A HIGHER-LEVEL BAILEY LEMMA

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April 1996

We propose a generalization of Bailey's lemma, useful for proving q-series identities. As an application, generalizations of Euler's identity, the Rogers–Ramanujan identities, and the Andrews–Gordon identities are derived. This generalized Bailey lemma also allows one to derive identities for the branching functions of higher-level $\mathrm{A}_1^{(1)}$ cosets.

1. The Bailey Lemma

In his famous 1949 paper,[1] W. N. Bailey notes the following seemingly trivial result.

Lemma 1 *Let* $\alpha = \{\alpha_L\}_{L\geq 0}, \ldots, \delta = \{\delta_L\}_{L\geq 0}$ *be sequences which satisfy*

$$\beta_L = \sum_{k=0}^{L} \frac{\alpha_k}{(q)_{L-k}(aq)_{L+k}} \tag{1}$$

and

$$\gamma_L = \sum_{k=L}^{\infty} \frac{\delta_k}{(q)_{k-L}(aq)_{k+L}}\,, \tag{2}$$

with $(a)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$*. Then*

$$\sum_{L=0}^{\infty} \alpha_L \gamma_L = \sum_{L=0}^{\infty} \beta_L \delta_L\,. \tag{3}$$

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A pair of sequences (α,β) which satisfies (1) is said to form a Bailey pair relative to a.

Bailey noted that (3) can be used to obtain identities of the Rogers–Ramanujan type, provided one finds an appropriate δ such that (2) can be summed to yield an explicit expression for γ. In particular, taking

$$\delta_L = \frac{q^{L^2} a^L}{(q)_{M-L}} \quad 0 \le L \le M \qquad \text{and} \qquad \delta_L = 0 \quad L > M \tag{4}$$

and applying the q-analogue of Saalschütz's theorem, yields

$$\gamma_L = \frac{q^{L^2} a^L}{(q)_{M-L}(aq)_{M+L}} \quad 0 \le L \le M \qquad \text{and} \qquad \gamma_L = 0 \quad L > M. \tag{5}$$

Substituted into (3) this gives

$$\sum_{L=0}^{M} \frac{q^{L^2} a^L}{(q)_{M-L}(aq)_{M+L}}\, \alpha_L = \sum_{L=0}^{M} \frac{q^{L^2} a^L}{(q)_{M-L}}\, \beta_L\,, \tag{6}$$

which, after taking $M \to \infty$, simplifies to

$$\frac{1}{(aq)_\infty} \sum_{L=0}^{\infty} q^{L^2} a^L\, \alpha_L = \sum_{L=0}^{\infty} q^{L^2} a^L\, \beta_L\,. \tag{7}$$

As an example of how this result may be used, we follow Andrews,[2] and take the following Bailey pair relative to a:

$$\begin{aligned} \alpha_L &= \frac{(-1)^L(1-aq^{2L})(a)_L q^{L(L-1)/2}}{(1-a)(q)_L} \\ \beta_L &= \delta_{L,0}\,. \end{aligned} \tag{8}$$

Substituting this into (7) and setting $a = q^\ell$, $\ell = 0,1,2,\ldots$, we arrive at Euler's identity

$$\frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{(3j+1)j/2} = 1, \tag{9}$$

independent of ℓ.

More complicated Bailey pairs were used by Bailey[1] and subsequently by Slater[3,4], who took the Bailey lemma as starting point for the derivation of her celebrated list of 130 Rogers–Ramanujan identities.

2. The Bailey Chain

A particularly important observation was made by Andrews,[2] who noted that if (α,β) forms a Bailey pair relative to a, then (6) allows one to construct a new pair of sequences (α',β') which again forms a Bailey pair relative to a. Specifically, from (6) we infer the following lemma.

Lemma 2 *If (α,β) forms a Bailey pair relative to a, then (α',β'), defined as*

$$\begin{aligned}\alpha'_L &= q^{L^2} a^L \alpha_L \\ \beta'_L &= \sum_{k=0}^{L} \frac{q^{k^2} a^k}{(q)_{L-k}} \beta_k,\end{aligned} \tag{10}$$

again forms a Bailey pair relative to a.
Since (10) can of course be iterated an arbitrary number of times, the above lemma gives rise to the so-called Bailey chain.[2]

For example, lemma 2 applied k times to the Bailey pair (8), gives a new Bailey pair $(\alpha^{(k)},\beta^{(k)})$ which substituted into (7) yields

$$\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}(-1)^j q^{\left((2k+3)j+1\right)j/2} a^{jk} = \sum_{n_1\geq\ldots\geq n_k\geq 0}\frac{q^{n_1^2+\cdots+n_k^2} a^{n_1+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}}, \tag{11}$$

with $a=1,q$. After rewriting the left-hand side into product form using Jacobi's triple product identity, these identities yield a subset of Andrews' analytic form of Gordon's identities.[5,6] For $k=1$ (11) corresponds to the Rogers–Ramanujan identities.[7,8]

3. A Higher-Level Bailey Lemma

As we have illustrated in the previous sections, Bailey's lemma and the Bailey chain are extremely powerful concepts, producing an infinite series of q-identities from a single Bailey pair.[a] In the following we generalize (γ,δ) of (4) and (5) to $(\gamma^{(N)},\delta^{(N)})$ where $(\gamma,\delta)=(\gamma^{(1)},\delta^{(1)})$. This provides, together with the known Bailey pairs (α,β) and Bailey's lemma (3), a vast number of new q-series identities. Inserting $(\gamma^{(N)},\delta^{(N)})$ in (3) gives the "higher-level Bailey lemma", so-called since among many identities, it gives rise to Rogers–Ramanujan type identities for the branching functions of the level-N coset conformal field theories $(\mathrm{A}_1^{(1)})_N\times(\mathrm{A}_1^{(1)})_L/(\mathrm{A}_1^{(1)})_{L+N}$.

Before we give our result some more notation is needed. First, we need the Gaussian polynomial or q-binomial coefficient

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} \dfrac{(q)_A}{(q)_B(q)_{A-B}} & \text{if } 0\leq B\leq A \\ 0 & \text{otherwise.}\end{cases} \tag{12}$$

Furthermore, we fix the integer $N\geq 1$, and denote C the Cartan matrix and $\mathcal{I}$ the incidence matrix of the Lie algebra A_{N-1}. That is, $\mathcal{I}_{j,k}=\delta_{j,k-1}+\delta_{j,k+1}$ and $C=2I-\mathcal{I}$ with I the $(N-1)\times(N-1)$ identity matrix. Finally, $\vec{k}$ (where $\vec{k}$ stands for $\vec{m},\vec{n},\vec{\mu},\vec{\eta}$) and $\vec{\mathrm{e}}_\ell$ are $(N-1)$-dimensional vectors with non-negative integer entries $(\vec{k})_j=k_j$ and $(\vec{\mathrm{e}}_\ell)_j=\delta_{\ell,j}$.

[a]By applying the so-called Bailey lattice,[9] even larger classes of identities can be derived from a given Bailey pair.

Our result can then be stated as follows.

Lemma 3 *Fix integers $M \geq 0$, $N \geq 1$ and $0 \leq \ell \leq N$, fix a in (2) to $a = q^\ell$, and choose $\delta^{(N)}$ as*

$$\delta_L^{(N)} = \frac{q^{L(L+\ell)/N}}{(q)_{M-L}} \sum_{\frac{L}{N}-(C^{-1}\vec{n})_1 \in Z} q^{\vec{n}\, C^{-1}(\vec{n}-\vec{e}_\ell)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix}, \tag{13}$$

with $0 \leq L \leq M$ ($\delta_L = 0$ for $L > M$) and with $\vec{m}$ fixed by $\vec{n}$ through the $(\vec{m},\vec{n})$-system

$$\vec{m} + \vec{n} = \frac{1}{2}(\mathcal{I}\,\vec{m} + (2L+\ell)\,\vec{e}_{N-1} + \vec{e}_\ell). \tag{14}$$

Then $\gamma^{(N)}$ is given by

$$\gamma_L^{(N)} = \frac{q^{L(L+\ell)/N}}{(q)_{M-L}(q^{\ell+1})_{M+L}} \sum_{\frac{L}{N}-(C^{-1}\vec{\eta})_1 \in Z} q^{\vec{\eta}\, C^{-1}(\vec{\eta}-\vec{e}_\ell)} \prod_{j=1}^{N-1} \begin{bmatrix} \mu_j + \eta_j \\ \eta_j \end{bmatrix}, \tag{15}$$

for $0 \leq L \leq M$ ($\gamma_L = 0$ for $L > M$), with $(\vec{\mu},\vec{\eta})$-system

$$\vec{\mu} + \vec{\eta} = \frac{1}{2}(\mathcal{I}\,\vec{\mu} + (M-L)\,\vec{e}_1 + (M+L+\ell)\,\vec{e}_{N-1} + \vec{e}_\ell). \tag{16}$$

We note that the sum $\sum_{A-(C^{-1}\vec{k})_1 \in Z}$ has to be interpreted as a sum over non-negative integers $k_1,\ldots,k_{N-1}$, such that $A - \vec{e}_1 C^{-1}\vec{k}$ is again integer. We further note that for $N=1$ we reproduce the results (4) and (5) with $a=1$ or $a=q$. An inductive proof of lemma 3 will be given in Ref. 10.

An immediate corollary of lemma 3 is the generalization of (7) to arbitrary N.

Corollary 1 *Let N and ℓ be defined as in lemma 3 and let (α,β) be a Bailey pair relative to q^ℓ. Then*

$$\begin{aligned} \frac{1}{(q^{\ell+1})_\infty} \sum_{L=0}^{\infty} q^{L(L+\ell)/N}\, \alpha_L \sum_{\frac{L}{N}-(C^{-1}\vec{\eta})_1 \in Z} \frac{q^{\vec{\eta}\, C^{-1}(\vec{\eta}-\vec{e}_\ell)}}{(q)_{\eta_1}\cdots(q)_{\eta_{N-1}}} \\ = \sum_{L=0}^{\infty} q^{L(L+\ell)/N}\, \beta_L \sum_{\frac{L}{N}-(C^{-1}\vec{n})_1 \in Z} q^{\vec{n}\, C^{-1}(\vec{n}-\vec{e}_\ell)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix}, \end{aligned} \tag{17}$$

with the $(\vec{m},\vec{n})$-system (14).

As a simple application of corollary 1, we substitute the Bailey pair (8) with $a=1$ ($\ell=0$) into (17). This yields the following generalization of Euler's identity:

$$\frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\left((1+2/N)j+1\right)j/2} \sum_{\frac{j}{N}-(C^{-1}\vec{\eta})_1 \in Z} \frac{q^{\vec{\eta}\, C^{-1}\vec{\eta}}}{(q)_{\eta_1}\cdots(q)_{\eta_{N-1}}} = 1. \tag{18}$$

As a more elaborate example, we substitute the Bailey pair $(\alpha^{(k)},\beta^{(k)})$ (obtained from (8) by k times iterating (10)) into (17). For $a=1$ ($\ell=0$), this leads to the following generalization of the first Rogers–Ramanujan ($N = 1$, $k = 1$) and first Andrews–Gordon identity ($N = 1$, $k \geq 2$):

$$\frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\left((2k+1+2/N)j+1\right)j/2} \sum_{\frac{j}{N}-(C^{-1}\vec{\eta})_1 \in Z} \frac{q^{\vec{\eta}\, C^{-1}\vec{\eta}}}{(q)_{\eta_1}\cdots(q)_{\eta_{N-1}}}$$

$$= \sum_{r_1\geq\ldots\geq r_k\geq 0} \frac{q^{r_1^2/N+r_2^2+\cdots+r_k^2}}{(q)_{r_1-r_2}\cdots(q)_{r_{k-1}-r_k}(q)_{r_k}} \sum_{\frac{r_1}{N}-(C^{-1}\vec{n})_1\in Z} q^{\vec{n}\, C^{-1}\vec{n}} \prod_{j=1}^{N-1} \begin{bmatrix} m_j+n_j \\ n_j \end{bmatrix}, \tag{19}$$

with $(\vec{m},\vec{n})$-system

$$\vec{m}+\vec{n} = \frac{1}{2}(\mathcal{I}\,\vec{m} + 2r_1\,\vec{e}_{N-1}). \tag{20}$$

These identities are closely related to the Göllnitz–Gordon identities[11,12,13] when $N = 2$. To the best of our knowledge (19) is new for $N \geq 3$.

4. Discussion

In this note we have presented a higher-level generalization of the well-known Bailey lemma. As an application, some new q-series identities of the Rogers–Ramanujan type have been derived.

Finally, we list some general remarks about the higher-level Bailey lemma.

- Besides the factor $1/(q)_{M-L}(aq)_{M+L}$, $\gamma_L^{(N)}$ in (15) also depends on M through the M-dependence of the $(\vec{\mu},\vec{\eta})$-system (16). Hence, for $N \geq 2$, equations (13) and (15) cannot be used to obtain a Bailey chain in the same way as (4) and (5) gave (10).

- Apart from the factor $q^{L(L+\ell)}/(q)_{M-L}$, the expression (13) for $\delta_L^{(N)}$ coincides with the fermionic polynomial expressions for the configuration sums of the level-2 $A_{N-1}^{(1)}$ Jimbo–Miwa–Okado models,[14] as calculated by Foda *et al.*[15] This suggests that other exactly solvable lattice models, in particular the level-2 $\mathcal{G}_r^{(1)}$ models with $\mathcal{G} = D$ and E,[16,17] can be used to obtain further generalizations of the Bailey lemma.

- In Ref. 18 it was pointed out that the polynomial identities for finitized Virasoro characters of the minimal models $M(2, 2k+1)$[19,20] and $M(p, p+1)$[21–25] give rise to Bailey pairs. The application of Bailey's original lemma to the Bailey pairs arising from the polynomial identities for the most general model $M(p, p')$ was discussed in Ref. 26 and 27 where the Bailey transformation was interpreted as a renormalization group flow between different minimal models. When substituted in (17), the $M(p, p')$ Bailey pairs yield Rogers–Ramanujan type identities for the branching functions of the level-N cosets

$$\frac{(A_1^{(1)})_N \times (A_1^{(1)})_L}{(A_1^{(1)})_{N+L}}\,, \tag{21}$$

where N is that of lemma 3, and L is the (in general) fractional level $p'/p-2$ or $-N-2-p'/p$. For example, from the Bailey pair arising from $M(1,p)$ we arrive at the unitary character identities with integer level $L=p-2$, previously obtained in Refs. 28 and 29. A more detailed discussion of the q-series identities for the level-N cosets will be given in Ref. 10.

- The original (γ,δ) pair of Bailey[1] depends on two continuous parameters ρ_1 and ρ_2

$$\delta_L = \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_M(aq/\rho_2)_M}\,\frac{(aq/\rho_1\rho_2)_{M-L}}{(q)_{M-L}} \qquad 0\le L\le M \tag{22}$$

and

$$\gamma_L = \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L}\,\frac{1}{(q)_{M-L}(aq)_{M+L}} \qquad 0\le L\le M, \tag{23}$$

which reduce to (4) and (5) as $\rho_1,\rho_2\to\infty$. Slater[3,4] exploits this dependence on ρ_1 and ρ_2 to obtain many character identities for cosets of the form (21) with $N=2$, by keeping ρ_1 finite and letting $\rho_2\to\infty$. Further examples of this construction were given in Ref. 27 where it was also shown that the characters of the unitary $N=2$ supersymmetric models follow by specializing both ρ_1 and ρ_2 to appropriate finite values. At present it is unclear to us how to generalize the sequences $(\delta^{(N)},\gamma^{(N)})$ of lemma 3 to include such additional parameters ρ_i.

- In Ref. 30 Milne and Lilly have given yet another generalization of the Bailey lemma by extending the definitions (1) and (2) to higher-rank groups ((1) and (2) correspond to A_1). Using the theory of higher-rank basic hypergeometric series they then found an appropriate generalization of δ in (22) that can be summed explicitly. An extremely challenging problem would be to generalize our higher-level Bailey lemma to the higher-rank cases of Milne and Lilly.

Acknowledgements

It is a pleasure to thank Barry McCoy for many valuable discussions. SOW thanks the Institute for Theoretical Physics at Stony Brook for kind hospitality. This work is (partially) supported by the Australian Research Council and NSF grant DMR9404747.

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