. For $[\omega(P)] \in \mathrm{LRT}(\cdot, \mu)$, however, ℓ can take the value one even if $P \neq P_{\min}$, due to point 1 of Remark 4.1. Hence (5.16) breaks down for $\ell = 1$, i.e., when there is a $P' \in \gamma$ such that the letter $|\mu|$ is contained in the first step. However, in this case we are saved by the following lemma. Therein, the height of a path $P = p_L \otimes \cdots \otimes p_1$ is defined as $\mathrm{height}(P) := \max_{1\leq i\leq L}\{\mathrm{height}(p_i)\}$.

Lemma 5.7. *Let* $P \in \overline{\mathcal{P}}_\mu$ *over* $\{1, 2, \ldots, |\mu|\}$. *Then there exists a path* $P' = p'_L \otimes \cdots \otimes p'_1 \in \mathcal{O}_P$ *such that* p'_1 *contains the letter* $|\mu|$ *if and only if* $\mathrm{height}(\omega(P)) = \mathrm{height}(P)$.

Proof. Let us first show that the existence of P' implies the condition on the height of $\omega(P)$. Since p'_1 contains $|\mu|$ the word $\omega(P')$ starts with $x_1^{(a_1)}$. By (ii) of Lemma 3.1 $\omega(P')$ is in row-representation. Hence the height of $\omega(P')$ equals the height of p'_1 and the first step is also (one of) the highest. Again by (ii) of Lemma 3.1 $\omega(P)$ and $\omega(P')$ have the same shape so that the height of $\omega(P)$ equals the height of P.

To prove the reverse, consider $P' \in \mathcal{O}_P$ such that the first step is highest. Employing again (ii) of Lemma 3.1 we see that the height of $\omega(P')$ equals the height of the first step. Now suppose that p'_1 does not contain $|\mu|$. This means that $\omega(P') = x_i^{(a_i)}u$ for some $u \in \mathcal{W}$ with $i > 1$. Since P' is in row representation $x_i^{(a_i)}$ must be above $x_1^{(a_1)}$ in $[\omega(P')]$. This contradicts the fact that the height of $\omega(P')$ is the height of p'_1. □

The previous lemma shows that there exist chains γ such that $\gamma = \{P_m \rightsquigarrow \cdots \rightsquigarrow P_1\}$ (so that (5.16) is violated) if and only if the modified initial charge $\overline{\mathcal{C}}_{\mathrm{p}}$ differs from $\mathcal{S}^{-1} \circ \mathcal{C}_{\mathrm{p}}$. This is the case because the dropping and insertion operators $\mathcal{D}_{\mathrm{p}} := \omega^{-1} \circ \mathcal{D} \circ \omega$ and $\mathcal{U}_{\mathrm{p}} := \omega^{-1} \circ \mathcal{U} \circ \omega$ in the relation $\overline{\mathcal{C}}_{\mathrm{p}} = \mathcal{U}_{\mathrm{p}} \circ \mathcal{S}^{-1} \circ \mathcal{C}_{\mathrm{p}} \circ \mathcal{D}_{\mathrm{p}}$ only act non-trivially when the height of $\omega(P)$ equals the height of P, or equivalently by Lemma 5.7, when there exists a chain $\gamma = \{P_m \rightsquigarrow \cdots \rightsquigarrow P_1\}$. The dropping operator, however, does not change the weight of a path as shown in the following lemma.

Lemma 5.8. *For* $\mu \in \mathcal{R}^L$ *let* $P \in \overline{\mathcal{P}}_\mu$ *such that* $\mathrm{height}(P) = \mathrm{height}(\omega(P))$. *Then* $H'(P) = H'(\mathcal{D}_{\mathrm{p}}(P))$.

Lemmas 5.6–5.8 imply (5.12) and hence Theorem 5.1 for the following reason. For $P \in \overline{\mathcal{P}}_\mu$, the path $\mathcal{D}_{\mathrm{p}}(P)$ does not contain any chains $\gamma = \{P_m \rightsquigarrow \cdots \rightsquigarrow P_1\}$ thanks to Lemma 5.7. Hence $H'(P) = H'(\mathcal{D}_{\mathrm{p}}(P)) = H'(\mathcal{C}_{\mathrm{p}} \circ \mathcal{D}_{\mathrm{p}}(P)) + 1$. On the other hand $H'(\mathcal{C}_{\mathrm{p}} \circ \mathcal{D}_{\mathrm{p}}(P)) = H'(\overline{\mathcal{C}}_{\mathrm{p}}(P))$ since $\mathcal{S}^{-1}$ does not change the energy of a path and because of Remark 4.2 and Lemma 5.8. This proves (5.12).

Proof of Lemma 5.8. For a path in $\mathcal{P}_{\lambda\mu}$ we refer to μ as its content. Now suppose the path P of Lemma 5.8 has k steps of shape $\nu \in \mathcal{R}$ where $k \geq 1$ and height(ν) = height(P). Then all $P' \in \mathcal{O}_P$ have k steps of shape ν and, by (ii) of Lemma 3.1, height$(\omega(P'))$ = height(ν). Define $\mathcal{D}_\nu(P')$ as the path obtained from P' by dropping all steps of shape ν. Let η be the content of $\mathcal{D}_\nu(P)$. Then for each permutation $\tilde{\eta}$ of η define the suborbit $\mathcal{S}_{\tilde{\eta}} \subset \mathcal{O}_P$ as

$$\mathcal{S}_{\tilde{\eta}} = \{\tilde{P} \in \mathcal{O}_P |\ \text{content}(\mathcal{D}_\nu(\tilde{P})) = \tilde{\eta}\}. \tag{5.17}$$

Then clearly $\mathcal{O}_P$ is the disjoint union of $\mathcal{S}_{\tilde{\eta}}$ over all permutations $\tilde{\eta}$ of η, and $|\mathcal{S}_{\tilde{\eta}}| = \binom{L}{k}$.

Let us now show that for any $Q \in \mathcal{S}_{\tilde{\eta}}$

$$\sum_{\tilde{P}\in\mathcal{S}_{\tilde{\eta}}} \sum_{i=1}^{L-1} i h_{L-i}(\tilde{P}) = \binom{L}{k} \sum_{i=1}^{L-k-1} i h_{L-k-i}(\mathcal{D}_\nu(Q)). \tag{5.18}$$

Since $\mathcal{D}_{\rm p}$ is a composition of $\mathcal{D}_\nu$'s, Equation (5.18) clearly implies $H'(P) = H'(\mathcal{D}_{\rm p}(P))$.

To prove (5.18) we first study some properties of the energy $h_i(\tilde{P})$ for $\tilde{P} \in \mathcal{S}_{\tilde{\eta}}$. For $\tilde{P} = \tilde{p}_L \otimes \cdots \otimes \tilde{p}_1 \in \mathcal{S}_{\tilde{\eta}}$, let $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_L)$ be the content of $\tilde{P}$ and define $L \geq m_1 > \cdots > m_k \geq 1$ such that $\tilde{\mu}_{m_i} = \nu$. Then $\tilde{P}' = \sigma_{m_i+1} \circ \sigma_{m_i}(\tilde{P})$ is also in $\mathcal{S}_{\tilde{\eta}}$ and, for $1 \leq i \leq k$,

$$h_{m_i}(\tilde{P}) = h_{m_i-1}(\tilde{P}) = 0, \tag{5.19}$$

$$h_{m_i}(\tilde{P}') = h_{m_i+1}(\tilde{P}). \tag{5.20}$$

The proof of (5.19) and (5.20) makes extensive use of Lemma 3.1. For two steps $p \in \mathcal{B}_\lambda$ and $p' \in \mathcal{B}_{\lambda'}$ let us call the shape $\lambda + \lambda'$ minimal because $h(p \otimes p') = 0$ if shape$(p \cdot p') = \lambda + \lambda'$. Equation (5.19) states that $\tilde{p}_{m_i+1} \cdot \tilde{p}_{m_i}$ and $\tilde{p}_{m_i} \cdot \tilde{p}_{m_i-1}$ have minimal shape, or equivalently by (i) of Lemma 3.1, that $\omega(\tilde{p}_{m_i+1} \otimes \tilde{p}_{m_i})$ and $\omega(\tilde{p}_{m_i} \otimes \tilde{p}_{m_i-1})$ have shapes $\tilde{\mu}_{m_i+1} + \nu$ and $\nu + \tilde{\mu}_{m_i-1}$, respectively. But since the height of $\omega(\tilde{P})$ is the height of ν, the heights of $\omega(\tilde{p}_{m_i+1} \otimes \tilde{p}_{m_i})$ and $\omega(\tilde{p}_{m_i} \otimes \tilde{p}_{m_i-1})$ equal the height of ν, and hence their shape has to be minimal.

We now turn to the proof of (5.20). Denote $\tilde{P}' = \tilde{p}'_L \otimes \cdots \otimes \tilde{p}'_1$. Since $\tilde{P}' = \sigma_{m_i+1} \circ \sigma_{m_i}(\tilde{P})$ we know by (ii) of Lemma 3.1 that $\omega(\tilde{p}_{m_i+2} \otimes \tilde{p}_{m_i+1} \otimes \tilde{p}_{m_i})$ and $\omega(\tilde{p}'_{m_i+2} \otimes \tilde{p}'_{m_i+1} \otimes \tilde{p}'_{m_i})$ have the same shape. But since by $\omega(\tilde{p}_{m_i+1} \otimes \tilde{p}_{m_i})$ and $\omega(\tilde{p}'_{m_i+2} \otimes \tilde{p}'_{m_i+1})$ have minimal shape by (5.19) we can conclude that $\omega(\tilde{p}_{m_i+2} \otimes \tilde{p}_{m_i+1})$ and $\omega(\tilde{p}'_{m_i+1} \otimes \tilde{p}'_{m_i})$ have the same shape. Hence by (i) of Lemma 3.1 also $\tilde{p}_{m_i+2} \cdot \tilde{p}_{m_i+1}$ and $\tilde{p}'_{m_i+1} \cdot \tilde{p}'_{m_i}$ have the same shape which implies (5.20).

Analogous to the proof of (5.20) we find that for $\tilde{P}' = \sigma_{m_i}(\tilde{P})$ the tableaux $\tilde{p}_{m_i+1} \cdot \tilde{p}_{m_i-1}$ and $\tilde{p}'_{m_i} \cdot \tilde{p}'_{m_i-1}$ have the same shape. Setting $\tilde{P}$ to Q in this argument shows that $h_{L-k-i}(\mathcal{D}_\nu(Q))$ is independent of $Q \in \mathcal{S}_{\tilde{\eta}}$. Hence we can restrict our attention to $Q \in \mathcal{S}_{\tilde{\eta}}$ with steps 1 to k of shape ν in the following.

If $k = L$ or $L-1$ the right-hand side of (5.18) is zero due to the empty sum. Equation (5.19) ensures that the left-hand side is zero as well. If $1 \leq k \leq L-2$ set $X_i := h_{L-k-i}(\mathcal{D}_\nu(Q))$ for $1 \leq i < L-k$. Define r_j as $r_j = L+1-m_j-j$ for $1 \leq j \leq k$ and $r_0 = 0$, $r_{k+1} = L-k$ for a given $\tilde{P}$ where, as above, the m_i are the positions of the steps of shape ν. Treating X_i as an indeterminate we see from (5.19) and (5.20) that the contribution to X_i from $\sum_{i=1}^{L-1} i h_{L-i}(\tilde{P})$ is given by

$$\begin{array}{ll} (i+j) & \text{for } r_j < i < r_{j+1} \text{ and } 0 \leq j \leq k, \\ 0 & \text{for } i = r_j \text{ and } 1 \leq j \leq k. \end{array}$$

Summing over all $\tilde{P} \in \mathcal{S}_{\tilde{\eta}}$ or, equivalently, over all possible r_i we find that

$$\begin{aligned}\sum_{\tilde{P}\in\mathcal{S}_{\tilde{\mu}}}\sum_{i=1}^{L-1} i h_{L-i}(\tilde{P}) &= \sum_{i=1}^{L-k-1} X_i \sum_{j=0}^{k}(i+j) \sum_{r_0\le\cdots\le r_j<i<r_{j+1}\le\cdots\le r_{k+1}} 1 \\ &= \sum_{i=1}^{L-k-1} X_i \sum_{j=0}^{k}(i+j)\binom{i+j-1}{j}\binom{L-i-j-1}{k-j} \\ &= \binom{L}{k}\sum_{i=1}^{L-k-1} iX_i\end{aligned}$$

where the last step follows from (a special case of) the ${}_2F_1$ Gauß sum. Recalling that $X_i = h_{L-k-i}(\mathcal{D}_\nu(Q))$ this proves Equation (5.18) and hence lemma 5.8. □

6. The poset structure on $\mathrm{LRT}(\cdot,\mu)$

As shown in Theorem 5.1, the weight and the cocharge are related as $H(P) = \mathrm{co}(\Omega\circ\omega(P))$. Since $H(P)$ is by its definition 3.4 finite and for each LR tableau $T \in \mathrm{LRT}(\cdot,\mu)$ there exists a path $P \in \mathcal{P}_{\cdot\mu}$ such that $T = [\omega(P)]$, Theorem 5.1 immediately implies that $\mathrm{co}(T)$ is finite for all $T \in \mathrm{LRT}(\cdot,\mu)$. This means that each $T \in \mathrm{LRT}(\cdot,\mu)$ reaches the minimal LR tableau $T_{\min}$ after a finite number of applications of $\overline{\mathcal{C}}$. This, in turn, ensures the following corollary.

Corollary 6.1. *The modified initial cyclage $\overline{\mathcal{C}}$ induces a ranked poset structure on* $\mathrm{LRT}(\cdot,\mu)$.

The statement of this corollary can be extended to more general cyclages which generalize the λ-cyclages of Lascoux and Schützenberger [32, 36] for Young tableaux $T \in \mathrm{Tab}(\cdot,\mu)$. We define λ-cyclages for LR tableaux $T \in \mathrm{LRT}(\cdot,\mu)$ in Section 6.1 and prove the analogue of Corollary 6.1 in Theorem 6.3. In Section 6.2 we deduce several important properties of the charge and cocharge which are needed in Section 7.2 to prove recurrences for the A_{n-1} supernomials and generalized Kostka polynomials.

6.1. **The λ-cyclage and λ-cocyclage.** The λ-cyclage is a generalization of the initial cyclage $\mathcal{C}$. Let us first define the cyclage operator $\mathcal{Z}$ on words $w = x_i^{(a_i)}u \in \mathcal{W}_\mu$ as $\mathcal{Z}(w) = w^{(1)}$ where $w^{(1)}$ is defined as in (4.1) by dropping the restriction that w is in row-representation. The only additional requirement is that all w' in the orbit of w (i.e., all w' such that $\omega^{-1}(w') \in \mathcal{O}_{\omega^{-1}(w)}$ with ω^{-1} as defined in (5.4)) start with a letter different from $x_1^{(a_1)}$. If w violates this condition we set $\mathcal{Z}(w) = 0$.

Similarly one can define a cocyclage $\mathcal{Z}^{-1}$ on a word $w = ux_i^{(1)} \in \mathcal{W}_\mu$. This time $\mathcal{Z}^{-1}(w) = 0$ if there exists a w' in the orbit of w ending with $x_1^{(1)}$. If not, set $w^{(1)} = w$ and construct the chain of transformations

$$w^{(1)} \to w^{(2)} \to \cdots \to w^{(a_i)}, \tag{6.1}$$

where $w^{(j+1)}$ is obtained from $w^{(j)}$ by exchanging the last letter $x_i^{(j)}$ with its inverted partner $x_i^{(j+1)}$ in the subword of $w^{(j)}$ consisting of the letters $x_i^{(j)}$ and $x_i^{(j+1)}$ only. Then the cocyclage $\mathcal{Z}^{-1}(w)$ is obtained by cycling the last letter $x_i^{(a_i)}$ in $w^{(a_i)}$ to the front of the word. Obviously, $\mathcal{Z}^{-1}$ is the inverse of $\mathcal{Z}$.

The cyclage $\mathcal{Z}$ and cocyclage $\mathcal{Z}^{-1}$ defined on the set of words have analogues on the set of LR tableaux. A λ-cyclage $\mathcal{Z}_\lambda$ on an LR tableau $T \in \mathrm{LRT}(\cdot,\mu)$ is defined as $\mathcal{Z}_\lambda(T) = \mathcal{Z}(w)$ where $w = x_i^{(a_i)}u \in \mathcal{W}_\mu$ is a word such that $T = [w]$ and $\mathrm{shape}(u) = \lambda$. Similarly, a λ-cocyclage $\mathcal{Z}_\lambda^{-1}$ is defined as $\mathcal{Z}_\lambda^{-1}(T) = \mathcal{Z}^{-1}(w)$ if $w = ux_i^{(1)}$ is a word such that $T = [w]$ and $\mathrm{shape}(u) = \lambda$. In both cases if no such w exists we set $\mathcal{Z}_\lambda(T) = 0$ and $\mathcal{Z}_\lambda^{-1}(T) = 0$, respectively. The λ-cyclage and λ-cocyclage obey

$$\begin{aligned} &\mathcal{Z}_\lambda \circ \mathcal{Z}_\lambda^{-1}(T) = T \qquad \text{if } \mathcal{Z}_\lambda^{-1}(T) \neq 0, \\ &\mathcal{Z}_\lambda^{-1} \circ \mathcal{Z}_\lambda(T) = T \qquad \text{if } \mathcal{Z}_\lambda(T) \neq 0. \end{aligned} \tag{6.2}$$

In analogy with (4.4) we also define the modified λ-cyclage as

$$\overline{\mathcal{Z}}_\lambda = \mathcal{U} \circ \mathcal{Z}_\lambda \circ \mathcal{D} \tag{6.3}$$

where $\mathcal{D}(T)$ was defined in Section 4 by successively dropping all $x_i^{(j)}$ $(1 \le j \le a_i)$ from T if $\mathrm{height}(T) = \mathrm{height}(\mu_i)$ (recall that $\mathrm{height}(\mu_i) = a_i$) and $\mathcal{U}$ was defined by reinserting all $x_i^{(j)}$ dropped by $\mathcal{D}$ in row j such that the Young tableau conditions are satisfied. Similarly we set

$$\overline{\mathcal{Z}}'_\lambda = \mathcal{U}' \circ \mathcal{Z}_\lambda^{-1} \circ \mathcal{D}', \tag{6.4}$$

where $\mathcal{D}'(T)$ is the tableau obtained by dropping all $x_i^{(j)}$ $(1 \le j \le a_i)$ if $\mathrm{width}(T) = \mathrm{width}(\mu_i)$ and $\mathcal{U}'$ reinserts all boxes dropped by $\mathcal{D}'$ such that there is one $x_i^{(j)}$ in each column and the resulting object is again in $\mathrm{LRT}(\cdot,\mu)$.

The initial cyclage $\mathcal{C}$ is a special λ-cyclage since for each $T \in \mathrm{LRT}(\cdot,\mu)$ there always exists a partition λ such that $\mathcal{C}(T) = \mathcal{Z}_\lambda(T)$. Hence also $\overline{\mathcal{C}}(T) = \overline{\mathcal{Z}}_\lambda(T)$ for this λ.

For ordinary Young tableaux $T \in \mathrm{Tab}(\cdot,\mu)$, the cyclages $\mathcal{Z}_\lambda$ and $\mathcal{Z}_\lambda^{-1}$ have been considered in [32, 36]. It was shown in [36] that the λ-cyclages induce a ranked poset structure on $\mathrm{Tab}(\cdot,\mu)$ with the cocharge of a Young tableau being its rank and the minimal element being $T_{\min} = [x_1^{\mu_1}\cdots x_L^{\mu_L}]$. Thanks to $\overline{\mathcal{Z}}_\lambda(T) = \mathcal{Z}_\lambda(T)$ for $T \in \mathrm{Tab}(\cdot,\mu)$ also $\overline{\mathcal{Z}}_\lambda$ induces a ranked poset structure on $\mathrm{Tab}(\cdot,\mu)$. Note, however, that $\overline{\mathcal{Z}}'_\lambda \neq \mathcal{Z}_\lambda^{-1}$ even on $\mathrm{Tab}(\cdot,\mu)$ since for example $\mathcal{Z}_\lambda^{-1}(T) = 0$, but $\overline{\mathcal{Z}}'_\lambda(T) = [3211]$ for $T = [2311]$ and $\lambda = (1)$.

From $\overline{\mathcal{Z}}_\lambda$ and $\overline{\mathcal{Z}}'_\lambda$ we may now define a cyclage- and cocyclage-graph.

Definition 6.1 (Cyclage- and cocyclage-graph)**.** *For $\mu \in \mathcal{R}^L$, the cyclage-graph $\mathcal{T}(\mu)$ is defined by connecting all $T, U \in \mathrm{LRT}(\cdot,\mu)$ as $T \to U$ if there exists a partition λ such that $U = \overline{\mathcal{Z}}_\lambda(T)$. Similarly, the cocyclage-graph $\mathcal{T}'(\mu)$ is obtained by connecting $T \to U$ if there exists a λ such that $U = \overline{\mathcal{Z}}'_\lambda(T)$.*

An example of a cyclage-graph is given in Appendix B. The cyclage- and cocyclage-graphs are related by an involution

$$\Lambda : \mathrm{LRT}(\lambda,\mu) \to \mathrm{LRT}(\lambda^\top, \mu^\star)$$

defined as follows. Let $T = [w_1 \cdots w_\ell] \in \mathrm{LRT}(\lambda,\mu)$ with $w_i \in X^a$. Then $\Lambda(T) = [w'_\ell \cdots w'_1]$ where $w'_m = x_i^{(k)}$ if $w_m = x_i^{(j)}$ is the kth $x_i^{(j)}$ in T from the left. One may easily check that Λ respects the Knuth equivalence relations (2.1) and is therefore indeed a function on LR tableaux. The ith row of T gets mapped to the ith column in $\Lambda(T)$ and hence the shape of $\Lambda(T)$ is indeed $\lambda^\top$.

For example

$$T = \begin{array}{|c|c|c|}\hline 2^{(2)} & 3^{(1)} & \multicolumn{1}{c}{}\\\hline 1^{(2)} & 2^{(2)} & \multicolumn{1}{c}{}\\\hline 1^{(1)} & 2^{(1)} & 2^{(1)}\\\hline\end{array} \overset{\Lambda}{\mapsto} \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{c}{} & 2^{(2)} & \multicolumn{2}{c}{}\\\hline 2^{(1)} & 2^{(2)} & 3^{(1)} & \multicolumn{1}{c}{}\\\hline 1^{(1)} & 1^{(1)} & 2^{(1)} & \multicolumn{1}{c}{}\\\hline\end{array}.$$

Lemma 6.2. *For $\mu \in \mathcal{R}^L$, $\mathcal{T}(\mu) = \Lambda\mathcal{T}'(\mu^\star)$ or, equivalently, $\mathcal{T}'(\mu) = \Lambda\mathcal{T}(\mu^\star)$.*

Proof. Observe that $\mathcal{D}' = \Lambda \circ \mathcal{D} \circ \Lambda$, $\mathcal{U}' = \Lambda \circ \mathcal{U} \circ \Lambda$ and $\mathcal{Z}_\lambda = \Lambda \circ \mathcal{Z}^{-1}_{\lambda^\top} \circ \Lambda$. This implies

$$\overline{\mathcal{Z}}_\lambda = \Lambda \circ \overline{\mathcal{Z}}'_{\lambda^\top} \circ \Lambda. \tag{6.5}$$

Hence, for $T, T' \in \mathrm{LRT}(\cdot, \mu)$ such that $T' = \overline{\mathcal{Z}}_\lambda(T)$ one finds $\Lambda(T') = \overline{\mathcal{Z}}'_{\lambda^\top} \circ \Lambda(T)$ which proves the lemma. □

We now wish to show that both $\mathcal{T}(\mu)$ and $\mathcal{T}'(\mu)$ induce a ranked poset structure on the set of LR tableaux $\mathrm{LRT}(\cdot, \mu)$. To prove this we extend the standardization embedding

$$\theta : \mathcal{T}(\mu) \hookrightarrow \mathcal{T}((1^{|\mu|})), \tag{6.6}$$

of Lascoux and Schützenberger [32, 36] (see also [9, Chapter 2.6]) when μ is a partition to the case when $\mu \in \mathcal{R}^L$. Define the map ϕ on LR tableaux as follows:

$$\text{change the rightmost } x_1^{(j)} \text{ to } x_2^{(j)} \text{ for all } 1 \le j \le a_1.$$

If $\mathrm{height}(\mu_1) = \mathrm{height}(\mu_2)$ and $\mathrm{width}(\mu_1) > \mathrm{width}(\mu_2)$ or $\mu_2 = 0$ then $\phi(T)$ is an LR tableau of the same shape as T and of content $\mu' = (\mu_1 - (1^{a_1}), \mu_2 + (1^{a_1}), \mu_3, \ldots, \mu_L)$. Denote by ϕ' the map ϕ restricted to the case when $\mu_2 = 0$. One can show that $\overline{\mathcal{Z}}_\lambda \circ \phi'(T) = 0$ if and only if $\overline{\mathcal{Z}}_\lambda(T) = 0$, and furthermore $[\phi', \overline{\mathcal{Z}}_\lambda] = 0$. (These statements can, for example, be proven by going over to paths using the map ω and noting that $\omega^{-1} \circ \phi' \circ \omega$ only acts on steps one and two –which is empty– and $S \circ \omega^{-1} \circ \overline{\mathcal{Z}}_\lambda \circ \omega$ only acts on the step containing the biggest entry in analogy to Lemma 5.4. For the first statement it is sufficient to consider a path of length three for which it can be explicitly varified. Assuming $\overline{\mathcal{Z}}_\lambda(T) \neq 0$ the second statement then follows trivially since the two operators act on different steps in the path).

Denote by $\mathcal{G}$ the group spanned by $\omega \circ \sigma_i \circ \omega^{-1}$, where σ_i is the isomorphism of Definition 3.2 and ω and ω^{-1} are defined in (3.16) and (5.4), respectively. Then, in analogy to (6.6), there exists an embedding

$$\theta : \mathcal{T}(\mu) \hookrightarrow \mathcal{T}(\nu) \qquad \nu^\star \text{ a partition.} \tag{6.7}$$

for $\mu \in \mathcal{R}^L$ by combining ϕ' with the action of $\mathcal{G}$. Since both $\tau \in \mathcal{G}$ and ϕ' are compatible with the cyclages (the proof of the first statement is analogous to the proof of (5.8)), we find that

$$[\theta, \overline{\mathcal{Z}}_\lambda] = 0. \tag{6.8}$$

Example 6.1. If T is the LR tableau of Equation (3.17) then under θ its content

$$\mu = (\ydiagram{2,1}, \ydiagram{2}, \ydiagram{2,2}) \quad \text{will be changed to} \quad \nu = (\ydiagram{2,1}, \ydiagram{2,1}, \ydiagram{2,1}, \ydiagram{1}, \ydiagram{1}).$$

The standardization $\theta(T)$ can be determined from

$$T \xrightarrow{\tau_1} \begin{array}{|c|c|c|c|}\hline 3^{(3)} & 4^{(2)} & \multicolumn{2}{c}{}\\\hline 3^{(2)} & 4^{(1)} & 4^{(2)} & \multicolumn{1}{c}{}\\\hline 1^{(1)} & 1^{(1)} & 3^{(1)} & 4^{(1)}\\\hline\end{array} \xhookrightarrow{\phi} \begin{array}{|c|c|c|c|}\hline 3^{(3)} & 4^{(2)} & \multicolumn{2}{c}{}\\\hline 3^{(2)} & 4^{(1)} & 4^{(2)} & \multicolumn{1}{c}{}\\\hline 1^{(1)} & 2^{(1)} & 3^{(1)} & 4^{(1)}\\\hline\end{array} \xrightarrow{\tau_2} \begin{array}{|c|c|c|c|}\hline 4^{(1)} & 5^{(3)} & \multicolumn{2}{c}{}\\\hline 1^{(2)} & 1^{(2)} & 5^{(2)} & \multicolumn{1}{c}{}\\\hline 1^{(1)} & 1^{(1)} & 3^{(1)} & 5^{(1)}\\\hline\end{array} \xhookrightarrow{\phi} \begin{array}{|c|c|c|c|}\hline 4^{(1)} & 5^{(3)} & \multicolumn{2}{c}{}\\\hline 1^{(2)} & 2^{(2)} & 5^{(2)} & \multicolumn{1}{c}{}\\\hline 1^{(1)} & 2^{(1)} & 3^{(1)} & 5^{(1)}\\\hline\end{array} \xrightarrow{\tau_3} \begin{array}{|c|c|c|c|}\hline 1^{(3)} & 5^{(1)} & \multicolumn{2}{c}{}\\\hline 1^{(2)} & 2^{(2)} & 3^{(2)} & \multicolumn{1}{c}{}\\\hline 1^{(1)} & 2^{(1)} & 3^{(1)} & 4^{(1)}\\\hline\end{array} = \theta(T),$$

where $\tau_1 = \omega\circ\sigma_2\circ\sigma_3\circ\sigma_1\circ\omega^{-1}$, $\tau_2 = \omega\circ\sigma_2\circ\sigma_3\circ\sigma_4\circ\sigma_1\circ\sigma_2\circ\sigma_3\circ\omega^{-1}$ and $\tau_3 = \omega\circ\sigma_2\circ\sigma_3\circ\sigma_1\circ\sigma_2\circ\omega^{-1}$.

Theorem 6.3. *Let $\mu \in \mathcal{R}^L$. Then the cyclage-graph $\mathcal{T}(\mu)$ imposes a ranked poset structure on* $\mathrm{LRT}(\cdot,\mu)$ *with minimal element $T_{\min} = [x_1^{\mu_1}\cdots x_L^{\mu_L}]$. Similarly, $\mathcal{T}'(\mu)$ imposes a ranked poset structure on* $\mathrm{LRT}(\cdot,\mu)$ *with minimal element $T_{\max} = [x_L^{\mu_L}\cdots x_1^{\mu_1}]$.*

Proof. Let us first consider μ to be a partition and show that in this case $\mathcal{T}'(\mu)$ is a ranked poset. For every $T \in \mathrm{Tab}(\cdot,\mu)$ with $T \neq T_{\max}$ there exists at least one partition λ such that $\overline{\mathcal{Z}}'_\lambda(T) \neq 0$ and one can show that

$$c(\overline{\mathcal{Z}}'_\lambda(T)) = c(T) - 1. \tag{6.9}$$

Namely, if $\mathcal{D}'(T) = T$, i.e., $\overline{\mathcal{Z}}'_\lambda(T) = \mathcal{Z}_\lambda^{-1}(T)$ then $c(\mathcal{Z}_\lambda^{-1}(T)) = \|\mu\| - \mathrm{co}(\mathcal{Z}_\lambda^{-1}(T)) = \|\mu\| - \mathrm{co}(T) - 1 = c(T) - 1$ by Equation (6.2) and the fact that the cocharge is the rank of $\mathcal{T}(\mu)$ for a partition μ as shown by Lascoux and Schützenberger [36]. From the explicit prescription for calculating the charge of a Young tableau $T \in \mathrm{Tab}(\cdot,\mu)$ via indices (see for example [37, page 242] [9, page 111]) one may easily check that $c(T) = c(\mathcal{D}'(T))$ which proves (6.9). This shows that for a partition μ, $\mathcal{T}'(\mu)$ is a poset ranked by the charge with minimal element $T_{\max}$.

From Lemma 6.2 and Equations (6.7) and (6.8) we deduce that also $\mathcal{T}(\mu)$ with $\mu \in \mathcal{R}^L$ is a ranked poset. Since for $T_{\max} = \left[x_L^{\mu_L^\top}\cdots x_1^{\mu_1^\top}\right]$ with $\mu \in \mathcal{R}^L$

$$\Lambda(T_{\max}) = T_{\min} = \left[x_1^{\mu_1}\cdots x_L^{\mu_L}\right], \tag{6.10}$$

the minimal element of $\mathcal{T}(\mu)$ is $T_{\min}$. According to Lemma 6.2 also $\mathcal{T}'(\mu)$ is a ranked poset for all $\mu \in \mathcal{R}^L$ with minimal element equal to $T_{\max}$. □

The standardization embedding (6.7) can be refined by combining ϕ with the action of $\mathcal{G}$ to obtain

$$\psi_{\nu\mu} : \mathcal{T}(\nu) \hookrightarrow \mathcal{T}(\mu), \qquad \nu \geq \mu \tag{6.11}$$

for $\mu,\nu \in \mathcal{R}^L$ with the ordering $\nu \geq \mu$ as defined in Section 2. Similar to (6.8)

$$[\psi_{\nu\mu}, \overline{\mathcal{Z}}_\lambda] = 0. \tag{6.12}$$

Certainly, $[\psi_{\nu\mu}, \overline{\mathcal{C}}] = 0$ thanks to (5.8) and $[\phi, \overline{\mathcal{C}}] = 0$ which can be varified explicitly. To establish (6.12) for general $\overline{\mathcal{Z}}_\lambda$ we are left to show $[\phi, \overline{\mathcal{Z}}_\lambda] = 0$. Let us briefly sketch the proof here. Firstly, $\psi_{\nu\mu}$ only depends on ν and μ, but not on its explicit composition in terms of ϕ and σ_i's. This can be shown by induction on the cocharge using $[\psi_{\nu\mu}, \overline{\mathcal{C}}] = 0$. Secondly, $\overline{\mathcal{Z}}_\lambda(T) = 0$ if and only if $\overline{\mathcal{Z}}_\lambda \circ \phi(T) = 0$. This can be seen as follows. For every LR tableaux there exists a standardization composed only of ϕ' and σ_i's. Denote by θ_1 and θ_2 such standardizations for T and $\phi(T)$, respectively. Since the standardization is independent of the composition of ϕ and σ_i's we conclude $\theta_1(T) = \theta_2 \circ \phi(T)$. Thanks to (6.8) this means that

$$\theta_1 \circ \overline{\mathcal{Z}}_\lambda(T) = \overline{\mathcal{Z}}_\lambda \circ \theta_1(T) = \overline{\mathcal{Z}}_\lambda \circ \theta_2 \circ \phi(T) = \theta_2 \circ \overline{\mathcal{Z}}_\lambda \circ \phi(T)$$

which proves the assertion. When $\overline{\mathcal{Z}}_\lambda(T) \neq 0$ the commutation relation $[\overline{\mathcal{Z}}_\lambda, \phi] = 0$ can again be explicitly shown on paths using the maps ω and ω^{-1}.

6.2. **Properties of charge and cocharge.** In this section we establish some properties of the charge and cocharge for LR tableaux. The cocharge of an LR tableau T is its rank in the poset induced by the modified initial cyclage $\overline{\mathcal{C}}$. Since the initial cyclage is a special λ-cyclage the cocharge is also the rank of the poset $\mathcal{T}(\mu)$. In Definition 4.3 the charge was defined as $c(T) = \|\mu\| - \mathrm{co}(T)$ where $\|\mu\| = \sum_{i<j} |\mu_i \cap \mu_j|$. We show now that $\|\mu\|$ is in fact the cocharge of $T_{\max} = \left[x_L^{\mu_L} \cdots x_1^{\mu_1}\right]$ and that the charge of an LR tableau is equal to its rank in the poset $\mathcal{T}'(\mu)$.

Lemma 6.4. *For $L \in \mathbb{Z}_{\geq 0}$ and $\mu \in \mathcal{R}^L$, $\|\mu\| = \mathrm{co}(T_{\max})$.*

Proof. Set $P_{\max} := \omega^{-1}(w_{T\max})$ with ω^{-1} as defined in (5.4). Using Theorem 5.1, $\|\mu\| = \mathrm{co}(T_{\max})$ is equivalent to $\|\mu\| = H'(P_{\max})$ with H' as in (5.13). Defining $L_i^{(a)}$ as the number of components of μ equal to (i^a) and setting $X_{ij}^{ab} := |(i^a) \cap (j^b)|$, we may rewrite $\|\mu\|$ as

$$\|\mu\| = \frac{1}{2} \sum_{\substack{1\leq i,j\leq N \\ 1\leq a,b\leq n}} L_i^{(a)}(L_j^{(b)} - \delta_{ij}\delta_{ab})X_{ij}^{ab}, \tag{6.13}$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

Let $P \in \mathcal{O}_{P_{\max}}$. Then $P \in \mathcal{P}_{\lambda\tilde{\mu}}$ where $\tilde{\mu}$ is a permutation of μ. One may easily show that $h(p_{k+1} \otimes p_k) = |\tilde{\mu}_k \cap \tilde{\mu}_{k+1}|$ for all $P \in \mathcal{O}_{P_{\max}}$. Hence

$$H'(P_{\max}) = \frac{1}{|\mathcal{O}_{P_{\max}}|} \sum_{\tilde{\mu}} \sum_{k=1}^{L-1} (L-k)|\tilde{\mu}_k \cap \tilde{\mu}_{k+1}|,$$

where the first sum is over all permutations of μ. Notice that $|\tilde{\mu}_k \cap \tilde{\mu}_{k+1}| = X_{ij}^{ab}$ if $\tilde{\mu}_k = (i^a)$ and $\tilde{\mu}_{k+1} = (j^b)$. We now wish to determine the coefficient of X_{ij}^{ab} in $H'(P_{\max})$. Since

$$|\mathcal{O}_{P_{\max}}| = \text{number of permutations of } \mu = \frac{L!}{\prod_{c,k\geq 1} L_k^{(c)}!},$$

the contribution to X_{ij}^{ab} in $H'(P_{\max})$ is

$$\frac{1}{|\mathcal{O}_{P_{\max}}|} \cdot \frac{(L-2)!}{\prod_{c,k\geq 1}(L_k^{(c)} - \delta_{ac}\delta_{ik} - \delta_{bc}\delta_{jk})!} \sum_{k=1}^{L-1}(L-k) = \frac{1}{2}L_i^{(a)}(L_j^{(b)} - \delta_{ij}\delta_{ab}).$$

Summing over all i, j, a, b yields (6.13) which completes the proof. □

The charge and cocharge are dual in the following sense.

Lemma 6.5. *For $T \in \mathrm{LRT}(\cdot, \mu)$, $c(T) = \mathrm{co}(\Lambda(T))$.*

Proof. Since $c(T) = \|\mu\| - \mathrm{co}(T)$, the lemma is equivalent to

$$\mathrm{co}(\Lambda(T)) + \mathrm{co}(T) = \|\mu\|. \tag{6.14}$$

For $T_{\min} = [x_1^{\mu_1} \cdots x_L^{\mu_L}]$ we have $\Lambda(T_{\min}) = T_{\max} = \left[x_L^{\mu_L^\top} \cdots x_1^{\mu_1^\top}\right]$. Since $\mathrm{co}(T_{\min}) = 0$ and $\mathrm{co}(T_{\max}) = \|\mu^\star\| = \|\mu\|$ by Lemma 6.4, Equation (6.14) holds for $T = T_{\min}$ and $T = T_{\max}$.

Now assume that (6.14) holds for some $T \in \mathrm{LRT}(\cdot, \mu)$ so that $\mathcal{D}(T) = T$. Then (6.14) also holds for $\overline{\mathcal{Z}}_\lambda(T) = \mathcal{Z}_\lambda(T)$ if we can show that $\mathrm{co}(\Lambda(T)) = \mathrm{co}(\Lambda \circ \mathcal{Z}_\lambda(T)) - 1$ because $\mathrm{co}(T) = \mathrm{co}(\mathcal{Z}_\lambda(T)) + 1$. Since $\Lambda \circ \mathcal{Z}_\lambda = \mathcal{Z}_{\lambda^\top}^{-1} \circ \Lambda$ and $\mathcal{D} \circ \mathcal{Z}_{\lambda^\top}^{-1} = \mathcal{Z}_{\lambda^\top}^{-1}$ this is fulfilled.

If on the other hand $\mathcal{D}'(T) = T$ or equivalently $\mathcal{D} \circ \Lambda(T) = \Lambda(T)$ then (6.14) also holds for $\overline{\mathcal{Z}}'_\lambda(T) = \mathcal{Z}_\lambda^{-1}(T)$ because $\mathrm{co}(T) = \mathrm{co}(\mathcal{Z}_\lambda^{-1}(T)) - 1$ and $\mathrm{co}(\Lambda(T)) = \mathrm{co}(\Lambda \circ \mathcal{Z}_\lambda^{-1}(T)) + 1$ thanks to $\Lambda \circ \mathcal{Z}_\lambda^{-1} = \mathcal{Z}_{\lambda^\top} \circ \Lambda$.

Since $\mathcal{D}(T) = T$ if $\mathcal{D}'(T) \neq T$ and vice versa, unless T is equal to both $T_{\min}$ and $T_{\max}$, this proves (6.14) for all $T \in \mathrm{LRT}(\cdot, \mu)$. □

As argued before the cocharge is the rank of the poset $\mathcal{T}(\mu)$ since the initial cyclage is a special λ-cyclage. Lemmas 6.5 and 6.2 show that the charge is the rank of the poset $\mathcal{T}'(\mu)$. This is summarized in the following corollary.

Corollary 6.6. *For $\mu \in \mathcal{R}^L$, the cocharge is the rank of the poset $\mathcal{T}(\mu)$ and the charge is the rank of the poset $\mathcal{T}'(\mu)$. In addition*

$$0 \leq \mathrm{co}(T) \leq \|\mu\| \quad \textit{and} \quad 0 \leq c(T) \leq \|\mu\|,$$

with $\mathrm{co}(T_{\min}) = c(T_{\max}) = 0$ *and* $\mathrm{co}(T_{\max}) = c(T_{\min}) = \|\mu\|$.

7. Properties of the supernomials and generalized Kostka polynomials

Several interesting properties of the supernomials (3.10) and generalized Kostka polynomials (3.20) are stated. In Section 7.1 a duality formula for the generalized Kostka polynomials as well as relations between the supernomials and the generalized (cocharge) Kostka polynomial are given. Recurrences for the A_{n-1} supernomials and the generalized Kostka polynomials are established in Section 7.2. These will be used in Section 8 to obtain a representation of the generalized Kostka polynomials of the Kirillov–Reshetikhin-type (2.5). In Section 7.3 we treat the A_1 supernomials in more detail and sketch an elementary proof of the Rogers–Ramanujan-type identities of [47].

7.1. General properties.

The results of the previous section imply the following duality formula for the generalized Kostka polynomials.

Theorem 7.1. *For λ a partition and $\mu \in \mathcal{R}^L$,*

$$K_{\lambda\mu}(q) = q^{\|\mu\|} K_{\lambda^\top \mu^\star}(1/q). \tag{7.1}$$

Proof. This follows from the charge representation of the generalized Kostka polynomials of Corollary 5.2, Lemma 6.5 and $c(T) = \|\mu\| - \mathrm{co}(T)$. □

The supernomial $S_{\lambda\mu}(q)$ and the generalized cocharge Kostka polynomial $\tilde{K}_{\lambda\mu}(q)$ satisfy linear relations as follows.

Theorem 7.2. *For $\lambda \in \mathbb{Z}^n_{\geq 0}$ and $\mu \in \mathcal{R}^L$,*

$$S_{\lambda\mu}(q) = \sum_{\eta \vdash |\lambda|} K_{\eta\lambda} \tilde{K}_{\eta\mu}(q), \tag{7.2}$$

where $K_{\eta\lambda} = K_{\eta\lambda}(1)$ is the Kostka number.

Proof. By definition the supernomial $S_{\lambda\mu}(q)$ is the generating function over all paths $P \in \mathcal{P}_{\lambda\mu}$ weighted by $H(P)$ and by (5.3) $\tilde{K}_{\eta\mu}(q)$ is the generating function over all LR tableaux $T \in \mathrm{LRT}(\eta, \mu)$ with cocharge statistic. Hence, since $[\omega(P)] \in \mathrm{LRT}(\cdot, \mu)$ and $H(P) = \mathrm{co}(\Omega \circ \omega(P))$

by Theorem 5.1 for $P \in \mathcal{P}_{\lambda\mu}$, Equation (7.2) is proven if we can show that for all partitions η of $|\lambda|$ and $T \in \mathrm{LRT}(\eta,\mu)$ there are $K_{\eta\lambda}$ paths such that $[\omega(P)] = T$.

To this end let us show that for all partitions η of $|\lambda|$ with $\eta \geq \lambda$ a pair (T,t) with $T \in \mathrm{LRT}(\eta,\mu)$ and $t \in \mathrm{Tab}(\eta,\lambda)$ uniquely specifies a path $P = p_L \otimes \cdots \otimes p_1 \in \mathcal{P}_{\lambda\mu}$ by requiring that $p_L \cdot \ldots \cdot p_1 = t$ and $[\omega(P)] = T$. Firstly, by point (i) of Lemma 3.1 indeed $\mathrm{shape}(p_L \cdot \ldots \cdot p_1) = \mathrm{shape}([\omega(P)])$. Let us now construct $P \in \mathcal{P}_{\lambda\mu}$ from a given pair (T,t). Set $a_i = \mathrm{height}(\mu_i)$ and define $p_i^{(k)}$ and $t_i^{(k)}$ $(1 \leq i \leq L; 1 \leq k \leq a_i)$ recursively as follows. Set $t_{L+1}^{(1)} = t$ and decompose for $1 \leq i \leq L$

$$t_{i+1}^{(1)} = p_i^{(a_i)} \cdot t_i^{(a_i)} \quad \text{and} \quad t_i^{(k+1)} = p_i^{(k)} \cdot t_i^{(k)} \quad (1 \leq k < a_i) \tag{7.3}$$

such that $\mathrm{shape}(p_i^{(k)}) = (\mathrm{width}(\mu_i))$ and $\mathrm{shape}(t_i^{(k)}) = \mathrm{shape}(T_i^{(k)})$, where $T_i^{(k)}$ is obtained from T by dropping all letters $x \geq x_i^{(k)}$. The decompositions in (7.3) are unique by the Pieri formula. The desired path is $P = p_L \otimes \cdots \otimes p_1$ where $p_i := p_i^{(a_i)} \cdot \ldots \cdot p_i^{(1)}$ $(1 \leq i \leq L)$ because p_i has shape μ_i since $T \in \mathrm{LRT}(\eta,\mu)$, $p_L \cdot \ldots \cdot p_1 = t$ and $[\omega(P)] = T$ by construction. □

From Equation (7.2) one can infer that the special cases of the supernomials for which μ or $\mu^\star$ is a partition have previously occurred in the literature. In the study of finite abelian subgroups, Butler [7, 8, 9] defines polynomials $\alpha_\mu(S;q)$, where μ is a partition and $S = \{a_1 < \cdots < a_{n-1}\}$ an ordered set of $n-1$ integers such that $a_{n-1} < |\mu|$, and shows that they satisfy

$$h_{a_1}(x)h_{a_2-a_1}(x)\cdots h_{m-a_{n-1}}(x) = \sum_{\mu\vdash m} \alpha_\mu(S;q^{-1})q^{\|\mu\|}P_\mu(x;q). \tag{7.4}$$

Here $h_k(x)$ is the kth homogeneous symmetric function and $P_\mu(x;q)$ is the Hall-Littlewood polynomial. Using $h_{\lambda_1}(x)\cdots h_{\lambda_n}(x) = \sum_\eta K_{\eta\lambda}s_\eta(x)$ and Equations (2.3) and (7.2) immediately yields that $\alpha_\mu(S;q) = S_{\lambda\mu}(q)$ where $\lambda = (a_1, a_2 - a_1, \ldots, |\mu| - a_{n-1})$. When $\mu^\star$ is a partition the supernomial has been studied by Hatayama *et al.* [15].

An immediate consequence of Theorem 7.2 is the inverse of relation (7.2).

Corollary 7.3. *For λ a partition with* $\mathrm{height}(\lambda) \leq n$ *and $\mu \in \mathcal{R}^L$*

$$\tilde{K}_{\lambda\mu}(q) = \sum_{\tau\in S_n} \epsilon(\tau) S_{(\lambda_1+\tau_1-1,\ldots,\lambda_n+\tau_n-n)\mu}(q), \tag{7.5}$$

where S_n is the permutation group on $1,2,\ldots,n$ and $\epsilon(\tau)$ is the sign of τ.

Proof. Substitute (7.2) into the right-hand side of (7.5) and use (see [13, page76])

$$\sum_{\tau\in S_n} \epsilon(\tau) K_{\eta(\lambda_1+\tau_1-1,\ldots,\lambda_n+\tau_n-n)} = \delta_{\eta\lambda}.$$ □

7.2. **Recurrences of the $\mathbf{A}_{n-1}$ supernomials and generalized Kostka polynomials.** We have seen in Equations (3.11) and (3.21) that the supernomials and generalized Kostka polynomials are independent of the ordering of μ. We may therefore label the supernomials and generalized Kostka polynomials by a matrix $\boldsymbol{L}$ with component $L_i^{(a)}$ in row a and column i where

$$L_i^{(a)} = L_i^{(a)}(\mu) := \text{number of components of } \mu \text{ equal to } (i^a). \tag{7.6}$$

If $N := \max\{\mathrm{width}(\mu_k)\}$ then $\boldsymbol{L}$ is an $n \times N$ matrix. We denote the supernomials and generalized Kostka polynomials with the label $\boldsymbol{L}$ by $S(\boldsymbol{L},\lambda)$ and $K(\boldsymbol{L},\lambda)$, respectively, and from now on we identify $S(\boldsymbol{L},\lambda)$ and $S_{\lambda\mu}(q)$ (similarly $K(\boldsymbol{L},\lambda)$ and $K_{\lambda\mu}(q)$) if μ and $\boldsymbol{L}$ are related as in (7.6). Define $\boldsymbol{e}_i^{(a)}$ as the $n \times N$ matrix with the only non-zero element in row a and column i equal to 1 and furthermore set $L = \sum_{i=1}^N \sum_{a=1}^n L_i^{(a)}$,

$$\ell_i^{(a)} = \sum_{j=1}^N \min\{i,j\} L_j^{(a)} \quad \text{and} \quad \overline{\ell}_i^{(a)} = \sum_{b=1}^n \min\{a,b\} L_i^{(b)}. \tag{7.7}$$

With this notation we can state the following recurrence relations.

Theorem 7.4 (Recurrences)**.** *Let $i,a,N,n \in \mathbb{Z}_{\geq 0}$ such that $1 \leq i < N$ and $1 \leq a < n$. Let $\boldsymbol{L}$ be an $n \times N$ matrix with non-negative integer components such that $L_i^{(a)} \geq 2$. Then for $\lambda \in \mathbb{Z}_{\geq 0}^n$*

$$S(\boldsymbol{L},\lambda) = S(\boldsymbol{L} + \boldsymbol{e}_{i-1}^{(a)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_{i+1}^{(a)},\lambda) + q^{\ell_i^{(a)}-i} S(\boldsymbol{L} + \boldsymbol{e}_i^{(a-1)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_i^{(a+1)},\lambda) \tag{7.8}$$

and for λ a partition

$$K(\boldsymbol{L},\lambda) = q^{\overline{\ell}_i^{(a)}-a} K(\boldsymbol{L} + \boldsymbol{e}_{i-1}^{(a)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_{i+1}^{(a)},\lambda) + K(\boldsymbol{L} + \boldsymbol{e}_i^{(a-1)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_i^{(a+1)},\lambda). \tag{7.9}$$

Proof. First we prove (7.8). Take $\mu \in \mathcal{R}^L$ corresponding to $\boldsymbol{L}$ such that $\mu_{L-1} = \mu_L = (i^a)$ (which is possible since $L_i^{(a)} \geq 2$). Define μ' and μ'' by $\mu'_L = ((i+1)^a)$, $\mu'_{L-1} = ((i-1)^a)$, $\mu''_L = (i^{a+1})$, $\mu''_{L-1} = (i^{a-1})$ and $\mu'_j = \mu''_j = \mu_j$ for $1 \leq j \leq L-2$. Recalling Definition 3.5 of the supernomials, it is obvious that $\mathcal{P}_{\lambda\mu'}$ and $\mathcal{P}_{\lambda\mu''}$ are the sets of paths underlying the two terms on the right-hand side of (7.8). Furthermore $\mathcal{P}_{\lambda\mu'}$ and $\mathcal{P}_{\lambda\mu''}$ are disjoint. We now wish to establish a bijection between $\mathcal{P}_{\lambda\mu}$ and $\mathcal{P}_{\lambda\mu'} \cup \mathcal{P}_{\lambda\mu''}$. To this end define $\tau(p_L \otimes p_{L-1}) = \tilde{p}_L \otimes \tilde{p}_{L-1}$ for $p_{L-1}, p_L \in \mathcal{B}_{\mu_L}$ such that

$$\tilde{p}_L \cdot \tilde{p}_{L-1} = p_L \cdot p_{L-1} \tag{7.10}$$

and either (a) $\tilde{p}_{L-1} \in \mathcal{B}_{\mu'_{L-1}}$, $\tilde{p}_L \in \mathcal{B}_{\mu'_L}$ if $\nu \cap ((i+1)^a) = ((i+1)^a)$ or (b) $\tilde{p}_{L-1} \in \mathcal{B}_{\mu''_{L-1}}$, $\tilde{p}_L \in \mathcal{B}_{\mu''_L}$ if $\nu \cap (i^{a+1}) = (i^{a+1})$ where $\nu = \mathrm{shape}(p_L \cdot p_{L-1})$. Indeed these conditions are mutually excluding and determine $\tilde{p}_{L-1}$ and $\tilde{p}_L$ uniquely, i.e., the Littlewood-Richardson coefficient $c^{\nu}_{\mu'_{L-1}\mu'_L} = 1$ if and only if $c^{\nu}_{\mu''_{L-1}\mu''_L} = 0$ and vice versa. Conversely, if $\tilde{p}_{L-1} \in \mathcal{B}_{\mu'_{L-1}}$, $\tilde{p}_L \in \mathcal{B}_{\mu'_L}$ (or $\tilde{p}_{L-1} \in \mathcal{B}_{\mu''_{L-1}}$, $\tilde{p}_L \in \mathcal{B}_{\mu''_L}$) one can find unique $p_L \otimes p_{L-1} = \tau^{-1}(\tilde{p}_L \otimes \tilde{p}_{L-1})$ with $p_{L-1}, p_L \in \mathcal{B}_{\mu_L}$ by requiring (7.10). Hence $\tau : \mathcal{P}_{\lambda\mu} \to \mathcal{P}_{\lambda\mu'} \cup \mathcal{P}_{\lambda\mu''}$ with $\tau(P) := \tau(p_L \otimes p_{L-1}) \otimes p_{L-2} \otimes \cdots \otimes p_1$ for each path $P = p_L \otimes \cdots \otimes p_1 \in \mathcal{P}_{\lambda\mu}$, is the desired bijection. This proves (7.8) at $q = 1$.

To prove (7.8) at arbitrary base q notice that if $\tau(P) \in \mathcal{P}_{\lambda\mu'}$, then the LR tableaux $T = [\Omega \circ \omega(P)]$ and $T' = [\Omega \circ \omega \circ \tau(P)]$ are related as

$$T' = \psi_{\mu\mu'}(T),$$

with $\psi_{\mu\mu'}$ defined in (6.11). Because of (6.12) we have $\mathrm{co}(T) = \mathrm{co}(T')$. Hence Theorem 5.1 implies that also $H(P) = H(\tau(P))$ for all P such that $\tau(P) \in \mathcal{P}_{\lambda\mu'}$. Therefore, the term $S(\boldsymbol{L} + \boldsymbol{e}_{i-1}^{(a)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_{i+1}^{(a)},\lambda)$ in (7.8) comes without a power of q.

Similarly, if $\tau(P) \in \mathcal{P}_{\lambda\mu''}$ then the LR tableaux $T = [\Omega \circ \omega(P)]$ and $T'' = [\Omega \circ \omega \circ \tau(P)]$ are related as $\Lambda(T'') = \psi_{\mu\mu''} \circ \Lambda(T)$ which implies

$$\mathrm{co}(\Lambda(T)) = \mathrm{co}(\Lambda(T'')). \tag{7.11}$$

Therefore, pulling all strings in our register, we derive

$$\begin{aligned} H(P) - H(\tau(P)) &= \mathrm{co}(T) - \mathrm{co}(T'') && \text{by Theorem 5.1} \\ &= \|\mu\| - \mathrm{co}(\Lambda(T)) - \|\mu''\| + \mathrm{co}(\Lambda(T'')) && \text{by Lemma 6.5} \\ &= \|\mu\| - \|\mu''\| && \text{by Equation (7.11)} \\ &= \ell_i^{(a)} - i && \text{recalling } \|\mu\| = \sum_{j<k} |\mu_j \cap \mu_k| \end{aligned}$$

which is the power of q in front of the second term in (7.8). This concludes the proof of (7.8).

To prove (7.9), recall Definition 3.9 of the generalized Kostka polynomials. The generalized cocharge Kostka polynomials (3.19) obey the same recurrences (7.8) as the supernomials. This follows from the fact that $\overline{\mathcal{P}}_{\lambda\mu} \subset \mathcal{P}_{\lambda\mu}$ and that $\tau(P)$ is in $\overline{\mathcal{P}}_{\lambda\mu'}$ or $\overline{\mathcal{P}}_{\lambda\mu''}$ if $P \in \overline{\mathcal{P}}_{\lambda\mu}$ by the same arguments as in the proof of point (ii) of Lemma 3.1. Using (3.20), $\|\mu\| - \|\mu'\| = \overline{\ell}_i^{(a)} - a$ and $\|\mu\| - \|\mu''\| = \ell_i^{(a)} - i$ yields (7.9). □

Rectangular Young tableaux over the alphabet $\{1, 2, \ldots, n\}$ of height n are often identified with the empty tableau. When this identification is made for the steps of the paths in the generating functions defining the supernomials and generalized Kostka polynomials one obtains the following properties.

Lemma 7.5. *Let $n \geq 2$, $N, i \geq 1$ be integers and $\boldsymbol{L}$ an $n \times N$ matrix with non-negative entries. Then for $\lambda \in \mathbb{Z}_{\geq 0}^n$*

$$S(\boldsymbol{L} + \boldsymbol{e}_i^{(n)}, \lambda + (i^n)) = S(\boldsymbol{L}, \lambda), \tag{7.12}$$

and for λ a partition with at most n parts

$$K(\boldsymbol{L} + \boldsymbol{e}_i^{(n)}, \lambda + (i^n)) = q^{\sum_a a\ell_i^{(a)}} K(\boldsymbol{L}, \lambda). \tag{7.13}$$

Proof. Writing $S(\boldsymbol{L} + \boldsymbol{e}_i^{(n)}, \lambda + (i^n))$ as a generating function over paths as in (3.10), each path P in the sum has at least one step of height n. Hence $\mathrm{height}(\omega(P)) = n$ by Lemma 3.1. Denoting by P' the path obtained from P by dropping the step p_k of shape (i^n), we find from Lemma 5.8 and $\Omega(p_k) = p_k$ that $H(P) = H(P')$. This proves (7.12).

The generalized cocharge Kostka polynomials $\tilde{K}$ obey the same relation (7.12) as S. Let μ and μ' be any of the arrays of rectangular partitions corresponding to $\boldsymbol{L} + \boldsymbol{e}_i^{(n)}$ and $\boldsymbol{L}$, respectively, by (7.6). Then using $\|\mu\| - \|\mu'\| = \sum_a a\ell_i^{(a)}$ and recalling (3.20) one finds (7.13). □

7.3. **The A_1 supernomials.** The A_1 supernomials are given by specializing Definition 3.5 to $n = 2$, i.e., $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}_{\geq 0}^2$. By Lemma 7.5 it is sufficient to label all A_1 supernomials by a vector $\boldsymbol{L} \in \mathbb{Z}_{\geq 0}^N$ instead of a two-row matrix. Recall that the supernomials vanish unless

$\lambda_1+\lambda_2=|\mu|=\ell_N$ where $\ell_i=\sum_{j=1}^N \min\{i,j\}L_j$. We therefore set

$$S_1(\boldsymbol{L},a):=S_{\lambda\mu}(q)=\sum_{P\in\mathcal{P}_{\lambda\mu}} q^{H(P)} \tag{7.14}$$

where $\boldsymbol{L}\in\mathbb{Z}^N_{\geq 0}$, $a+\frac{1}{2}\ell_N=0,1,\ldots,\ell_N$ and

$$\begin{aligned}\mu&=(1^{L_1}2^{L_2}\cdots N^{L_N}),\\ \lambda&=(\tfrac{1}{2}\ell_N+a,\tfrac{1}{2}\ell_N-a).\end{aligned} \tag{7.15}$$

For $S_1(\boldsymbol{L},a)$ the recurrences (7.8) read

$$S_1(\boldsymbol{L},a)=S_1(\boldsymbol{L}+\boldsymbol{e}_{i-1}-2\boldsymbol{e}_i+\boldsymbol{e}_{i+1},a)+q^{\ell_i-i}S_1(\boldsymbol{L}-2\boldsymbol{e}_i,a), \tag{7.16}$$

for $1\leq i<N$. The $\boldsymbol{e}_i$ are the canonical basis vectors of $\mathbb{Z}^N$ and $\boldsymbol{e}_0=0$. The above equation is in fact part of a larger family of recursion relations.

Lemma 7.6. *Let A,B,N be integers such that $1\leq A\leq B<N$ and let $\boldsymbol{L}\in\mathbb{Z}^N_{\geq 0}$ such that $L_1=\cdots=L_{B-1}=0$ if $A<B$. Then for $a+\frac{1}{2}\ell_N=0,1,\ldots,\ell_N$*

$$S_1(\boldsymbol{L}+\boldsymbol{e}_A+\boldsymbol{e}_B,a)=S_1(\boldsymbol{L}+\boldsymbol{e}_{A-1}+\boldsymbol{e}_{B+1},a)+q^{\ell_A+A}S_1(\boldsymbol{L}+\boldsymbol{e}_{B-A},a). \tag{7.17}$$

Proof. When $A=B$ Equation (7.17) reduces to (7.16) with $\boldsymbol{L}$ replaced by $\boldsymbol{L}+2\boldsymbol{e}_i$. For $A<B$ Equation (7.17) follows from the recurrences

$$S(\boldsymbol{L}+\boldsymbol{e}^{(1)}_A+\boldsymbol{e}^{(1)}_B,\lambda)=S(\boldsymbol{L}+\boldsymbol{e}^{(1)}_{A-1}+\boldsymbol{e}^{(1)}_{B+1},\lambda)+q^{\ell^{(1)}_A+A}S(\boldsymbol{L}+\boldsymbol{e}^{(1)}_{B-A}+\boldsymbol{e}^{(2)}_A,\lambda) \tag{7.18}$$

where $n=2$ and $\boldsymbol{L}$ is a $2\times N$ matrix such that $L^{(1)}_1=\cdots=L^{(1)}_{B-1}=0$. This can be seen by dropping all entries of the matrix in the second row using Lemma 7.5 and then replacing the matrix $\boldsymbol{L}$ by a vector $\boldsymbol{L}$.

Equation (7.18) can be proven in complete analogy to the proof of Theorem 7.4 as follows. Take μ corresponding to $\boldsymbol{L}+\boldsymbol{e}^{(1)}_A+\boldsymbol{e}^{(1)}_B$, replace the components $\mu_{L-1},\ldots,\mu''_L$ in the proof of Theorem 7.4 by $\mu_{L-1}=(B),\mu_L=(A),\mu'_{L-1}=(B+1),\mu'_L=(A-1),\mu''_{L-1}=(B-A),\mu''_L=(A,A)$ and set $\tau(p_L\otimes p_{L-1})=\tilde{p}_L\otimes\tilde{p}_{L-1}$ where again $\tilde{p}_L\cdot\tilde{p}_{L-1}=p_L\cdot p_{L-1}$ and now (a) $\tilde{p}_L\in\mathcal{B}_{\mu'_L}$, $\tilde{p}_{L-1}\in\mathcal{B}_{\mu'_{L-1}}$ if $\mathrm{shape}(p_L\cdot p_{L-1})\neq(B,A)$ and (b) $\tilde{p}_L\in\mathcal{B}_{\mu''_L}$, $\tilde{p}_{L-1}\in\mathcal{B}_{\mu''_{L-1}}$ if $\mathrm{shape}(p_L\cdot p_{L-1})=(B,A)$. Note that we have used here that $n=2$ which ensures that the shape of the product of two steps has at most height 2. One may explicitly check that $\mathrm{co}(T)=\mathrm{co}(T')$ and $\mathrm{co}(T)=\mathrm{co}(T'')+\ell^{(1)}_A+A$ with T,T',T'' as defined in the proof of Theorem 7.4 which proves (7.18). □

Using $S_{\lambda\mu}(q)=\alpha_\mu(S;q)$ for μ a partition (see the discussion after Theorem 7.2) and the explicit representations for $\alpha_\mu(S;q)$ in [7, 8, 9], one finds that

$$S_1(\boldsymbol{L},a)=\begin{bmatrix}\boldsymbol{L}\\ a\end{bmatrix} \tag{7.19}$$

with $\left[\begin{smallmatrix}\boldsymbol{L}\\ a\end{smallmatrix}\right]$ given in Equation (1.3).

We now recall some identities of [47] involving the A_1 supernomial and show how the recurrences of Lemma 7.6 yield an elementary proof. The identities unify and extend many of the known Bose–Fermi or Rogers–Ramanujan-type identities for one-dimensional configuration

sums of solvable lattice models. Below we only quote the result of [47] corresponding to the Andrews–Baxter–Forrester model and its fusion hierarchy. Set

$$T_1(\boldsymbol{L},a)=q^{\frac14\sum_{i=1}^N L_i\ell_i-\frac{a^2}{N}}\begin{bmatrix}\boldsymbol{L}\\ a\end{bmatrix}_{1/q}. \tag{7.20}$$

Theorem 7.7. *Let a,b,p,N be integers such that $N<p-2$, $1\le a\le p-1$ and $1\le b\le p-N-1$ and let $\boldsymbol{L}\in\mathbb{Z}_{\ge0}^N$. Then*

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\Big\{q^{\frac{j}{N}(p(p-N)j+pb-(p-N)a)}T_1\big(\boldsymbol{L},\tfrac{b-a}{2}+pj\big)-q^{\frac1N(pj+a)((p-N)j+b)}T_1\big(\boldsymbol{L},\tfrac{b+a}{2}+pj\big)\Big\}\\ =q^{\frac{1}{4N}(b-a)(a-b-N)}\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_{\ge0}^{p-3}\\ \boldsymbol{m}\equiv\boldsymbol{Q}_{ab}\ (\mathrm{mod}\ 2)}}q^{\frac14\boldsymbol{m}C\boldsymbol{m}-\frac12 m_{a-1}}\prod_{j=1}^{p-3}\begin{bmatrix}m_j+n_j\\ m_j\end{bmatrix},\end{aligned} \tag{7.21}$$

where C is the Cartan matrix of A_{p-3} and $\boldsymbol{Q}_{ab}=\boldsymbol{Q}^{(a-1)}+\boldsymbol{Q}^{(p-b-1)}+\boldsymbol{Q}^{(p-2)}+\sum_{i=2}^N L_i\boldsymbol{Q}^{(i)}$ with $\boldsymbol{Q}^{(i)}=\boldsymbol{e}_{i-1}+\boldsymbol{e}_{i-3}+\cdots$. The expression $\boldsymbol{m}\equiv\boldsymbol{Q}$ (mod2) stands for $m_i\equiv Q_i$ (mod2) and $\boldsymbol{m}C\boldsymbol{m}=\sum_{i,j=1}^{p-3}C_{ij}m_im_j$. The variable $m_0=0$ and $\boldsymbol{n}$ is determined by

$$\boldsymbol{n}=\frac12(\boldsymbol{e}_{a-1}+\boldsymbol{e}_{p-b-1}+\sum_{i=1}^N L_i\boldsymbol{e}_i-C\boldsymbol{m}).$$

For $\boldsymbol{L}=L\boldsymbol{e}_{N'}$, $1\le N'\le N$, in the limit $L\to\infty$ the identity (7.21) yields an identity for branching functions of $A_1^{(1)}$ cosets.

If we can show that the (suitably normalized) $q\to1/q$ forms of both sides of (7.21) satisfy the recurrence

$$X(\boldsymbol{L}+\boldsymbol{e}_A+\boldsymbol{e}_B)=X(\boldsymbol{L}+\boldsymbol{e}_{A-1}+\boldsymbol{e}_{B+1})+q^{\ell_A+A}X(\boldsymbol{L}+\boldsymbol{e}_{B-A}) \tag{7.22}$$

then the identity is proven if it holds for the trivial initial conditions $\boldsymbol{L}=\boldsymbol{e}_i$ $(i=0,1,\dots,N)$. The $q\to1/q$ version of the left-hand side of (7.21) satisfies (7.22) by Lemma 7.6. For the right-hand side of (7.21) with $q\to1/q$ it is readily shown [47] that (7.22) holds for $A=B$ and all $\boldsymbol{L}\in\mathbb{Z}^N$ by using modified q-binomials obtained by extending the range of λ_2 in the top line of (1.1) to $\lambda_2\in\mathbb{Z}$. This implies (7.22) thanks to the following lemma.

Lemma 7.8. *Let X be a function of $\boldsymbol{L}\in\mathbb{Z}^N$ satisfying the recurrences*

$$X(\boldsymbol{L}+2\boldsymbol{e}_A)=X(\boldsymbol{L}+\boldsymbol{e}_{A-1}+\boldsymbol{e}_{A+1})+q^{\ell_A+A}X(\boldsymbol{L}) \tag{7.23}$$

for all $A=1,2,\dots,N-1$ and $\boldsymbol{L}\in\mathbb{Z}^N$. Then X fulfills the more general recurrences

$$X(\boldsymbol{L}+\boldsymbol{e}_A+\boldsymbol{e}_B)=X(\boldsymbol{L}+\boldsymbol{e}_{A-1}+\boldsymbol{e}_{B+1})+q^{\ell_A+A}X(\boldsymbol{L}+\boldsymbol{e}_{B-A}) \tag{7.24}$$

for all $1\le A\le B<N$ and $\boldsymbol{L}\in\mathbb{Z}^N$ such that $L_1=\cdots=L_{B-1}=0$ if $A<B$.

Proof. Assume that (7.24) is proven for all $1\le A'\le B'<B$. This is certainly true for $A'=B'=1$ thanks to (7.23). Using (7.23) successively (with A replaced by i) for $i=B,B-1,\dots,A$

yields

$$X(\boldsymbol{L}+\boldsymbol{e}_A+\boldsymbol{e}_B)=X(\boldsymbol{L}+\boldsymbol{e}_{A-1}+\boldsymbol{e}_{B+1})+\sum_{i=A}^{B}q^{\ell_i+A}X(\boldsymbol{L}+\boldsymbol{e}_A-\boldsymbol{e}_i-\boldsymbol{e}_{i+1}+\boldsymbol{e}_{B+1}). \tag{7.25}$$

Applying (7.23) with A replaced by B to the second term on the right-hand side in (7.25) one obtains

$$\begin{aligned}\sum_{i=A}^{B-1}q^{\ell_i+A}\big\{X(\boldsymbol{L}+\boldsymbol{e}_A-\boldsymbol{e}_i-\boldsymbol{e}_{i+1}-\boldsymbol{e}_{B-1}+2\boldsymbol{e}_B)-q^{\ell_B-2i+A}X(\boldsymbol{L}+\boldsymbol{e}_A-\boldsymbol{e}_i-\boldsymbol{e}_{i+1}-\boldsymbol{e}_{B-1})\big\}\\ +q^{\ell_B+A}X(\boldsymbol{L}+\boldsymbol{e}_A-\boldsymbol{e}_B).\end{aligned} \tag{7.26}$$

Telescoping the last term with the negative terms in the sum at $i=B-1,B-2,\dots,A$ using (7.23) with $A\to i-1$ yields $q^{\ell_B+A}X(\boldsymbol{L}+\boldsymbol{e}_{A-1}-\boldsymbol{e}_{B-1})$. The positive terms in the sum in (7.26) can be simplified to $q^{\ell_A+A}X(\boldsymbol{L}+\boldsymbol{e}_{B-A-1}-\boldsymbol{e}_{B-1}+\boldsymbol{e}_B)$ by combining successively the term $i=A$ with $i=A+1,\dots,B-1$ using (7.24) with $A\to i-A$ and $B\to i$. Therefore (7.25) becomes

$$\begin{aligned}X(\boldsymbol{L}+\boldsymbol{e}_A+\boldsymbol{e}_B)=X(\boldsymbol{L}+\boldsymbol{e}_{A-1}+\boldsymbol{e}_{B+1})\\ +q^{\ell_A+A}X(\boldsymbol{L}+\boldsymbol{e}_{B-A-1}-\boldsymbol{e}_{B-1}+\boldsymbol{e}_B)+q^{\ell_B+A}X(\boldsymbol{L}+\boldsymbol{e}_{A-1}-\boldsymbol{e}_{B-1}).\end{aligned}$$

The last two terms can be combined to $q^{\ell_A+A}X(\boldsymbol{L}+\boldsymbol{e}_{B-A})$ employing (7.24) with $A\to B-A$ and $B\to B-1$ and using that $L_1=\cdots=L_{B-1}=0$. This yields (7.24). □

Let us make some comments about the above outlined proof of Theorem 7.7. First, Lemma 7.6 requires $N>1$. However, thanks to $T_1\big((L_1,\dots,L_N,0,\dots,0),a\big)=q^{\frac{M-N}{MN}a^2}T_1\big((L_1,\dots,L_N),a\big)$, where the dimension of the vector on the left-hand side is M, one can derive the identities (7.21) for all $N\geq 1$ except when $p=4$. Second, we note that for $\boldsymbol{L}\in\mathbb{Z}_{\geq 0}^N$ the polynomials on the right-hand side of (7.21) indeed remain unchanged by replacing the q-binomial with the modified q-binomial. Finally, the proof given in [47] used the identities at $\boldsymbol{L}=L\boldsymbol{e}_1$ as initial conditions. The knowledge of these non-trivial identities is not necessary in the above proof.

In the discussion section we will conjecture higher-rank analogues of (7.21).

8. Fermionic representation of the generalized Kostka polynomials

In this section we give a fermionic representation of the generalized Kostka polynomials generalizing the Kirillov–Reshetikhin expression (2.5). Recalling the Definitions (7.7) we introduce the following function.

Definition 8.1. *Let $n\geq 2$ and $N\geq 1$ be integers, λ a partition with* $\mathrm{height}(\lambda)\leq n$ *and $\boldsymbol{L}$ an $n\times N$ matrix with entry $L_i^{(a)}\in\mathbb{Z}_{\geq 0}$ in row a and column i. Then set $F(\boldsymbol{L},\lambda)=0$ if $|\lambda|\neq\sum_{a,i\geq 1}aiL_i^{(a)}$ and otherwise*

$$F(\boldsymbol{L},\lambda)=\sum_{\alpha}q^{C(\alpha)}\prod_{a,i\geq 1}\begin{bmatrix}P_i^{(a)}+\alpha_i^{(a)}-\alpha_{i+1}^{(a)}\\ \alpha_i^{(a)}-\alpha_{i+1}^{(a)}\end{bmatrix}, \tag{8.1}$$

where the sum is over sequences $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots)$ *of partitions such that* $|\alpha^{(a)}| = \sum_{j\geq 1} j\overline{\ell}_j^{(a)} - (\lambda_1 + \cdots + \lambda_a)$. *Furthermore, with the convention that* $\alpha_i^{(0)} = 0$,

$$P_i^{(a)} = \sum_{k=1}^{i} (\alpha_k^{(a-1)} - 2\alpha_k^{(a)} + \alpha_k^{(a+1)}) + \ell_i^{(a)}, \tag{8.2}$$

$$C(\alpha) = \sum_{a,i\geq 1} \binom{A_i^{(a)} + \alpha_i^{(a-1)} - \alpha_i^{(a)}}{2}, \quad A_i^{(a)} = \sum_{\substack{k\geq i \\ b\geq a}} L_k^{(b)}. \tag{8.3}$$

Recalling that $K_{\lambda\mu}(q) = 0$ unless $|\lambda| = |\mu| = \sum_{a,i\geq 1} aiL_i^{(a)}$ we find that $F(\boldsymbol{L}, \lambda) = K(\boldsymbol{L}, \lambda)$ if $L_i^{(a)} = 0$ for $a > 1$ by comparing (8.1) with (2.5). We wish to show that $F(\boldsymbol{L}, \lambda)$ equals the generalized Kostka polynomial $K(\boldsymbol{L}, \lambda)$ for more general $\boldsymbol{L}$. We begin by showing that F obeys the same recurrence relation as K.

Lemma 8.1. *Let* $i, a, N, n \in \mathbb{Z}_{\geq 0}$ *such that* $1 \leq i < N$ *and* $1 \leq a < n$ *and let* λ *be a partition with* $\mathrm{height}(\lambda) \leq n$. *Let* $\boldsymbol{L}$ *be an* $n \times N$ *matrix with non-negative integer entries such that* $L_i^{(a)} \geq 2$. *Then*

$$F(\boldsymbol{L}, \lambda) = q^{\overline{\ell}_i^{(a)} - a} F(\boldsymbol{L} + \boldsymbol{e}_{i-1}^{(a)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_{i+1}^{(a)}, \lambda) + F(\boldsymbol{L} + \boldsymbol{e}_i^{(a-1)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_i^{(a+1)}, \lambda). \tag{8.4}$$

Proof. Under the substitution $\boldsymbol{L} \to \boldsymbol{L} + \boldsymbol{e}_{i-1}^{(a)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_{i+1}^{(a)}$ the variable $P_j^{(b)}$ and the function $C(\alpha)$ transform as

$$\begin{aligned} P_j^{(b)} &\to P_j^{(b)} - \delta_{ij}\delta_{ab}, \\ C(\alpha) &\to C(\alpha) - \overline{\ell}_i^{(a)} + a + \alpha_i^{(a)} - \alpha_{i+1}^{(a)}. \end{aligned} \tag{8.5}$$

On the other hand, replacing $\boldsymbol{L} \to \boldsymbol{L} + \boldsymbol{e}_i^{(a-1)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_i^{(a+1)}$ induces the changes

$$\begin{aligned} P_j^{(b)} &\to P_j^{(b)} + \min\{i, j\}(\delta_{a-1,b} - 2\delta_{ab} + \delta_{a+1,b}), \\ C(\alpha) &\to C(\alpha) - m_i^{(a)} + i. \end{aligned} \tag{8.6}$$

Now apply the q-binomial recurrence

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix} + \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}, \tag{8.7}$$

to the (a, i)th term in the product in (8.1) (this term cannot be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ because of the condition $L_i^{(a)} \geq 2$). Thanks to (8.5) one can immediately recognize the first term of the resulting expression as $q^{\overline{\ell}_i^{(a)} - a} F(\boldsymbol{L} + \boldsymbol{e}_{i-1}^{(a)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_{i+1}^{(a)}, \lambda)$. In the second term we perform the variable change $\alpha_j^{(b)} \to \alpha_j^{(b)} + \chi(j \leq i)\delta_{ab}$ where recall that $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Since this leads to exactly the same change in $P_j^{(b)}$ and $C(\alpha)$ as in (8.6), the second term indeed yields $F(\boldsymbol{L} + \boldsymbol{e}_i^{(a-1)} - 2\boldsymbol{e}_i^{(a)} + \boldsymbol{e}_i^{(a+1)}, \lambda)$. □

Theorem 8.2. *Let $N \geq 1, n \geq 2$ be integers, λ a partition with* $\mathrm{height}(\lambda) \leq n$ *and $\boldsymbol{L}$ an $n \times N$ matrix with components $L_i^{(a)} \in \mathbb{Z}_{\geq 0}$ in row a and column i. If either $L_i^{(a)} \geq L_i^{(a+2)}$ for all $1 \leq a \leq n-2$ and $1 \leq i \leq N$ or $L_i^{(a)} \geq L_{i+2}^{(a)}$ for all $1 \leq a \leq n$ and $1 \leq i \leq N-2$, then $F(\boldsymbol{L}, \lambda) = K(\boldsymbol{L}, \lambda)$.*

Proof. We use $F(\boldsymbol{L}, \lambda) = K(\boldsymbol{L}, \lambda)$ for $\boldsymbol{L}$ such that $L_i^{(a)} = 0$ when $a > 1$ as initial condition. Since K and F both satisfy the recurrences

$$X(\boldsymbol{L} + \boldsymbol{e}_{i+1}^{(a+1)}) = X(\boldsymbol{L} - \boldsymbol{e}_i^{(a-1)} + 2\boldsymbol{e}_i^{(a)}) - q^{\bar{\ell}_i^{(a)}+1} X(\boldsymbol{L} - \boldsymbol{e}_i^{(a-1)} + \boldsymbol{e}_{i-1}^{(a)} + \boldsymbol{e}_{i+1}^{(a)}) \tag{8.8}$$

(compare with (7.9) and (8.4), respectively) the theorem follows immediately for the first set of restrictions on $\boldsymbol{L}$. The second set of restrictions comes about by using the symmetry (7.1) of the generalized Kostka polynomials. □

The recurrences (8.8) are not sufficient to prove Theorem 8.2 for $\boldsymbol{L}$ with arbitrary entries $L_i^{(a)} \in \mathbb{Z}_{\geq 0}$. However, we nevertheless believe the theorem to be true for this case as well.

Conjecture 8.3. *Let $N \geq 1, n \geq 2$ be integers, λ a partition with* $\mathrm{height}(\lambda) \leq n$ *and $\boldsymbol{L}$ an $n \times N$ matrix with nonnegative integer entries. Then $F(\boldsymbol{L}, \lambda) = K(\boldsymbol{L}, \lambda)$.*

9. Discussion

We believe that there exist many further results for the generalized Kostka polynomials and supernomials. For example, (7.21) admits higher-rank analogues in terms of

$$T(\boldsymbol{L}, \lambda) = q^{\frac{1}{2}\sum_{i=1}^{N}\sum_{a,b=1}^{n-1} L_i^{(a)} C_{ab}^{-1} \ell_i^{(b)} - \frac{1}{2N}\sum_{i=1}^{n}(\lambda_i - \frac{1}{n}|\lambda|)^2} S(\boldsymbol{L}, \lambda)|_{1/q}, \tag{9.1}$$

where C^{-1} is the inverse of the Cartan matrix of A_{n-1}. For integers $n \geq 2, N, p \geq 1$ such that $N < p-n$ and any $n \times N$ matrix $\boldsymbol{L}$ with non-negative integer entries such that $\sum_{b=1}^{n-1} C_{ab}^{-1} L_i^{(b)} \in \mathbb{Z}$ for all i and a, we conjecture

$$\sum_{k_1+\cdots+k_n=0} \sum_{\tau \in S_n} \epsilon(\tau) q^{\sum_{i=1}^{n}\{\frac{1}{2N}(pk_i+\tau_i-i)^2 - \frac{p}{2}k_i^2 + ik_i\}} T(\boldsymbol{L}, \lambda(\boldsymbol{k}, \tau)) = \sum_{\boldsymbol{m}} q^{\frac{1}{2}\boldsymbol{m}(C^{-1}\otimes C)\boldsymbol{m}} \begin{bmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{m} \end{bmatrix}, \tag{9.2}$$

where the following notation has been used. On the left-hand side the components of $\lambda(\boldsymbol{k}, \tau)$ are given by $\lambda_j(\boldsymbol{k}, \tau) = \frac{1}{n}\sum_{a,i\geq 1} aiL_i^{(a)} + pk_j + \tau_j - j$. On the right-hand side the sum runs over all $\boldsymbol{m} = \sum_{a=1}^{n-1}\sum_{i=1}^{p-n-1} m_i^{(a)}(\boldsymbol{e}_a \otimes \boldsymbol{e}_i)$ with $m_i^{(a)} \in \mathbb{Z}_{\geq 0}$ such that $\sum_{b=1}^{n-1} C_{ab}^{-1} m_i^{(b)} \in \mathbb{Z}$ for all $a = 1, \ldots, n-1$ and $i = 1, \ldots, p-n-1$. The variable $\boldsymbol{n}$ is fixed by

$$(C \otimes I)\boldsymbol{n} + (I \otimes C)\boldsymbol{m} = \sum_{a=1}^{n-1}\sum_{i=1}^{N} L_i^{(a)}(\boldsymbol{e}_a \otimes \boldsymbol{e}_i) \tag{9.3}$$

where I is the identity matrix and C is the Cartan matrix of an A-type Lie algebra. The dimension of the first space in the tensor product is $n-1$ and that of the second space is

$p-n-1$. Finally we used the notation

$$(A\otimes B)\boldsymbol{m}=\sum_{a,b=1}^{n-1}\sum_{i,j=1}^{p-n-1}A_{ab}B_{ij}m_j^{(b)}(\boldsymbol{e}_a\otimes\boldsymbol{e}_i),$$

$$\boldsymbol{n}(A\otimes B)\boldsymbol{m}=\sum_{a,b=1}^{n-1}\sum_{i,j=1}^{p-n-1}A_{ab}B_{ij}n_i^{(a)}m_j^{(b)}$$

and

$$\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}=\prod_{a=1}^{n-1}\prod_{i=1}^{p-n-1}\begin{bmatrix}m_i^{(a)}+n_i^{(a)}\\ m_i^{(a)}\end{bmatrix}.$$

The identities (9.2) are polynomial analogues of branching function identities of the Rogers–Ramanujan type for $A_{n-1}^{(1)}$ cosets. For $n=2$ they follow from Theorem 7.7 with $a=b=1$ and for $\boldsymbol{L}=L(\boldsymbol{e}_1\otimes\boldsymbol{e}_1)$ they were claimed in [10] [1]. Unfortunately, the recurrences of Theorem 7.4 are not sufficient to prove (9.2) for general n and $\boldsymbol{L}$. A proof would require a more complete set of recurrences for the A_{n-1} supernomials analogous to those stated in Lemma 7.6 for $n=2$.

The left-hand side of Equation (9.2) can be interpreted in terms of paths of a level-$(p-n)$ $A_{n-1}^{(1)}$ lattice model. Denote by Λ_k $(0\le k\le n-1)$ the dominant integral weights of $A_{n-1}^{(1)}$. Then the states a of the lattice model underlying (9.2) are given by the level-$(p-n)$ dominant integral weights, i.e., $a=\sum_{k=0}^{n-1}a_k\Lambda_k$ such that $\sum_{k=0}^{n-1}a_k=p-n$. Define the adjacency matrices $\mathcal{A}$ labelled by two states a,b and a Young tableau as $\mathcal{A}_{a,b}^{\emptyset}=\chi(a=b)$, $\mathcal{A}_{a,b}^{[i]}=\chi(b=a+\Lambda_i-\Lambda_{i-1})$ $(i=1,\dots,n;\Lambda_n=\Lambda_0)$ and recursively $\sum_b\mathcal{A}_{a,b}^T\mathcal{A}_{b,c}^{[i]}=\mathcal{A}_{a,c}^{T\cdot[i]}$. Call a path $P=p_L\otimes\cdots\otimes p_1\in\mathcal{P}_{\lambda\mu}$ admissible with initial state $a^{(1)}$ if $\prod_{i=1}^{L}\mathcal{A}_{a^{(i)},a^{(i+1)}}^{p_i}=1$ where $a^{(i+1)}=a^{(i)}+\sum_{k=0}^{n-1}\Lambda_k(\lambda_k^{(i)}-\lambda_{k+1}^{(i)})$ for $i=1,\dots,L$ and $\lambda^{(i)}=\mathrm{content}(p_i)$. Then, up to an overall factor, (9.2) is an identity for the generating function of admissible paths $P\in\mathcal{P}_{\lambda\mu}$ starting at $a^{(1)}=(p-n)\Lambda_0$ with μ and $\boldsymbol{L}$ related as in (7.6) and $\lambda=(\frac{|\mu|}{n},\cdots,\frac{|\mu|}{n})$. The weights of the paths are given by $-H(P)$ with H as defined in (3.7).

Our initial motivation for studying the A_{n-1} supernomials is their apparent relevance to a higher-rank generalization of Bailey's Lemma [6]. Indeed, a Bailey-type lemma involving the supernomials $S_{\lambda\mu}(q)$ such that $\mu^\star$ (or any permutation thereof) is a partition can be formulated. Here we briefly sketch some of our findings. Further details about a Bailey lemma and Bailey chain for A_2 supernomials are given in [5], whereas we hope the report more on the general A_{n-1} case in a future publication.

Set

$$f(\boldsymbol{L},\lambda)=S_{\lambda\mu}(q)/(q)_{\boldsymbol{L}}\tag{9.4}$$

for $\boldsymbol{L}\in\mathbb{Z}_{\ge0}^{n-1}$ and $\lambda\in\mathbb{Z}_{\ge0}^{n}$ and zero otherwise where $(q)_{\boldsymbol{L}}=(q)_{L_1}\cdots(q)_{L_{n-1}}$ and $\mu=((1^{n-1})^{L_{n-1}},\dots,(1)^{L_1})$. Here $(1^i)^{L_i}$ denotes L_i components (1^i). Note that $\mu^\star=(1^{L_1}2^{L_2}\cdots(n-1)^{L_{n-1}})$ is indeed a partition.

[1] The proof in [10] seems to be incomplete.

Let $\boldsymbol{L}=(L_1,\dots,L_{n-1})$ and $\boldsymbol{k}=(k_1,\dots,k_n)$ such that $k_1+\cdots+k_n=0$ denote arrays of integers and let $\alpha=\{\alpha_{\boldsymbol{k}}\}_{k_1\geq\cdots\geq k_n}$, $\gamma=\{\gamma_{\boldsymbol{k}}\}_{k_1\geq\cdots\geq k_n}$, $\beta=\{\beta_{\boldsymbol{L}}\}$ and $\delta=\{\delta_{\boldsymbol{L}}\}$ be sequences. Then (α,β) and (γ,δ) such that

$$\beta_{\boldsymbol{L}}=\sum_{\substack{k_1+\cdots+k_n=0\\ k_1\geq\cdots\geq k_n}}\alpha_{\boldsymbol{k}}f(C\boldsymbol{L}+\ell\boldsymbol{e}_1,L_{n-1}\boldsymbol{\rho}-\boldsymbol{k}+\ell\boldsymbol{e}_n), \tag{9.5}$$

and

$$\gamma_{\boldsymbol{k}}=\sum_{\boldsymbol{L}\in\mathbb{Z}^{n-1}}\delta_{\boldsymbol{L}}f(C\boldsymbol{L}+\ell\boldsymbol{e}_1,L_{n-1}\boldsymbol{\rho}-\boldsymbol{k}+\ell\boldsymbol{e}_n) \tag{9.6}$$

are called an A_{n-1} Bailey pair relative to q^ℓ and an A_{n-1} conjugate Bailey pair relative to q^ℓ, respectively. Here $\ell\in\mathbb{Z}_{\geq 0}$, C is the Cartan matrix of A_{n-1} and $\boldsymbol{\rho}$ is the n-dimensional Weyl vector $\boldsymbol{\rho}=\boldsymbol{e}_1+\cdots+\boldsymbol{e}_n$.

When $n=2$, $f(L,\lambda)=1/(q)_{\lambda_1}(q)_{\lambda_2}$ for $L=\lambda_1+\lambda_2$ and zero otherwise, and (9.5) and (9.6) reduce (up to factors of $(q)_\ell$) to the usual definition [6] of a Bailey pair and conjugate Bailey pair (after identifying $\boldsymbol{k}=(k_1,k_2)=(k,-k)$),

$$\beta_L=\sum_{k=0}^{L}\frac{\alpha_k}{(q)_{L-k}(q)_{L+k+\ell}}\quad\text{and}\quad\gamma_k=\sum_{L=k}^{\infty}\frac{\delta_L}{(q)_{L-k}(q)_{L+k+\ell}}.$$

Analogous to the A_1 case the A_{n-1} Bailey pair and conjugate Bailey pair satisfy

$$\sum_{\substack{k_1+\cdots+k_n=0\\ k_1\geq\cdots\geq k_n}}\alpha_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}=\sum_{\boldsymbol{L}\in\mathbb{Z}^{n-1}}\beta_{\boldsymbol{L}}\delta_{\boldsymbol{L}}. \tag{9.7}$$

For $n\geq 2$, $N\geq 1$ we now claim the following A_{n-1} conjugate Bailey pair relative to q^ℓ. Choose integers $\lambda_j^{(a)}\geq 0$ $(a=1,\dots,n-1,\ j=1,\dots N-1)$ and σ such that

$$\ell-\sum_{a=1}^{n-1}\sum_{i=1}^{N-1}ai\lambda_i^{(a)}+\sigma N\equiv 0\pmod n. \tag{9.8}$$

Setting $\lambda=\sum_{a=1}^{n-1}\sum_{i=1}^{N-1}\lambda_i^{(a)}(\boldsymbol{e}_a\otimes\boldsymbol{e}_i)$ and $\boldsymbol{k}=\boldsymbol{k}(\boldsymbol{L})$, such that $k_i(\boldsymbol{L})=L_i-L_{i+1}$ ($L_n=0$, $L_{n+1}=L_1$, so that $\sum_{i=1}^{n}k_i=0$) the (γ,δ) pair

$$\begin{aligned}\gamma_{\boldsymbol{k}(\boldsymbol{L})}&=\frac{q^{\frac{1}{2N}(\boldsymbol{L}C\boldsymbol{L}+2\ell L_1)}}{(q)_\infty^{n-1}}\sum_{\boldsymbol{n}}\frac{q^{\frac{1}{2}\boldsymbol{n}(C\otimes C^{-1})\boldsymbol{n}-\boldsymbol{n}(I\otimes C^{-1})\lambda}}{(q)_{\boldsymbol{n}}}\\ \delta_{\boldsymbol{L}}&=q^{\frac{1}{2N}(\boldsymbol{L}C\boldsymbol{L}+2\ell L_1)}\sum_{\boldsymbol{n}}q^{\frac{1}{2}\boldsymbol{n}(C\otimes C^{-1})\boldsymbol{n}-\boldsymbol{n}(I\otimes C^{-1})\lambda}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix}\end{aligned} \tag{9.9}$$

satisfies (9.6). The summations in (9.9) run over all $\boldsymbol{n}=\sum_{a=1}^{n-1}\sum_{i=1}^{N-1}n_i^{(a)}(\boldsymbol{e}_a\otimes\boldsymbol{e}_i)$ such that

$$\frac{L_a+\ell C^{-1}_{a1}-\sum_{b=1}^{n-1}\sum_{k=1}^{N-1}kC^{-1}_{ab}\lambda_k^{(b)}}{N}-\sum_{j=1}^{N-1}C^{-1}_{1j}n_j^{(a)}\in\mathbb{Z}+\frac{a\sigma}{n}\qquad a=1,\dots,n-1. \tag{9.10}$$

In the expression for $\delta_{\boldsymbol{L}}$ the variable $\boldsymbol{m}$ is related to $\boldsymbol{n}$ by

$$(C \otimes I)\boldsymbol{n} + (I \otimes C)\boldsymbol{m} = (C\boldsymbol{L} + \ell \boldsymbol{e}_1) \otimes \boldsymbol{e}_{N-1} + \lambda. \tag{9.11}$$

Inserting (9.9) into (9.7) yields a rank $n-1$ and level N version of Bailey's lemma. Indeed, when $\lambda = \boldsymbol{e}_a \otimes \boldsymbol{e}_i$, $\gamma_{\boldsymbol{k}}$ is proportional to the level-N $A^{(1)}_{n-1}$ string function in the representation given by Georgiev [14]. When $n = 2$ the pair (γ, δ) of Equation (9.9) reduces to the conjugate Bailey pair of [45, 46]. The identities in (9.2) provide A_{n-1} Bailey pairs relative to 1. We remark that Milne and Lilly [38, 39] also considered higher-rank generalizations of Bailey's lemma. However, their definition of an A_{n-1} Bailey is different from the one above, and in particular we note that the function f is not q-hypergeometric for $n > 2$.

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Note added

After submission, several papers [25, 29] [49]-[52] with considerable overlap with this work have appeared. The generalized Kostka polynomials studied in this paper were also introduced in [52] as special types of Poincaré polynomials and further studied in [49, 50, 51]. In [25, 29] it was conjectured that the generalized Kostka polyonomials coincide with special cases of spin generating functions of ribbon tableaux [34] and that the fermionic representation (8.1) of the generalized Kostka polynomials is the generating function of rigged configurations. This last conjecture has now been established in [28].

We are indebted to Mark Shimozono for his questions and comments which led to several refinements of the paper. We also thank him for pointing out that the analogues of the recurrences of Lemma 7.6 for the (cocharge) Kostka polynomials have occurred in [42].

Appendix

Appendix A. Proof of Lemma 5.6

Obviously, the following lemma implies Lemma 5.6.

Lemma A.1. *Let $P \in \overline{\mathcal{P}}_\mu$ be a path over $\{1, 2, \ldots, M\}$ where $M = |\mu|$ and set $Q = \mathcal{C}_{\mathrm{p}}(P)$. Suppose M is contained in step p_i of $P = p_L \otimes \cdots \otimes p_1$.*

(i) If M is contained in the $(i-1)$th step of $\sigma_{i-1}(P)$ then $h_{i-1}(Q) - h_{i-1}(P) = 0$.

(ii) If M is contained in the ith step of $\sigma_{i-1}(P)$ then $h_{i-1}(Q) - h_{i-1}(P) = -1$.

(iii) If M is contained in the $(i+1)$th step of $\sigma_i(P)$ then $h_i(Q) - h_i(P) = 0$.

(*iv*) *If M is contained in the ith step of $\sigma_i(P)$ then $h_i(Q) - h_i(P) = 1$.*

This lemma, in turn, follows from the next lemma. The height of an entry M in a Young tableau is defined to be i if M is in the ith row from the bottom.

Lemma A.2. *For $\mu, \mu' \in \mathcal{R}$, let $p \in \mathcal{B}_\mu$ and $p' \in \mathcal{B}_{\mu'}$ such that each entry in $p \cdot p'$ occurs at most once and p contains the largest entry M. Then*

(*i*) $\mathrm{shape}(\mathcal{C}_{\mathrm{p}}(p) \cdot p') \neq \mathrm{shape}(p \cdot p')$ *if and only if the height of M in $p \cdot p'$ is bigger than* $\mathrm{height}(p)$, *and*

(*ii*) $0 \leq h(p \otimes p') - h(\mathcal{C}_{\mathrm{p}}(p) \otimes p') \leq 1$.

Before we prove Lemma A.2 let us first show that it indeed implies Lemma A.1.

Proof of Lemma A.1.

(*i*) Let $\tilde{p}_i \otimes \tilde{p}_{i-1} := \sigma(p_i \otimes p_{i-1})$. The steps p_i and $\tilde{p}_{i-1}$ have the same shape and by assumption they both contain M. Since $p_i \cdot p_{i-1} = \tilde{p}_i \cdot \tilde{p}_{i-1}$ we conclude that the height of M in $p_i \cdot p_{i-1}$ has to be $\mathrm{height}(p_i)$. By (*i*) of Lemma A.2 it follows that $\mathrm{shape}(\mathcal{C}_{\mathrm{p}}(p_i) \cdot p_{i-1}) = \mathrm{shape}(p_i \cdot p_{i-1})$ which proves $h_{i-1}(Q) - h_{i-1}(P) = 0$.

(*ii*) Again we denote $\tilde{p}_i \otimes \tilde{p}_{i-1} := \sigma(p_i \otimes p_{i-1})$. By assumption p_i and $\tilde{p}_i$ contain M. Equation $p_i \cdot p_{i-1} = \tilde{p}_i \cdot \tilde{p}_{i-1}$ can only hold if the box with entry M has been bumped at least once. But this implies that the height of M in $p_i \cdot p_{i-1}$ is bigger than $\mathrm{height}(p_i)$. By (*i*) of Lemma A.2 this means that $h_{i-1}(Q) \neq h_{i-1}(P)$, and by (*ii*) of Lemma A.2 the difference has to be -1.

(*iii*) This point can be proven analogous to (*i*).

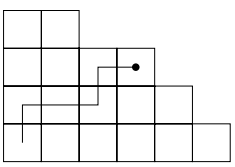
(*iv*) Let us show that this case follows from (*ii*) by considering $P' = \mathcal{C}_{\mathrm{p}}^{-1} \circ \Omega_{\mathrm{p}}(P)$. The path P' satisfies the conditions of case (*ii*) with $i \to L+1-i$ since σ_i commutes with $\mathcal{C}_{\mathrm{p}}^{-1}$ due to (5.8) and (5.9) and since (5.6) holds. Hence $h_{L-i}(\mathcal{C}_{\mathrm{p}}(P')) - h_{L-i}(P') = -1$ which is equivalent to $h_{L-i}(\Omega_{\mathrm{p}}(P)) - h_{L-i}(\Omega_{\mathrm{p}} \circ \mathcal{C}_{\mathrm{p}}(P)) = -1$ by inserting the definition of P' and using (5.9) and $\Omega_{\mathrm{p}}^2 = \mathrm{Id}$. Finally employing (5.5) proves (*iv*). □

Proof of Lemma A.2. Let $p' = [w]$ with $w = w_N \cdots w_1$ in row-representation. Define $p^{(0)} = p$ and $p^{(i+1)} = p^{(i)} \cdot [w_{N-i}]$ for $i = 0, 1, \ldots, N-1$. Then obviously $p^{(N)} = p \cdot p'$.

We will show inductively that either M got bumped in $p^{(i)}$ (which implies that the height of M is bigger than $\mathrm{height}(p)$) or the action of $\mathcal{C}_{\mathrm{p}}$ on $p^{(i)}$ is still described by the inverse sliding mechanism starting at the largest element M and ending in the bottom left corner.

We prove this claim by induction on i. The initial condition is satisfied since $\mathcal{C}_{\mathrm{p}}$ acts on $p^{(0)} = p$ by the inverse sliding mechanism by definition. To prove the induction step suppose that M did not yet get bumped in $p^{(i)}$ (if it has been bumped in $p^{(i)}$ then this is also true for $p^{(k)}$ with $i \leq k \leq N$ and we are finished). By the induction hypothesis the action of $\mathcal{C}_{\mathrm{p}}$ on $p^{(i)}$ is still given by the inverse sliding mechanism. It is useful to denote the boxes in $p^{(i)}$ affected by the inverse sliding pictorially by drawing  and  for $a > b$ and $a < b$, respectively, if the corresponding boxes of $p^{(i)}$ are

. For example

$p^{(i)} =$  .

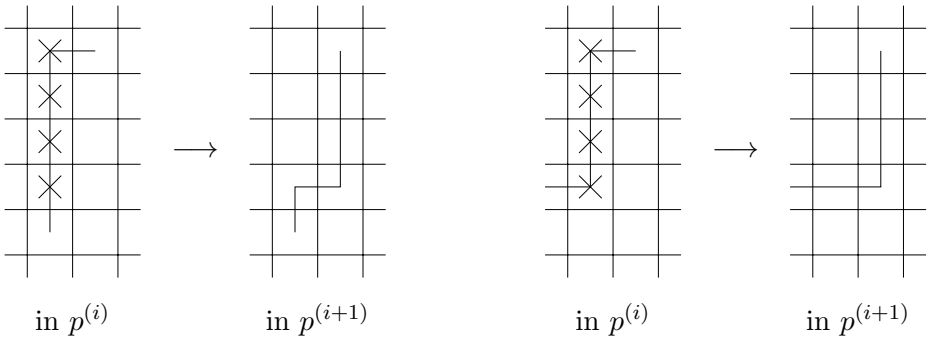


FIGURE 3. Change of inverse sliding route from $p^{(i)}$ to $p^{(i+1)}$

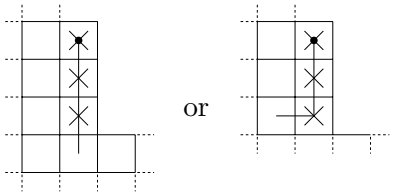
The dot indicates the position of M in $p^{(i)}$. Comparing with Figure 2 we see that the line traces exactly the movement of an empty box under the inverse sliding mechanism. We now wish to insert $[w_{N-i}]$ by the Schensted bumping algorithm to obtain $p^{(i+1)}$, i.e., $[w_{N-i}]$ gets inserted in the first row of $p^{(i)}$ and possibly bumps another box to the second row and so on. Let us label the boxes of $p^{(i)}$ which get bumped when inserting $[w_{N-i}]$ by a cross. Two things may happen:

(1) None of the boxes  and  contain a cross. (We include  in the set of boxes depicted by).

(2) There are boxes  or  which contain a cross.

If (1) occurs there can be at most one box containing both a line and a cross. One may easily see that the line of $p^{(i)}$ also describes the route of the inverse sliding mechanism in $p^{(i+1)}$ and that M does not get bumped. Hence we are finished in this case. If (2) occurs all boxes vertically above or up to and including either (a)  or (b) must also contain a cross. In case (a) the line indicating the inverse sliding changes from $p^{(i)}$ to $p^{(i+1)}$ as illustrated in Figure 3. In case (b) contains a cross and hence M got bumped. This concludes the proof of the claim.

Observe that, as long as cases (1) or $(2a)$ occur, M does not get bumped and $\text{shape}(p^{(i+1)}) = \text{shape}(\mathcal{C}_{\text{p}}(p^{(i+1)})$ since $\mathcal{C}_{\text{p}}$ is still described by the inverse sliding mechanism. If, however, case $(2b)$ occurs for $p^{(i)}$ which implies that M got bumped in $p^{(i+1)}$, then $\text{shape}(p^{(i+1)})$ and $\text{shape}(\mathcal{C}_{\text{p}}(p^{(i+1)}))$ differ. This is so since $p^{(i)}$ must contain



where the dashed lines indicate possible other boxes and the number of the vertically aligned crossed boxes may of course vary (but at least the box containing the dot must also contain a cross). Suppose the lowest crossed box in the vertical line below is in row k. In comparison with $p^{(i)}$, the shape of $p^{(i+1)}$ has one more box above the $\text{height}(p)$. In the shape of $\mathcal{C}_{\text{p}}(p^{(i+1)}) =$

$\mathcal{C}_{\rm p}(p^{(i)}) \cdot [w_{N-i}]$, this box has been moved to row k. One may also easily see that shape$(p^{(j)})$ and shape$(\mathcal{C}_{\rm p}(p^{(j)}))$ for $j > i$ differ by moving exactly one box from above height(p) to the kth row. This proves Lemma A.2. □

Appendix B. Example of a cyclage-graph

Figure 4 shows the poset structure of LRT$(\cdot, \mu)$ for $\mu = ((2), (2), (1^2))$. A black arrow from LR tableau T to LR tableau T' means $T' = \overline{\mathcal{C}}(T)$. A white arrow indicates that T' and T are related by a modified λ-cyclage (as defined in Section 6) other than the modified initial cyclage, i.e. $T' = \overline{\mathcal{Z}}_\lambda(T)$ for some shape λ but $T' \neq \overline{\mathcal{C}}(T)$.

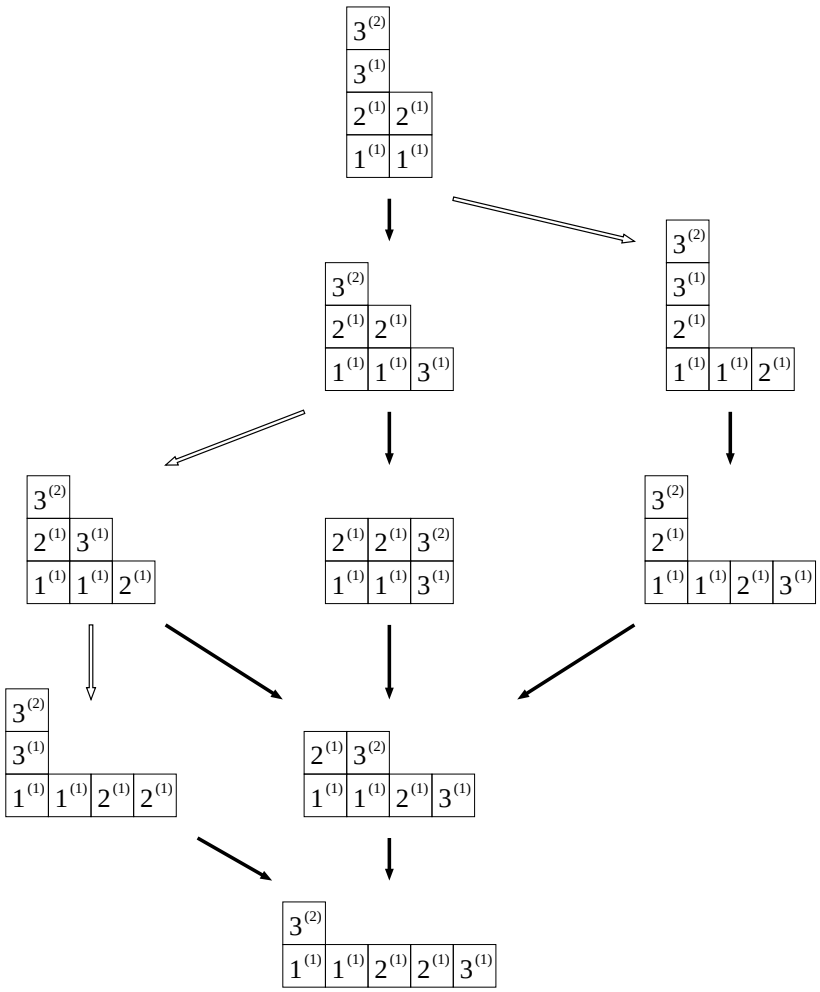


FIGURE 4. The cyclage-graph $\mathcal{T}(\mu)$ for $\mu = ((2), (2), (1^2))$

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