REMARKS ON THE CONJECTURES OF CAPPARELLI, MEURMAN, PRIMC AND PRIMC

SHASHANK KANADE, MATTHEW C. RUSSELL, SHUNSUKE TSUCHIOKA, S. OLE WARNAAR

Dedicated to George Andrews and Bruce Berndt in celebration of their 85th birthdays

Abstract. In a sequence of two papers, S. Capparelli, A. Meurman, A. Primc, M. Primc (CMPP) and then M. Primc put forth three remarkable sets of conjectures, stating that the generating functions of coloured integer partition in which the parts satisfy restrictions on the multiplicities admit simple infinite product forms. While CMPP related one set of conjectures to the principally specialised characters of standard modules for the affine Lie algebra $\mathrm{C}_n^{(1)}$, finding a Lie-algebraic interpretation for the remaining two sets remained an open problem. In this paper, we use the work of Griffin, Ono and the fourth author on Rogers–Ramanujan identities for affine Lie algebras to solve this problem, relating the remaining two sets of conjectures to non-standard specialisations of standard modules for $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$. We also use their work to formulate conjectures for the bivariate generating function of one-parameter families of CMPP partitions in terms of Hall–Littlewood symmetric functions. We make a detailed study of several further aspects of CMPP partitions, obtaining (i) functional equations for bivariate generating functions which generalise the well-known Rogers–Selberg equations, (ii) a partial level-rank duality in the $\mathrm{A}_{2n}^{(2)}$ case, and (iii) (conjectural) identities of the Rogers–Ramanujan type for $\mathrm{D}_3^{(2)}$.

1. Introduction

Gordon's partition theorem [20] is one of the deepest and most beautiful results in the theory of integer partitions, generalising the combinatorial version of the Rogers–Ramanujan identities to arbitrary odd moduli. To describe Gordon's theorem, some basic partition-theoretic notions are needed. Let λ be an integer partition, that is, $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of nonnegative integers such that only finitely many λ_i are strictly positive [3]. Such positive λ_i are known as the parts of λ, and it is standard convention to suppress the trailing sequence of zeros in a partition, so that $(5,3,3,2,2,2,1,0,\dots)$ is written as $(5,3,3,2,2,2,1)$. If $|\lambda| := \lambda_1 + \lambda_2 + \cdots = N$ then λ is said to be a partition of N, written as $\lambda \vdash N$. The number of parts of λ equal to i is known as the multiplicity or frequency of the part i, and is denoted by $f_i = f_i(\lambda)$. In the previous example, $f_1 = f_5 = 1$, $f_2 = 3$, $f_3 = 2$ and $f_i = 0$ for $i = 4$ and $i \geqslant 6$. Clearly, if $\lambda \vdash N$, then $\sum_{i\geqslant 1} i f_i = N$.

For integers a, k, N such that $0 \leqslant a \leqslant k$ and $N \geqslant 0$, let $A_{k,a}(N)$ be the set of partitions of N into parts not congruent to $0, \pm(a+1)$ modulo $2k+3$ and $B_{k,a}(N)$ the set of partitions of N such that

$$f_i + f_{i+1} \leqslant k \text{ and } f_1 \leqslant a. \tag{1.1}$$

(Gordon's original description of the set $B_{k,a}(N)$ is slightly different. He defined it as the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots) \vdash N$ such that $\lambda_i - \lambda_{i+k} \geqslant 2$ for all $i \geqslant 1$ and $f_1 \leqslant a$. It is not

---

2020 *Mathematics Subject Classification.* 05A17, 05E05, 05E10, 11P84, 12B67.

This work is supported by the Simons Foundation Travel Support for Mathematicians grant #636937 (Kanade), the JSPS Kakenhi Grant 23K03051 and JST CREST Grant JPMJCR2113 (Tsuchioka) and the Australian Research Council Discovery Project DP200102316 (Warnaar).

difficult to see that $\lambda_i - \lambda_{i+k} \geqslant 2$ for all i if and only if $f_i + f_{i+1} \leqslant k$ for all i.) Gordon's theorem states that $A_{k,a}(N)$ and $B_{k,a}(N)$ have equal cardinality.

Theorem 1.1 (Gordon [20])**.** *For integers a, k, N such that $0 \leqslant a \leqslant k$ and $N \geqslant 0$,*

$$|A_{k,a}(N)| = |B_{k,a}(N)|.$$

For $k = 1$ these are the famous Rogers–Ramanujan identities in their combinatorial incarnation, as first stated by MacMahon [36] and Schur [48]. Let

$$B_{k,a} := \bigcup_{N \geqslant 0} B_{k,a}(N)$$

be the set of all Gordon partitions. Since the generating function of partitions in $A_{k,a}(N)$ admits a simple product form, an equivalent form of Gordon's theorem is

$$\begin{aligned} \sum_{\lambda \in B_{k,a}} q^{|\lambda|} &= \sum_{N=0}^{\infty} |B_{k,a}(N)| q^N \\ &= \prod_{\substack{m=0 \\ m \not\equiv 0, \pm(a+1) \ (\mathrm{mod}\ 2k+3)}}^{\infty} \frac{1}{1-q^m} = \frac{(q^{a+1}, q^{2k-a+2}, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty}, \end{aligned} \tag{1.2}$$

where $(a_1, \dots, a_r; q)_\infty := \prod_{i=1}^r \prod_{j \geqslant 0} (1 - a_i q^j)$. For a more detailed introduction to Gordon's partition theorem the reader is referred to [3, Chapter 7] and [50, Chapter 3].

In a remarkable paper [12], Capparelli, Meurman, Primc and Primc (CMPP) recently conjectured a beautiful generalisation of (1.2) for coloured partitions of N. In their partitions, each part is assigned one of n possible colours, where the ordering among colours is immaterial. For example, if $n = 3$ with colour-set {red, blue, black}, the partitions $(5, 3, 3, 2, 2, 2, 1)$ and $(5, 3, 3, 2, 2, 2, 1)$ are considered distinct partitions of 18, but $(5, 3, 3, 2, 2, 2, 1)$ and $(5, 3, 3, 2, 2, 2, 1)$ represent one and the same partition. (Alternatively, one can choose to order the colour-set as in {red > blue > black} and require that parts of the same size and different colour are ordered accordingly. This would make $(5, 3, 3, 2, 2, 2, 1)$ inadmissible.) Given a coloured partition $\lambda \vdash N$, the multiplicity or frequency of parts of size i and colour c is denoted by $f_i^{(c)} = f_i^{(c)}(\lambda)$, so that $\sum_{i \geqslant 1} \sum_{c=1}^n i f_i^{(c)} = N$. The generalisation of (1.1) to CMPP partitions is most conveniently expressed in terms of lattice paths. Let $\mathcal{P}^{(n)}$ denote the set of n-coloured partitions. Given $\lambda \in \mathcal{P}^{(n)}$ and nonnegative integers $k_0, k_1, \dots, k_n \in \mathbb{N}_0$ which play the role of initial or boundary conditions, arrange the frequencies of λ in a semi-infinite array $\mathscr{F}_n(\lambda)$ of $2n$ rows as follows:

$$\begin{array}{cccccccccccc}
k_n & & f_1^{(n)} & & f_3^{(n)} & & f_5^{(n)} & & f_7^{(n)} & & \cdots \\
& 0 & & f_2^{(n)} & & f_4^{(n)} & & f_6^{(n)} & & f_8^{(n)} & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
k_2 & & f_1^{(2)} & & f_3^{(2)} & & f_5^{(2)} & & f_7^{(2)} & & \cdots \\
& 0 & & f_2^{(2)} & & f_4^{(2)} & & f_6^{(2)} & & f_8^{(2)} & \\
k_1 & & f_1^{(1)} & & f_3^{(1)} & & f_5^{(1)} & & f_7^{(1)} & & \cdots \\
& k_0 & & f_2^{(1)} & & f_4^{(1)} & & f_6^{(1)} & & f_8^{(1)} &
\end{array}$$

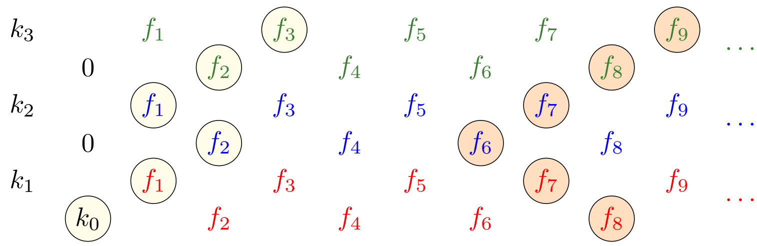
For $c \in \{1, \dots, n\}$, let $f_0^{(c)} = k_0 \delta_{c,1}$ (with $\delta_{i,j}$ the Kronecker delta) and $f_{-1}^{(c)} = k_c$. Then a path P on $\mathscr{F}_n(\lambda)$ is a sequence

$$(p_1, p_2, \dots, p_{2n})$$

such that

$$p_{2c-1} \in \{f_0^{(c)}, f_2^{(c)}, f_4^{(c)}, \dots\}, \quad p_{2c} \in \{f_{-1}^{(c)}, f_1^{(c)}, f_3^{(c)}, \dots\},$$

and, dropping the colour labels, such that f_i is followed by either f_{i-1} (which requires that $i \geqslant 0$) or f_{i+1}. Two examples of paths on $\mathscr{F}_3(\lambda)$ are shown below, where the (superfluous) colour labels have been omitted:



By slight abuse of notation, we write $P \in \mathscr{F}_n(\lambda)$ if P is a path on $\mathscr{F}_n(\lambda)$. A partition $\lambda \in \mathcal{P}^{(n)}$ is a CMPP-partition if for all $P = (p_1, \dots, p_{2n}) \in \mathscr{F}_n(\lambda)$,

$$\sum_{i=1}^{2n} p_i \leqslant k_0 + \cdots + k_n.$$

The set of all CMPP-partitions for given fixed $k_0, \dots, k_n$ will be denoted by $\mathscr{A}_{k_0,\dots,k_n}$.[1] If $n = 1$ then the set of paths on $\mathscr{F}_1(\lambda)$ is given by

$$\{(k_0, k_1), (k_0, f_1)\} \cup \{(f_{2i}, f_{2i-1}) : i \geqslant 1\} \cup \{(f_{2i}, f_{2i+1}) : i \geqslant 1\}.$$

Hence

$$\begin{aligned} \mathscr{A}_{k_0,k_1} &= \{\lambda \in \mathcal{P}^{(1)} : f_1(\lambda) \leqslant k_1 \text{ and } f_i(\lambda) + f_{i+1}(\lambda) \leqslant k_0 + k_1 \text{ for all } i \geqslant 1\} \\ &= B_{k_0+k_1,k_1}. \end{aligned} \tag{1.3}$$

CMPP conjectured the following generalisation of Gordon's partition theorem in the form (1.2). For $a, a_1, \dots, a_r \neq 0$, let $\theta(a, q) := (a, q/a; q)_\infty$ be a modified theta function and $\theta(a_1, \dots, a_r; q) := \prod_{i=1}^r \theta(a_i; q)$.

Conjecture 1.2 (CMPP [12, Conjecture 4.1])**.** *For k, n nonnegative integers and $k_0, \dots, k_n$ nonnegative integers such that $k_0 + \cdots + k_n = k$,*

$$\begin{aligned} \sum_{\lambda \in \mathscr{A}_{k_0,\dots,k_n}} q^{|\lambda|} &= \frac{(q^{2k+2n+1}; q^{2k+2n+1})_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^{\lambda_i+n-i+1}; q^{2k+2n+1}\big) \\ &\quad \times \prod_{1 \leqslant i < j \leqslant n} \theta\big(q^{\lambda_i - \lambda_j - i + j}, q^{\lambda_i+\lambda_j+2n-i-j+2}; q^{2k+2n+1}\big), \end{aligned}$$

where $\lambda_i := k_i + \cdots + k_n$ for $1 \leqslant i \leqslant n$.

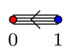
It should be remarked that CMPP do not state the product on the right in the explicit form as shown above, but instead provide a prescription for obtaining the product for fixed $k_0, \dots, k_n$ using what they refer to as 'congruence triangles'. The rationale for including the trivial $n = 0$ case is that from a representation-theoretic perspective it is natural to include $k = 0$ (for which the product trivialises to 1) in the conjecture. Since Conjecture 1.2 exhibits a partial 'level-rank duality' which interchanges the roles of k and n, it is natural to let k and n have the same range.

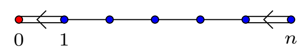
[1] The choice for the letter A reflects the fact that these partitions are related to $\mathrm{A}_{2n}^{(2)}$. Later we will describe variants of these partitions for the affine Lie algebras $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ which will be denoted by $\mathscr{C}_{k_0,\dots,k_n}$ and $\mathscr{D}_{k_0,\dots,k_n}$ respectively.

CMPP write about their conjecture that "for $n > 1$ there is no obvious connection [...] with representation theory of affine Lie algebras." The first purpose of this paper is to point out that there is a connection between Conjecture 1.2 and the representation theory of affine Lie algebras, very different from the well-known interpretation of the $n = 1$ case in terms of standard modules for $\mathrm{A}_1^{(1)}$ at level $2k + 1$, see e.g., [31–33]. Adopting standard notation and terminology (see Section 2.1 for details), let $L(\Lambda)$ be the $\mathrm{A}_{2n}^{(2)}$-standard module of highest weight Λ. Assuming the same relation between the k_i and λ_i as in Conjecture (1.2), parametrise Λ in terms of the fundamental weights $\Lambda_0, \dots, \Lambda_n$ of $\mathrm{A}_{2n}^{(2)}$ as

$$\begin{aligned}\Lambda &= 2(k - \lambda_1)\Lambda_0 + (\lambda_1 - \lambda_2)\Lambda_1 + \cdots + (\lambda_{n-1} - \lambda_n)\Lambda_{n-1} + \lambda_n\Lambda_n \\ &= 2k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_{n-1}\Lambda_{n-1} + k_n\Lambda_n.\end{aligned} \tag{1.4}$$

Here the usual labelling of the vertices of the $\mathrm{A}_{2n}^{(2)}$ Dynkin diagram is assumed:

$\mathrm{A}_2^{(2)}$: 

$\mathrm{A}_{2n}^{(2)}$: 

where, for now, the colour coding of the vertices may be ignored. Since $\mathrm{lev}(\Lambda_0) = 1$ and $\mathrm{lev}(\Lambda_i) = 2$ for all $i \geqslant 1$, the level of Λ is even and given by $\mathrm{lev}(\Lambda) = 2k$. Denote the character of $L(\Lambda)$ by $\mathrm{ch}\, L(\Lambda)$, and define the normalised character $\chi_\Lambda := \mathrm{e}(-\Lambda)\,\mathrm{ch}\, L(\Lambda)$, where $\mathrm{e}(\cdot)$ is a formal exponential. Then

$$\chi_\Lambda \in \mathbb{Z}[[\mathrm{e}(-\alpha_0), \dots, \mathrm{e}(-\alpha_n)]],$$

where $\alpha_0, \dots, \alpha_n$ are the simple roots of $\mathrm{A}_{2n}^{(2)}$. If $\varphi_n : \mathbb{Z}[[\mathrm{e}(-\alpha_0), \dots, \mathrm{e}(-\alpha_n)]] \to \mathbb{Z}[[q]]$ is the specialisation

$$\varphi_n(\mathrm{e}(-\alpha_0)) = -1 \quad \text{and} \quad \varphi_n(\mathrm{e}(-\alpha_i)) = q \ \text{ for } i \in \{1, \dots, n\},$$

then it follows from [21, Equation (3.31)] (see the appendix for details) that

$$\begin{aligned}\varphi_n(\chi_\Lambda) = \frac{(q^{2k+2n+1}; q^{2k+2n+1})_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^{\lambda_i+n-i+1}; q^{2k+2n+1}\big) \\ \times \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{\lambda_i-\lambda_j-i+j}, q^{\lambda_i+\lambda_j+2n-i-j+2}; q^{2k+2n+1}\big).\end{aligned} \tag{1.5}$$

(If the coefficient of Λ_0 in Λ is odd, so that Λ has odd level, then $\varphi_n(\chi_\Lambda) = 0$.) Importantly, the specialisation φ_n does not correspond to the much-studied principal gradation of $L(\Lambda)$ and hence the product (1.5) does not follow from Lepowsky's numerator formula [32]. By (1.5) the CMPP conjecture can be expressed in terms of $\mathrm{A}_{2n}^{(2)}$ as follows.

Conjecture 1.3 (Representation theoretic form of Conjecture 1.2)**.** *For nonnegative integers $k_0, \dots, k_n$, let $L(\Lambda)$ be the $\mathrm{A}_{2n}^{(2)}$-standard module of highest weight $\Lambda = 2k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_n\Lambda_n$. Then*

$$\sum_{\lambda \in \mathscr{A}_{k_0,\dots,k_n}} q^{|\lambda|} = \varphi_n(\chi_\Lambda). \tag{1.6}$$

As remarked above, the set $A_{k,a}(N)$ has a simple generating function given by the infinite product on the right of (1.2). In [2], Andrews showed that the two-variable generating function of Gordon partitions is expressible as a k-fold multisum:

$$\sum_{\lambda \in B_{k,a}} z^{l(\lambda)} q^{|\lambda|} = \sum_{r_1,\dots,r_k \geqslant 0} \frac{z^{r_1+\cdots+r_k} q^{r_1^2+\cdots+r_k^2+r_{a+1}+\cdots+r_k}}{(q;q)_{r_1-r_2} \cdots (q;q)_{r_{k-1}-r_k} (q;q)_{r_k}}, \tag{1.7}$$

where $(a;q)_n := (a;q)_\infty/(aq^n;q)_\infty = \prod_{i=0}^{n-1}(1-aq^i)$. Note in particular that $1/(q;q)_n$ vanishes for n a nonnegative integer so that the summand is nonzero for $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_k$ only. Equating the $z=1$ case of (1.7) with the right-hand side of (1.2) results in what are known as the Andrews–Gordon identities [2]:

$$\sum_{r_1,\dots,r_k\geqslant 0} \frac{q^{r_1^2+\cdots+r_k^2+r_{a+1}+\cdots+r_k}}{(q;q)_{r_1-r_2}\cdots(q;q)_{r_{k-1}-r_k}(q;q)_{r_k}} = \frac{(q^{a+1},q^{2k-a+2},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}. \tag{1.8}$$

The two $k=1$ cases are the Rogers–Ramanujan identities in their original analytic form [43–45].

The second purpose of this paper is to give a conjectural and highly incomplete generalisation of (1.7) to CMPP partitions or, as we may now call them, $\mathrm{A}_{2n}^{(2)}$-partitions. Let $P_\lambda(x_1,x_2,\dots;t)$ be the Hall–Littlewood symmetric function in countably-many variables, indexed by the partition λ, see Section 2.2 for details. Furthermore, for $\lambda=(\lambda_1,\lambda_2,\dots)$ a partition, let 2λ denote the partition $(2\lambda_1,2\lambda_2,\dots)$. Finally, if

$$k_0,\dots,k_n = \underbrace{i_1,\dots,i_1}_{m_1 \text{ times}},\underbrace{i_2,\dots,i_2}_{m_2 \text{ times}},\dots\dots,\underbrace{i_r,\dots,i_r}_{m_r \text{ times}},$$

where $m_1+\cdots+m_r=n+1$, we more succinctly write this as $k_0,\dots,k_n = i_1^{m_1},i_2^{m_2},\dots,i_r^{m_r}$, typically omitting those exponents m_i that are equal to 1. For example, if $k_0,\dots,k_8 = 1,0,0,0,2,1,3,3,0$ this would be shortened to $k_0,\dots,k_8=1,0^3,2,1,3^2,0$.

Conjecture 1.4. *For k a nonnegative integer and n a positive integer*

$$\sum_{\lambda\in\mathscr{A}_{0^n,k}} z^{l(\lambda)}q^{|\lambda|} = \sum_{\substack{\lambda\\ \lambda_1\leqslant k}} (zq)^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big) \tag{1.9a}$$

and

$$\sum_{\lambda\in\mathscr{A}_{k,0^n}} z^{l(\lambda)}q^{|\lambda|} = \sum_{\substack{\lambda\\ \lambda_1\leqslant k}} (zq^2)^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big). \tag{1.9b}$$

Given a partition, let λ' be its conjugate, i.e., $\lambda_i' = f_i(\lambda)-f_{i+1}(\lambda)$. Making the identification $\lambda_i'=r_i$, we have [35, page 213]

$$(zq)^{|\lambda|}P_{2\lambda}(1,q,q^2,\dots;q) = \prod_{i\geqslant 1}\frac{z^{r_i}q^{r_i^2}}{(q;q)_{r_i-r_{i+1}}}. \tag{1.10}$$

Recalling (1.3) and using that $\lambda_1\leqslant k$ implies that $r_i=\lambda_i'=0$ for $i>k$, it follows that (1.9a) and (1.9b) for $n=1$ are the $a=k$ and $a=0$ instances of (1.7) respectively. By [21, Theorem 1.1], (1.9a) and (1.9b) for $z=1$ are equivalent to Conjecture 1.3 for $k_0,\dots,k_n$ given by $0^n,k$ and $k,0^n$ respectively.

The remainder of this paper is organised as follows. In the next section we introduce the necessary terminology and notation pertaining to affine Lie algebras and Hall–Littlewood symmetric functions needed for this paper. This section includes a proof, using the Bailey lemma, of an identity of [21] for $P_{(2^r)}(1,q,q^2,\dots;q^N)$ for arbitrary positive integers N (see Proposition 2.1) and a conjectural Andrews–Gordon-type sum for $P_{2\lambda}(1,q,q^2,\dots;q^2)$ (see Conjecture 2.2). In Section 3 we establish a partial level-rank duality for the CMPP conjecture and discuss some known or provable cases of Conjectures 1.2–1.4. We also derive a set of functional equations for the two-variable generating function of CMPP partitions (see Proposition 3.2), generalising the well-known Rogers–Selberg equations. There are two important variants of the CMPP conjecture for coloured partitions whose frequency arrays have an odd number of rows. One of these, related to the principal specialisation of characters of $\mathrm{C}_n^{(1)}$-standard modules, is also due to CMPP, while the other, due to Primc [38], is related to a non-standard specialisation of characters of standard modules for $\mathrm{D}_{n+1}^{(2)}$. Both these conjectures, which turn out to be closely related, are the topic of Section 4. For almost all of our results and conjectures for $\mathrm{A}_{2n}^{(2)}$, with the exception of level-rank duality, we formulate analogues for $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$. Using the abovementioned Conjecture 2.2 for Hall–Littlewood symmetric functions, this also leads to a number of new conjectures of Rogers–Ramanujan type, including

$$\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}} \prod_{i=1}^{k} \frac{q^{(r_i+s_i)^2+s_i^2+s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}} = \frac{(q,q^{k+1},q^{k+2};q^{k+2})_\infty}{(q;q^2)_\infty(q;q)_\infty},$$

which is a q-series identity for the principally specialised character of the $\mathrm{A}_1^{(1)}$-standard module $L(k\Lambda_0)$ very different from those in [5, 14, 17, 28, 52, 53], and the very similar

$$\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}} \prod_{i=1}^{k} \frac{q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}} = \frac{(q^{k+1},q^{k+2},q^{k+2},q^{k+3},q^{2k+4},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty(q;q)_\infty},$$

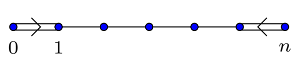
which is a q-series identity related to a non-standard specialisation of the $\mathrm{D}_3^{(2)}$-standard module $L(k\Lambda_1)$. The close resemblance of these formulas to the Andrews–Gordon identities (1.7) seems quite remarkable. In Section 5 we state some preliminary results, mostly conjectural, towards the seemingly very hard problem of completing Conjecture 1.4 (and its analogues for $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$) to arbitrary $k_0,\dots,k_n$. Finally, in the appendix we provide the details of the proof of the non-standard specialisation formula (1.5) and its $\mathrm{D}_{n+1}^{(2)}$-analogue.

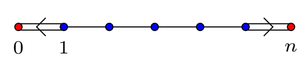
2. Preliminaries

2.1. **Affine Lie algebras.** For n a positive integer, fix the index set $I := \{0,1,\dots,n\}$ and let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra with generalised Cartan matrix $A = (a_{ij})_{i,j\in I}$ and Cartan subalgebra $\mathfrak{h}$, see e.g., [26]. We choose bases $\{\alpha_0,\dots,\alpha_n,\Lambda_0\}$ and $\{\alpha_0^\vee,\dots,\alpha_n^\vee,d\}$ of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, such that

$$\langle\alpha_i^\vee,\alpha_j\rangle = a_{ij}, \quad \langle\alpha_i^\vee,\Lambda_0\rangle = \langle d,\alpha_i\rangle = \delta_{i,0}, \quad \langle d,\Lambda_0\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. In the following it will be assumed that $\mathfrak{g}$ is one of the affine Lie algebras $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$, $\mathrm{D}_{n+1}^{(2)}$, and that the labelling of the simple roots α_i is in accordance with the labelling of the corresponding Dynkin diagrams as shown on page 4 or in the diagrams below:

$\mathrm{C}_n^{(1)}$: 

$\mathrm{D}_{n+1}^{(2)}$: 

Denoting the set of simple roots of $\mathfrak{g}$ by Δ, i.e., $\Delta = \{\alpha_i\}_{i\in I}$, we define the subset Δ^* of marked simple roots by

$$\Delta^* = \begin{cases} \{\alpha_0\} & \text{for } \mathfrak{g} = \mathrm{A}_{2n}^{(2)}, \\ \emptyset & \text{for } \mathfrak{g} = \mathrm{C}_n^{(1)}, \\ \{\alpha_0,\alpha_n\} & \text{for } \mathfrak{g} = \mathrm{D}_{n+1}^{(2)}. \end{cases}$$

Hence the marked simple roots correspond to the vertices of the Dynkin diagrams coloured red while the unmarked simple roots corresponds to the blue vertices.

The marks and comarks (or labels and colabels) a_i and $a_i^\vee$ for $i\in I$ are positive integers such that $\sum_{i\in I} a_{ij}a_j = \sum_{i\in I} a_i^\vee a_{ij} = 0$ and $\gcd(a_i)_{i\in I} = \gcd(a_i^\vee)_{i\in I} = 1$. In particular, for the three cases of interest the comarks are given by

$$(a_0^\vee, a_1^\vee, \dots, a_{n-1}^\vee, a_n^\vee) = \begin{cases} (1,2,\dots,2,2) & \text{for } \mathfrak{g} = \mathrm{A}_{2n}^{(2)}, \\ (1,1,\dots,1,1) & \text{for } \mathfrak{g} = \mathrm{C}_n^{(1)}, \\ (1,2,\dots,2,1) & \text{for } \mathfrak{g} = \mathrm{D}_{n+1}^{(2)}. \end{cases}$$

The null or imaginary root δ is defined as $\delta = \sum_{i\in I} a_i\alpha_i$ and together with the fundamental weights Λ_i $(i \in I)$, given by $\langle \Lambda_i, \alpha_j^\vee\rangle = \delta_{ij}$ and $\langle \Lambda_i, d\rangle = 0$, yields an alternative basis of $\mathfrak{h}^*$. Deviating slightly from [26] by dropping the δ-part, we define the set of dominant integral weights of $\mathfrak{g}$ as

$$P_+ := \sum_{i\in I} \mathbb{N}_0\Lambda_i.$$

The one-dimensional center of $\mathfrak{g}$ is spanned by the canonical central element $K = \sum_{i\in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h}$. In terms of K, the level of $\Lambda \in \mathfrak{h}^*$ is defined as $\mathrm{lev}(\Lambda) := \langle \Lambda, K\rangle$, and thus $\mathrm{lev}(\Lambda_i) = a_i^\vee$. For k a nonnegative integer, we further let $P_+^k := \{\Lambda \in P_+ : \mathrm{lev}(\Lambda) = k\}$ be the set of level-k dominant integral weight. In our discussion of the CMPP and Primc conjectures, it will be convenient to define a set of scaled fundamental weights $\{\mathbf{\Lambda}_i\}_{i\in I}$ as follows:

$$(\mathbf{\Lambda}_0, \mathbf{\Lambda}_1, \dots, \mathbf{\Lambda}_{n-1}, \mathbf{\Lambda}_n) = \begin{cases} (2\Lambda_0, \Lambda_1, \dots, \Lambda_{n-1}, \Lambda_n) & \text{for } \mathfrak{g} = \mathrm{A}_{2n}^{(2)}, \\ (\Lambda_0, \Lambda_1, \dots, \Lambda_{n-1}, \Lambda_n) & \text{for } \mathfrak{g} = \mathrm{C}_n^{(1)}, \\ (2\Lambda_0, \Lambda_1, \dots, \Lambda_{n-1}, 2\Lambda_n) & \text{for } \mathfrak{g} = \mathrm{D}_{n+1}^{(2)}. \end{cases}$$

Note in particular that for all $i \in I$ we have $\mathrm{lev}(\mathbf{\Lambda}_i) = 2$ for $\mathfrak{g} = \mathrm{A}_{2n}^{(2)}$ or $\mathfrak{g} = \mathrm{D}_{n+1}^{(2)}$ and $\mathrm{lev}(\mathbf{\Lambda}_i) = 1$ for $\mathfrak{g} = \mathrm{C}_n^{(1)}$, and that it is exactly the fundamental weights corresponding to the marked vertices of the Dynkin diagram that have been scaled by a factor of two in going from Λ_i to $\mathbf{\Lambda}_i$.

The standard modules (also known as integrable highest weight modules) of $\mathfrak{g}$ are indexed by dominant integral weights, and in the following $L(\Lambda)$ will denote the unique standard module of highest weight $\Lambda \in P_+$. The character of $L(\Lambda)$ is defined as

$$\mathrm{ch}\, L(\Lambda) = \sum_{\mu\in\mathfrak{h}^*} \dim(V_\mu)\, \mathrm{e}(\mu),$$

where $\mathrm{e}(\cdot)$ is a formal exponential and $\dim(V_\mu)$ is the dimension of the weight space V_μ in the weight-space decomposition of $L(\Lambda)$. As in the introduction, we define the normalised character $\chi_\Lambda := \mathrm{e}(-\Lambda)\,\mathrm{ch}\, L(\Lambda)$, so that

$$\chi_\Lambda \in \mathbb{Z}[[\mathrm{e}(-\alpha_0), \dots, \mathrm{e}(-\alpha_n)]].$$

According to the Weyl–Kac character formula [26],

$$\chi_\Lambda = \frac{\sum_{w\in W} \mathrm{sgn}(w)\, \mathrm{e}(w(\Lambda+\rho) - \Lambda - \rho)}{\prod_{\alpha>0}(1-\mathrm{e}(-\alpha))^{\mathrm{mult}(\alpha)}}, \tag{2.1}$$

where W is the Weyl group of $\mathfrak{g}$, $\mathrm{sgn}(w)$ the signature of $w \in W$ and $\rho = \sum_{i\in I} \Lambda_i$ is the Weyl vector. The product over $\alpha > 0$ is a product over the positive roots of the root system of $\mathfrak{g}$ and $\mathrm{mult}(\alpha)$ is the dimension of the root space corresponding to α. Rather than the full characters of $\mathfrak{g}$ we require specialisations. In all three cases under consideration we use the same notation for this specialisation, despite its $\mathfrak{g}$-dependence. Following [21], we define $\varphi_n : \mathbb{Z}[[\mathrm{e}(-\alpha_0), \dots, \mathrm{e}(-\alpha_n)]] \to \mathbb{Z}[[q]]$ by

$$\varphi_n(\mathrm{e}(-\alpha)) = \begin{cases} -1 & \text{if } \alpha \in \Delta^*, \\ q & \text{if } \alpha \in \Delta \setminus \Delta^*. \end{cases} \tag{2.2}$$

For $\mathfrak{g}=\mathrm{C}_n^{(1)}$, since there are no marked simple roots, this is the well-known principal specialisation, but for the other two types the above specialisation is non-standard. From Lepowsky's numerator formula [32] for $\mathrm{C}_n^{(1)}$ (see also [11]) and results from [21] for $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$, it follows that in all three cases the φ_n-specialisation of χ_λ admits a product form. For $\mathfrak{g}=\mathrm{A}_{2n}^{(2)}$ and $\Lambda\in P_+^{2k}$ parametrised as in (1.4), i.e., as

$$\Lambda=(k-\lambda_1)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n, \tag{2.3}$$

the specialisation $\varphi_n(\chi_\Lambda)$ is given by (1.5). Similarly, for $\mathfrak{g}=\mathrm{C}_n^{(1)}$ and $\Lambda\in P_+^k$ parametrised as in (2.3) we have

$$\begin{aligned}\varphi_n(\chi_\Lambda)&=\frac{(q^{k+n+1};q^{2k+2n+2})_\infty(q^{2k+2n+2};q^{2k+2n+2})_\infty^n}{(q;q^2)_\infty(q;q)_\infty^n}\\&\quad\times\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};q^{k+n+1}\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};q^{2k+2n+2}\big).\end{aligned} \tag{2.4}$$

Finally, for $\mathfrak{g}=\mathrm{D}_{n+1}^{(2)}$,

$$\varphi_n(\chi_\Lambda)=\frac{(q^{2k+2n};q^{2k+2n})_\infty^n}{(q^2;q^2)_\infty(q;q)_\infty^{n-1}}\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+1};q^{2k+2n}\big), \tag{2.5}$$

where again Λ is given by (2.3). Since the proofs of the two non-principal product forms were not included in paper [21], we present the full details of their derivation in the appendix.

2.2. **Hall–Littlewood polynomials.** The Hall–Littlewood polynomials $P_\lambda(t)$ are an important family of symmetric functions, interpolating between the Schur functions s_λ and monomial symmetric functions m_λ. They have long been known to be related to characters of affine Lie algebras and identities of the Rogers–Ramanujan type, see [8, 18, 21–23, 25, 29, 42, 52, 55, 56, 58].

For λ a partition of length at most k, the Hall–Littlewood polynomial is defined as [35]

$$P_\lambda(t)=P_\lambda(x_1,\dots,x_k;t):=\sum_{w\in S_k/S_k^\lambda}w\bigg(x_1^{\lambda_1}\cdots x_k^{\lambda_k}\prod_{\lambda_i>\lambda_j}\frac{x_i-tx_j}{x_i-x_j}\bigg),$$

where S_k is the symmetric group of degree k. $P_\lambda(x_1,\dots,x_k;t)$ is symmetric in the x_i (with coefficients in $\mathbb{Z}[t]$) and homogeneous of degree $|\lambda|$. By the stability property

$$P_\lambda(x_1,\dots,x_k,0;t)=\begin{cases}P_\lambda(x_1,\dots,x_k;t) & \text{if } l(\lambda)\leqslant k,\\ 0 & \text{if } l(\lambda)=k+1,\end{cases}$$

the Hall–Littlewood polynomials extend to symmetric functions in infinitely many variables in the usual fashion, see [35] for details. Well-known special cases of $P_\lambda(t)$ are $P_\lambda(0)=s_\lambda$, $P_\lambda(1)=m_\lambda$ and $P_{(1^r)}=e_r$, with e_r the rth elementary symmetric function.

An important result for Hall–Littlewood polynomials is the principal specialisation formula [35, page 213]

$$P_\lambda(1,t,\dots,t^{k-1};t)=\frac{t^{n(\lambda)}(t;t)_k}{(t;t)_{k-l(\lambda)}\prod_{i\geqslant1}(t;t)_{\lambda'_i-\lambda'_{i+1}}}=\frac{t^{n(\lambda)}(t;t)_k}{\prod_{i\geqslant0}(t;t)_{f_i(\lambda)}}, \tag{2.6}$$

where $l(\lambda)\leqslant k$, $n(\lambda):=\sum_{i\geqslant1}(i-1)\lambda_i$ and $f_0(\lambda):=k-l(\lambda)$. In order to prove Conjecture 1.4 for $k=1$, we require the following generalisation of the large-k limit of (2.6) for $\lambda=(2^r)$, which was stated without proof in [21, Equation (2.7)].

Proposition 2.1. *For r, n nonnegative integers and $\delta \in \{0, 1\}$,*

$$P_{(2^r)}\big(1, q, q^2, \dots; q^{2n+\delta}\big) = \sum_{r_1,\dots,r_n \geqslant 0} \frac{q^{r^2-r+r_1^2+\dots+r_n^2+r_1+\dots+r_n}}{(q;q)_{r-r_1}(q;q)_{r_1-r_2}\cdots(q;q)_{r_{n-1}-r_n}(q^{2-\delta};q^{2-\delta})_{r_n}}. \tag{2.7}$$

Proof. For integers k, n, let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

be a q-binomial coefficient. Our starting point for proving (2.7) is a formula for the Hall–Littlewood polynomials due to Lassalle and Schlosser stated in [30, Theorem 7.1]. For $\lambda = (2^r, 1^s)$ (the partition with r parts of size 2 and s parts of size 1) the Lassalle–Schlosser result simplifies to

$$P_{(2^r,1^s)}(t) = \sum_{i=0}^{r} (-1)^i t^{\binom{i}{2}} \frac{1-t^{2i+s}}{1-t^{i+s}} \begin{bmatrix} i+s \\ s \end{bmatrix}_t e_{r-i} e_{r+s+i},$$

where $(1-t^{2i+s})/(1-t^{i+s})$ should be interpreted as 1 if $i = 0$ for all $s \geqslant 0$ (i.e., including when $s = 0$). By [35, page 27]

$$e_r(1, q, \dots, q^{k-1}) = q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix},$$

(which is (2.6) for $\lambda = (1^r)$), this yields

$$\begin{aligned} &P_{(2^r,1^s)}(1, q, \dots, q^{k-1}; t) \\ &\quad = \sum_{i=0}^{r} (-1)^i t^{\binom{i}{2}} q^{\binom{r-i}{2}+\binom{r+s+i}{2}} \frac{1-t^{2i+s}}{1-t^{i+s}} \begin{bmatrix} i+s \\ s \end{bmatrix}_t \begin{bmatrix} k \\ r-i \end{bmatrix} \begin{bmatrix} k \\ r+s+i \end{bmatrix}. \end{aligned} \tag{2.8}$$

We note that for $t = q$ the sum over i can be carried out by the very-well poised ${}_6\phi_5$ summation [19, Equation (II.21)] with $(a, b, c, n) \mapsto (q^s, q^{-(k-r-s)}, \infty, r)$ to give

$$P_{(2^r,1^s)}(1, q, \dots, q^{k-1}; q) = \frac{q^{\binom{r}{2}+\binom{r+s}{2}}(q;q)_k}{(q;q)_{k-r-s}(q;q)_r(q;q)_s}, \tag{2.9}$$

in accordance with (2.6).

A pair of sequences $(\alpha, \beta) = \big((\alpha_r)_{r\geqslant 0}, (\beta_r)_{r\geqslant 0}\big)$ such that

$$\beta_r = \sum_{i=0}^{r} \frac{\alpha_i}{(q;q)_{r-i}(aq;q)_{r+i}} \tag{2.10}$$

is known as a Bailey pair relative to a, see [4, 6, 54]. The identity that arises after taking the large-k limit of (2.8) is equivalent to the statement that

$$\alpha_i = (-1)^i t^{\binom{i}{2}} q^{i(i+s)} \frac{1-t^{2i+s}}{1-t^{i+s}} \begin{bmatrix} i+s \\ s \end{bmatrix}_t, \tag{2.11a}$$

$$\beta_r = q^{-\binom{r}{2}-\binom{r+s}{2}} (q;q)_s \, P_{(2^r,1^s)}(1, q, q^2, \dots; t) \tag{2.11b}$$

forms a Bailey pair relative to q^s. Specialising $s = 0$ and $t = q^{2n+\delta}$ for n a nonnegative integer and $\delta \in \{0, 1\}$, the resulting α-sequence corresponds to a known Bailey pair relative to 1 (given by [47, Equation (4.13)] with $(k, i) \mapsto (n+2, 2)$), with corresponding β-sequence

$$\beta_r = \sum_{r_1,\dots,r_n \geqslant 0} \frac{q^{r_1^2+\dots+r_n^2+r_1+\dots+r_n}}{(q)_{r-r_1}(q)_{r_1-r_2}\cdots(q)_{r_{n-1}-r_n}(q^{2-\delta};q^{2-\delta})_{r_n}}.$$

Since the α-sequence determines the β-sequence uniquely, the above expression me be equated with (2.11b) for $s = 0$ and $t = q^{2n+\delta}$, completing the proof.

We remark that, by (2.9) for $k \to \infty$, the Bailey pair (2.11) for $s = 1$ and $t = q$ is equivalent to the Bailey pair B(3) in Slater's famous list of Bailey pairs [51]. Unfortunately, in the general $t = q^{2n+\delta}$ case the α-sequence (2.11a) for $s = 1$ does not correspond to anything known. ☐

From (1.10) (which is a special case of (2.6)),

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant k}} (zq)^{|\lambda|} P_{2\lambda}(1, q, q^2, \dots; q) = \sum_{r_1,\dots,r_k \geqslant 0} \prod_{i=1}^{k} \frac{z^{r_i} q^{r_i^2}}{(q;q)_{r_i - r_{i+1}}}.$$

Conjecturally, this generalises to Hall–Littlewood functions of base q^2.

Conjecture 2.2. *For k a nonnegative integer,*

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant k}} (zq)^{|\lambda|} P_{2\lambda}(1, q, q^2, \dots; q^2) = \sum_{\substack{r_1,\dots,r_k \geqslant 0 \\ s_1,\dots,s_k \geqslant 0}} \prod_{i=1}^{k} \frac{z^{r_i+s_i} q^{(r_i+s_i)^2 + s_i^2 + s_i}}{(q;q)_{r_i - r_{i+1}} (q^2;q^2)_{s_i - s_{i-1}}},$$

where $r_{k+1} = s_0 := 0$.

Apart from the trivial case $k = 0$ this also holds for $k = 1$ by (2.7) for $n = 1$, $\delta = 0$ and $(r, r_1) \mapsto (r_1 + s_1, s_1)$.

3. Special cases of the CMPP conjecture

There is a 'rough' level-rank duality at play in Conjecture 1.2, corresponding to the interchange of k and n. It is not the case, however, that for every identity corresponding to a level-$2k$ standard module of $\mathrm{A}_{2n}^{(2)}$ there is a counterpart in terms of a level-$2n$ standard module of $\mathrm{A}_{2k}^{(2)}$, and a one-to-one correspondence only occurs for $n = 1$ and $n = 2$ (or, by level-rank duality, for $k = 1$ and $k = 2$). For $n = 1$, let i, k be integers such that $0 \leqslant i \leqslant k$. Then

$$\varphi_1\big(\chi_{(k-i)\Lambda_0 + i\Lambda_1}\big) = \begin{cases} \varphi_k\big(\chi_{\Lambda_{i/2}}\big) & \text{if } i \text{ is even,} \\ \varphi_k\big(\chi_{\Lambda_{k-(i-1)/2}}\big) & \text{if } i \text{ is odd.} \end{cases} \tag{3.1}$$

Similarly, for $n = 2$, let i, j, k be integers such that $0 \leqslant i \leqslant j \leqslant k$. Then

$$\varphi_2\big(\chi_{(k-j)\Lambda_0 + (j-i)\Lambda_1 + i\Lambda_2}\big) = \begin{cases} \varphi_k\big(\chi_{\Lambda_{(j-i)/2} + \Lambda_{(i+j)/2}}\big) & \text{if } i+j \text{ is even,} \\ \varphi_k\big(\chi_{\Lambda_{k-(i+j-1)/2} + \Lambda_{k-(j-i-1)/2}}\big) & \text{if } i+j \text{ is odd.} \end{cases} \tag{3.2}$$

As mentioned above, for general n and k there only is a partial level-rank duality, depending in a non-trivial manner on the choice of highest weight. There are, however, some general patterns, the simplest of which are

$$\begin{aligned} \varphi_n\big(\chi_{k\Lambda_0}\big) &= \varphi_k\big(\chi_{n\Lambda_0}\big), && k, n \geqslant 1, \\ \varphi_n\big(\chi_{(k-2)\Lambda_0 + 2\Lambda_1}\big) &= \varphi_k\big(\chi_{(n-2)\Lambda_0 + 2\Lambda_1}\big), && k, n \geqslant 2, \\ \varphi_n\big(\chi_{(k-3)\Lambda_0 + 2\Lambda_1 + \Lambda_2}\big) &= \varphi_k\big(\chi_{(n-3)\Lambda_0 + 2\Lambda_1 + \Lambda_2}\big), && k, n \geqslant 3, \end{aligned}$$

where, in accordance with (3.1) and (3.2), the second duality holds for all $k, n \geqslant 1$ and the third duality for all $k, n \geqslant 2$ provided $-\Lambda_0 + 2\Lambda_1$ is interpreted as Λ_1. In terms of Conjectures 1.2 and 1.3, the duality (3.1) may be expressed as follows. For $a \in \{0, \dots, n\}$, let

$$\mathscr{A}_a^{(n)}(N) := \mathscr{A}_{0^a,1,0^{n-a}}(N) \quad \text{and} \quad \mathscr{A}_a^{(n)} := \bigcup_{N \geqslant 0} \mathscr{A}_a^{(n)}(N).$$

Then (1.6) for $k_0 + \cdots + k_n = 1$ gives

$$\sum_{\lambda \in \mathscr{A}_a^{(n)}} q^{|\lambda|} = \varphi_n\big(\chi_{\Lambda_a}\big).$$

By (3.1) with $k \mapsto n$ this yields

$$\sum_{\lambda \in \mathscr{A}_a^{(n)}} q^{|\lambda|} = \begin{cases} \varphi_1\big(\chi_{(n-2a)\Lambda_0+2a\Lambda_1}\big) & \text{for } 0 \leqslant a \leqslant \lfloor n/2 \rfloor, \\ \varphi_1\big(\chi_{(2a-n-1)\Lambda_0+(2n-2a+1)\Lambda_1}\big) & \text{for } \lfloor n/2 \rfloor < a \leqslant n. \end{cases}$$

Finally, by (1.5) with $n = 1$, $\kappa = 2n + 3$ and $(\lambda_0, \lambda_1) = (2n - 2a, 2a)$ in the first case and $(\lambda_0, \lambda_1) = (2a - 1, 2n - 2a + 1)$ in the second case, we find

$$\sum_{\lambda \in \mathscr{A}_a^{(n)}} q^{|\lambda|} = \frac{(q^{2a+1}, q^{2n-2a+2}, q^{2n+3}; q^{2n+3})_\infty}{(q;q)_\infty}. \tag{3.3}$$

With some effort this may be seen to corresponds to a coloured partition theorem due to Jing, Misra and Savage, stated as [24, Theorem 1.2; M odd]. The exact correspondence between $\mathscr{A}_a^{(n)}(N)$ and the set of coloured partitions $C_N(M, r)$ defined by Jing, Misra and Savage is as follows:

$$M = 2n + 3, \quad r = \begin{cases} 2a + 1 & \text{for } 0 \leqslant a \leqslant \lfloor n/2 \rfloor, \\ 2n - 2a + 2 & \text{for } \lfloor n/2 \rfloor < a \leqslant n, \end{cases}$$

where in the second case, the colour labelling of [24] needs to be reversed, i.e., the colour c should be mapped to $n + 1 - c$. (This also affects the actual definition of coloured partitions in [24] since an order on the colour labels is assumed.) Jing, Misra and Savage prove their result by showing that the set of coloured partition $C_N(2n + 3, r)$ is in bijection with the set of ordinary partitions $\lambda \vdash N$ such that $\lambda_i - \lambda'_i \in \{2 - r, \dots, 2n - r + 1\}$ for all $1 \leqslant i \leqslant d$ where $d = \max\{i \geqslant 1 : \lambda_i \geqslant i\}$ is the size of the Durfee square of λ. By the work of Andrews on successive rank partitions [1, Theorem 4.1], this gives the claimed product form. An alternative proof of (3.3) was given in [46], based on the fact that if

$$A_a^{(n)}(z) = A_a^{(n)}(z, q) := \sum_{\lambda \in \mathscr{A}_a^{(n)}} z^{l(\lambda)} q^{|\lambda|},$$

then the following system of functional equations holds:

$$A_a^{(n)}(z) - A_{n-a}^{(n)}(zq) = \sum_{i=1}^{a} zq^{2i-1} A_{a-i+1}^{(n)}(zq^{2i}) + \sum_{i=1}^{a} zq^{2i} A_{n-a+i}^{(n)}(zq^{2i+1}), \tag{3.4a}$$

$$A_{n-a}^{(n)}(z) - A_{a+1}^{(n)}(zq) = \sum_{i=1}^{a+1} zq^{2i-1} A_{n-a+i-1}^{(n)}(zq^{2i}) + \sum_{i=1}^{a} zq^{2i} A_{a-i+1}^{(n)}(zq^{2i+1}), \tag{3.4b}$$

where $0 \leqslant a \leqslant \lfloor n/2 \rfloor$ in (3.4a) and $0 \leqslant a \leqslant \lfloor (n-1)/2 \rfloor$ in (3.4b). Since these are equivalent to the Corteel–Welsh equations [15] for cylindric partitions of rank 2 and level $2n+1$, we have the known solution [46, 57][2]

$$A_a^{(n)}(z, q) = \begin{cases} F_{2a,1}^{(n)}(z, q) & \text{for } 0 \leqslant a \leqslant \lfloor n/2 \rfloor, \\ F_{2n-2a+1,1}^{(n)}(z, q) & \text{for } \lfloor n/2 \rfloor < a \leqslant n, \end{cases} \tag{3.5}$$

[2]To be precise, $A_a^{(n)}(z, q)$ corresponds to the normalised generating function for cylindric partitions with profile $(2n - a + 1, a)$, denoted by $G_{(2n-a+1,a)}(z, q)$ in [15]. The Corteel–Welsh functional equations for the function $G_c(z, q)$, for more general profiles c is given by [15, Equation (3.5)].

where, for $0 \leqslant a \leqslant n$ and $\delta \in \{0,1\}$,

$$F_{a,\delta}^{(n)}(z,q) := \sum_{r_1,\dots,r_n \geqslant 0} \frac{z^{r_1} q^{r_1^2+\dots+r_n^2+r_{a+1}+\dots+r_n}}{(q;q)_{r_1-r_2}\cdots(q;q)_{r_{n-1}-r_n}(q^{2-\delta};q^{2-\delta})_{r_n}}. \tag{3.6}$$

Since for $z=1$ and $\delta=1$ this is exactly the left-hand side of (1.8) with $k \mapsto n$, this once again implies (3.3).

Also Conjecture 1.4 can be shown to hold for $k=1$.

Proposition 3.1. *Conjecture 1.4 holds for $k=1$ and all positive integers n. That is,*

$$A_n^{(n)}(z,q) = \sum_{\lambda \in \mathscr{A}_n^{(n)}} z^{l(\lambda)} q^{|\lambda|} = \sum_{r=0}^{\infty} (zq)^r P_{(2^r)}\big(1,q,q^2,\dots;q^{2n-1}\big) \tag{3.7a}$$

and

$$A_0^{(n)}(z,q) = \sum_{\lambda \in \mathscr{A}_0^{(n)}} z^{l(\lambda)} q^{|\lambda|} = \sum_{r=0}^{\infty} (zq^2)^r P_{(2^r)}\big(1,q,q^2,\dots;q^{2n-1}\big). \tag{3.7b}$$

Proof. Let $\sigma \in \{0,1\}$. By Proposition 2.1 with $n \mapsto n-1$, $\delta=1$ and $(r,r_1,\dots,r_{n-1}) \mapsto (r_1,r_2,\dots,r_n)$,

$$\begin{aligned}
&\sum_{r=0}^{\infty} (zq^{2-\sigma})^r P_{(2^r)}\big(1,q,q^2,\dots;q^{2n-1}\big) \\
&\quad = \sum_{r_1,\dots,r_n \geqslant 0} \frac{z^{r_1} q^{r_1^2+\dots+r_n^2+(1-\sigma)r_1+r_2+\dots+r_n}}{(q)_{r_1-r_2}\cdots(q)_{r_{n-1}-r_n}(q;q)_{r_n}} = F_{\sigma,1}^{(n)}(z,q).
\end{aligned}$$

By (3.5) for $a=\sigma n$, this yields $A_{\sigma n}^{(n)}(z,q)$. □

To express the functional equations for the two-variable generating function of CMPP partitions for arbitrary level and rank it is more convenient to use dominant integral weights instead of the coefficients $k_0,\dots,k_n$ of the respective fundamental weights $\Lambda_0,\dots,\Lambda_n$ as indexing set. This motivates the definition

$$\mathcal{A}_{k_0\Lambda_0+\dots+k_n\Lambda_n}^{(n)}(z) = \mathcal{A}_{k_0\Lambda_0+\dots+k_n\Lambda_n}^{(n)}(z,q) := \sum_{\lambda \in \mathscr{A}_{k_0,\dots,k_n}} z^{l(\lambda)} q^{|\lambda|},$$

so that $A_a^{(n)}(z) = \mathcal{A}_{\Lambda_a}^{(n)}(z)$. We expect that for all fixed k,n, the generating functions in

$$\{\mathcal{A}_{\Lambda}^{(n)}(z)\}_{\mathrm{lev}(\Lambda)=2k}$$

satisfy a system of functional equations generalising (3.4), uniquely determining all such functions. For $n=1$ this system is given by the well-known Rogers–Selberg functional equations [45, 49]:

$$\mathcal{A}_{(k-a)\Lambda_0+a\Lambda_1}^{(1)}(z) - \mathcal{A}_{(k-a+1)\Lambda_0+(a-1)\Lambda_1}^{(1)}(z) = (zq)^a \mathcal{A}_{a\Lambda_0+(k-a)\Lambda_1}^{(1)}(zq), \tag{3.8}$$

where $0 \leqslant a \leqslant k$ with $\mathcal{A}_{(k+1)\Lambda_0-\Lambda_1}^{(1)}(z) := 0$. It is exactly these equations that were solved by Andrews in [2] to show that $\mathcal{A}_{(k-a+1)\Lambda_0+(a-1)\Lambda_1}^{(1)}(z)$ (for $1 \leqslant a \leqslant k+1$) is given by the multisum expression in (1.7). Below we give an incomplete set of functional equations for arbitrary k and n.

Proposition 3.2. *For a, k integers such that $0 \leqslant a \leqslant k$,*

$$\mathcal{A}^{(n)}_{(k-a)\Lambda_0+a\Lambda_n}(z) = \sum_{i=0}^{a} (zq)^i \mathcal{A}^{(n)}_{i\Lambda_0+(a-i)\Lambda_1+(k-a)\Lambda_n}(zq). \tag{3.9a}$$

Moreover, for $k \geqslant 1$,

$$\begin{aligned}\mathcal{A}^{(n)}_{(k-1)\Lambda_0+\Lambda_1}(z) &= \mathcal{A}^{(n)}_{\Lambda_{n-1}+(k-1)\Lambda_n}(zq) + (zq^2)^k \mathcal{A}^{(n)}_{k\Lambda_0}(zq^2) \\ &\quad + zq \sum_{i=0}^{k-1} (zq^2)^i \mathcal{A}^{(n)}_{i\Lambda_0+(k-i)\Lambda_1}(zq^2).\end{aligned} \tag{3.9b}$$

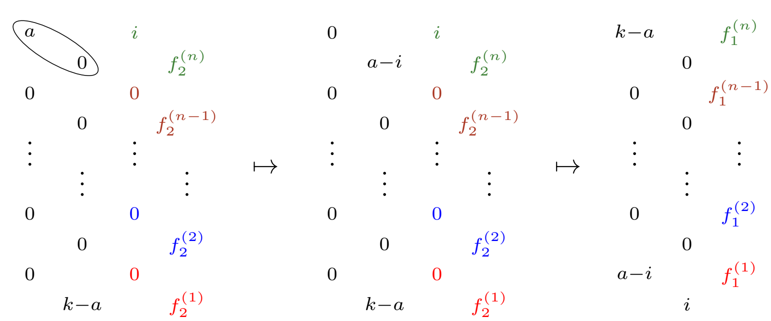
For $n = 1$ the functional equation (3.9a) simplifies to

$$\mathcal{A}^{(1)}_{(k-a)\Lambda_0+a\Lambda_1}(z) = \sum_{i=0}^{a} (zq)^i \mathcal{A}^{(1)}_{i\Lambda_0+(k-i)\Lambda_1}(zq),$$

which is easily seen to be equivalent to the Rogers–Selberg equations (3.8). For arbitrary n but $k = 1$ the $a = 0$ case of (3.9a) is (3.4a) for $a = 0$ and the $a = 1$ case of (3.9a) is a combination of the $a = 0$ cases of (3.4a) and (3.4b). Similarly, the $k = 1$ and $a = 0$ case of (3.9b) is a combination of the $a = 0$ case of (3.4a) and the $a = 1$ case of (3.4b). By (3.9a) with $a = k$ and $z \mapsto zq$, the functional equation (3.9b) can be simplified to

$$\mathcal{A}^{(n)}_{(k-1)\Lambda_0+\Lambda_1}(z) = \mathcal{A}^{(n)}_{\Lambda_{n-1}+(k-1)\Lambda_n}(zq) + (1-zq)(zq^2)^k \mathcal{A}^{(n)}_{k\Lambda_0}(zq^2) + zq\mathcal{A}^{(n)}_{k\Lambda_n}(zq).$$

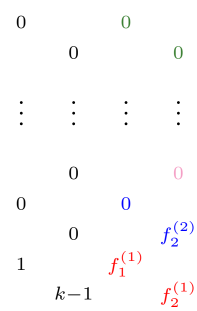
Proof. We first prove (3.9a), which is the simplest of the two claims. The task is to show that the generating function $\mathcal{A}^{(n)}_{(k-a)\Lambda_0+a\Lambda_n}(z)$ may be expressed as the sum given on the right-hand side of (3.9a). We begin by noting that if $k_0 = k - a$, $k_n = a$ and $k_2 = \cdots = k_{n-1} = 0$, then a necessary condition for a partition $\lambda \in \mathcal{P}^{(n)}$ to have an admissible frequency-array is that $f^{(c)}_1 = 0$ for $1 \leqslant c \leqslant n-1$ and $f^{(n)}_1 \leqslant a$. (More generally, $f^{(c)}_i = 0$ for $1 \leqslant i \leqslant n-1$ and $\lfloor i/2 \rfloor < c \leqslant n - \lceil i/2 \rceil$.) Replacing $f^{(n)}_1$ by i, where $0 \leqslant i \leqslant a$, the first four columns of the frequency array of any partition contributing to $\mathcal{A}^{(n)}_{(k-a)\Lambda_0+a\Lambda_n}(z)$ take the form as shown in the left-most of the following three (partial) frequency arrays:



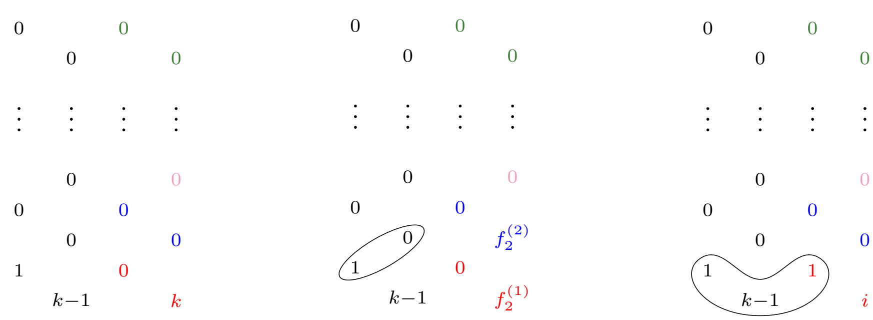
where, to emphasise that the proof does not rely on the fact that $f^{(2)}_2, \dots, f^{(n-1)}_2$ all need to be zero for admissibility, we have not placed any zeros in the fourth column. The sole reason for including this column is to visualise a relabelling of the frequencies $f^{(c)}_i$ in the final step of the proof.

The objective now is to eliminate the first column. To achieve this we note that any path P on the left-most array terminating at the vertex labelled a must be of the form $P = (p_1, \dots, p_{2n-2}, 0, a)$, where the final two entries correspond to the encircled pair of vertices. For each such path P there is a companion $Q = (p_1, \dots, p_{2n-2}, 0, i)$. Since $i \leqslant a$, if P is admissible then so is Q. Replacing the encircled vertices a and 0 by 0 and $a-i$ respectively, P and Q are mapped to $P'=(p_1,\dots,p_{2n-2},a-i,0)$ and $Q'=(p_1,\dots,p_{2n-2},a-i,i)$. This time the admissibility of Q' guarantees the admissibility of P'. Moreover, Q' is admissible if and only if P is admissible. Hence we may replace the left-most array by the array shown in the middle of the above diagram as it allows for the exact same set of admissible paths on the full array. Since the left-most column in the second array contains only zeros it is redundant and may thus be deleted. After redrawing the resulting array upside-down and relabelling $f_j^{(c)}$ by $f_{j-1}^{(n-c+1)}$ for $j\geqslant 2$ and $1\leqslant c\leqslant n$, this yields the third of the above three arrays. The contribution of the full set of arrays of this form to the generating function is $(zq)^i\mathcal{A}^{(n)}_{i\Lambda_0+(a-i)\Lambda_1+(k-a)\Lambda_n}(zq)$. Here the prefactor $(zq)^i$ accounts for the fact that the vertex labelled i in the second column arose from $f_1^{(n)}=i$ in the original array, thus contributing $(zq)^i$ to the generating function. The argument zq instead of z in the generating function accounts for the fact that f_j has been replaced by f_{j-1}. Adding all the contributions from $i\in\{0,1,\dots,a\}$ results in the right-hand side of (3.9a).

Next we prove the more complicated (3.9b). The initial conditions $k_0=k-1$ and $k_1=1$ force $f_i^{(c)}=0$ for $1\leqslant i\leqslant 2$ and $2\leqslant c\leqslant n$, so that the first four columns of the frequency array of any admissible partition take the form



Considering $f_1^{(1)}$ and $f_2^{(1)}$, there are three possible scenarios. The first is $f_1^{(1)}=0$ and $f_2^{(1)}=k$ (which forces $f_2^{(2)}=0$), the second is $f_1^{(1)}=0$ and $f_2^{(1)}<k$ and the third is $f_1^{(1)}=1$ and $f_2^{(1)}=i$ for $1\leqslant i\leqslant k-1$ (which again forces $f_2^{(2)}=0$). These three cases lead to the following three partial frequency arrays:

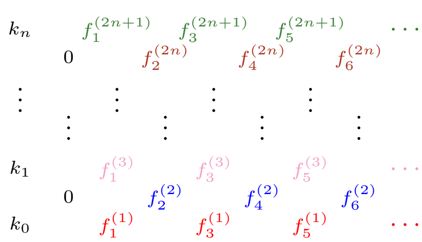


where in the second array it is assumed that $f_2^{(1)}<k$. The first two columns in the left-most array can be deleted without impacting the admissibility of any of the paths. Relabelling $f_i^{(c)}$ as $f_{i-2}^{(c)}$ for $i\geqslant 2$ this yields the contribution $(zq^2)^k\mathcal{A}^{(n)}_{k\Lambda_0}(zq^2)$ to the generating function. In the middle array, if we swap the positions of the encircled 0 and 1, no paths are affected except for paths of the form $(f_2^{(1)},0,0,\dots)$ where the second of the two zeros corresponds to the encircled zero. Such paths map to $(f_2^{(1)},0,1,\dots)$ by the swap, which is exactly what is needed to ensure

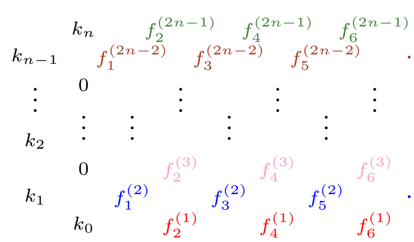
that $f_2^{(1)} < k$. After the swap, the first column can be deleted. Drawing the resulting array upside down and relabelling $f_i^{(c)}$ as $f_{i-1}^{(n-c+1)}$ for $i \geqslant 1$ and $1 \leqslant c \leqslant n$, yields the contribution $\mathcal{A}^{(n)}_{\Lambda_{n-1}+(k-1)\Lambda_n}(zq)$ to the generating function. Finally, in the right-most array we can replace the encircled triple $1, k-1, 1$ by $0, 0, k-i$. Prior to the replacement there are two types of paths through at least one of the vertices in the first two columns labelled $k-1$ and 1; paths of the form $(k-1, 1, 0, p_4, p_5, \dots)$ and paths of the form $(k-1, 1, \bar{p}_3, \bar{p}_4, \dots)$. Since the first two entries sum to k in both cases, this forces all the p_i and $\bar{p}_i$ to be zero. By the replacement, these paths become $(0, 0, 0, p_4, p_5, \dots)$ and $(0, 1, \bar{p}_3, \bar{p}_4, \dots)$, which imposes no constraints on the p_i and the minor constraint $\sum_i \bar{p}_i \leqslant k-1$ on the $\bar{p}_i$. However, prior to the replacement we also have the companion paths $(i, 1, 0, p_4, p_5, \dots)$ and $(i, 1, \bar{p}_4, \bar{p}_5, \dots)$ (imposing weaker constraints on the p_i and $\bar{p}_i$ than vanishing, unless $i = k-1$) which map to $(i, k-i, 0, p_4, p_5, \dots)$ and $(i, k-i, \bar{p}_3, \bar{p}_4, \dots)$, once again forcing all p_i and $\bar{p}_i$ to be zero. This justifies the above change in the encircled triple. After the change, the first two columns can again be deleted Once again relabelling $f_i^{(c)}$ as $f_{i-2}^{(c)}$ for $i \geqslant 2$ this yields the contribution $zq(zq^2)^i \mathcal{A}^{(n)}_{i\Lambda_0+(k-i)\Lambda_1}(zq^2)$ for each $0 \leqslant i \leqslant k-1$. Adding up all the various contributions to the generating function completes the proof. □

4. The CMPP conjectures for $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$

There is a second conjecture in the work of CMPP that was subsequently complemented by Primc in [38]. These two conjectures concern (a) partitions in $\mathcal{P}^{(n+1)}$ in which the parts of the $(n+1)$th colour are all odd ([12, Conjecture 3.3]) or (b) partitions in $\mathcal{P}^{(n)}$ in which the parts of the nth colour are all even ([38, Conjecture 2.1]). For reasons of symmetry we will adopt somewhat different coloured-partition models (along the lines of [16]) to describe these two additional conjectures, instead considering (a') partitions in $\mathcal{P}^{(2n+1)}$ in which all parts with an odd colour label are odd and all parts with an even colour label are even (i.e., $f_i^{(c)} = 0$ unless $c+i$ is even) and (b') partitions in $\mathcal{P}^{(2n-1)}$ in which all parts with an even colour label are odd and all parts with an odd colour label are even (i.e., $f_i^{(c)} = 0$ unless $c+i$ is odd). In the corresponding frequency arrays for these two sets of coloured partitions the "forbidden" $f_i^{(c)}$ are omitted rather than represented as zeros. For fixed nonnegative integers $k_0, \dots, k_n$, which are again to be viewed as initial or boundary conditions, the frequency arrays then take the form



(a')



(b')

where $n \geqslant 1$ in case (a') and $n \geqslant 2$ in case (b'). In order to test some of the conjectures below, we extend the above to $n = 0$ in case (a') and $n = 1$ in case (b'), both corresponding to one-row frequency arrays. In the case of (b') the correct labelling of the single left-boundary vertex is given by $k_0 + k_1$.

In case (a'), set $m := 2n+1$, $f_{-1}^{(2c+1)} := k_c$ for $0 \leqslant c \leqslant n$ and $f_0^{(2c)} := 0$ for $1 \leqslant c \leqslant n$. Similarly, in case (b'), set $m = 2n-1$, $f_{-1}^{(2c)} := k_c$ for $1 \leqslant c \leqslant n-1$ and $f_0^{(2c-1)} := k_0\delta_{c,1} + k_n\delta_{c,n}$ for $1 \leqslant c \leqslant n$. Then a path on the frequency array of type (a') (resp. (b')) is a sequence

$P=(p_1,p_2,\dots,p_m)=(f_{i_1}^{(1)},f_{i_2}^{(2)},\dots,f_{i_m}^{(m)})$ such that for all $1\leqslant c\leqslant m$, $i_c\geqslant -1$, i_c+c is even (resp. odd) and $|i_r-i_{r+1}|=1$. A coloured partition of type (a') or (b') is admissible if for all paths on its frequency array

$$\sum_{i=1}^m p_i\leqslant k_0+\dots+k_n.$$

In both cases, frequency arrays of coloured partitions exhibit a $\mathbb{Z}_2$-symmetry, corresponding to $f_i^{(c)}\mapsto f_i^{(m-c+1)}$. As we shall see shortly, this is a reflection of the diagram automorphisms of the Dynkin diagrams of the underlying affine Lie algebras: $\mathrm{C}_n^{(1)}$ for the coloured partitions of type (a') and $\mathrm{D}_{n+1}^{(2)}$ for the coloured partitions of type (b').

To state the further conjectures of CMPP and Primc we denote the set of all admissible partitions of type (a') (resp. (b)) by $\mathscr{C}_{k_0,\dots,k_n}$ (resp. $\mathscr{D}_{k_0,\dots,k_n}$).

Conjecture 4.1 (CMPP [12, Conjecture 3.3])**.** *For k,n nonnegative integers and $k_0,\dots,k_n$ nonnegative integers such that $k_0+\dots+k_n=k$,*

$$\begin{aligned}\sum_{\lambda\in\mathscr{C}_{k_0,\dots,k_n}} q^{|\lambda|}&=\frac{(q^{k+n+1};q^{2k+2n+2})_\infty(q^{2k+2n+2};q^{2k+2n+2})_\infty^n}{(q;q^2)_\infty(q;q)_\infty^n}\\&\quad\times\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};q^{k+n+1}\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};q^{2k+2n+2}\big),\end{aligned}$$

where $\lambda_i:=k_i+\dots+k_n$ for $1\leqslant i\leqslant n$.

For $k_0=k$ and $k_1=\dots=k_n=0$ this was first stated by Primc and Šikić in the form of a conjectural basis for the $\mathrm{C}_n^{(1)}$-standard module $L(k\Lambda_0)$, see [40, Conjecture 1] (and [39, Theorem 12.1] for the $L(\Lambda_0)$-case). Building on results from [7], this basis-conjecture was recently proved by Primc and Trupčević in [41, Theorem 2.1] by establishing a connection with Feigin–Stoyanovsky-type subspaces for $\mathrm{C}_{2n}^{(1)}$. Hence for this special case Conjecture (4.1) is now a theorem.

Once again CMPP do not give the product on the right in the above explicit form, instead using a description in terms of congruence triangles, see also [11]. Unlike Conjecture 1.2, however, CMPP do identify the product as the principally specialised character of the standard $\mathrm{C}_n^{(1)}$-module of highest weight $\Lambda=k_0\Lambda_0+\dots+k_n\Lambda_n$, i.e., as $\phi_n(\chi_\Lambda)$ in accordance with (2.4). Here $\mathrm{C}_1^{(1)}$ should be interpreted as $\mathrm{A}_1^{(1)}$. Specifically, Conjecture 4.1 for $n=1$ amounts to

$$\sum_{\lambda\in\mathscr{C}_{k_0,k_1}} q^{|\lambda|}=\frac{(q^{k_0+1},q^{k_1+1},q^{k_0+k_1+2};q^{k_0+k_1+2})_\infty}{(q;q^2)_\infty(q;q)_\infty},$$

where the right-hand side exactly is the principally specialised $\mathrm{A}_1^{(1)}$-standard module of highest weight $\Lambda=k_0\Lambda_0+k_1\Lambda_1$, see [31, Theorem 5.9]. Mapping the frequencies of the three-colour partitions to frequencies of two-colour partitions as follows:

$$\begin{array}{cccccccc} k_1 & & f_1^{(3)} & & f_3^{(3)} & & f_5^{(3)} & \cdots\\ & 0 & & f_2^{(2)} & & f_4^{(2)} & & f_6^{(2)}\\ k_0 & & f_1^{(1)} & & f_3^{(1)} & & f_5^{(1)} & \cdots\end{array}\quad\longmapsto\quad\begin{array}{cccccccc} k_1 & & f_1^{(1)} & & f_3^{(2)} & & f_5^{(1)} & \cdots\\ & 0 & & f_2^{(1)} & & f_4^{(1)} & & f_6^{(1)}\\ k_0 & & f_1^{(2)} & & f_3^{(1)} & & f_5^{(2)} & \cdots\end{array}$$

where the second (red) colour on the right has only odd parts and where the condition on the paths is unchanged, i.e., for every path $P=(p_1,p_2,p_3)$, $p_1+p_2+p_3\leqslant k_0+k_1$, we obtain the two-colour partition model of [37, Equations (11.2.10) & (11.2.11)]. By results of that same paper, this establishes Conjecture 4.1 for $n=1$. The case $n=0$, which has no Lie-algebraic interpretation, is even simpler. Since $\mathscr{C}_k$ is the set of ordinary partitions into odd parts such that no part occurs more than k times,

$$\sum_{\substack{\lambda\in\mathscr{C}_k \\ \lambda_1\leqslant 2\ell-1}} \Big(\frac{z}{q}\Big)^{l(\lambda)} q^{|\lambda|} = \frac{(z^{k+1};q^{2k+2})_\ell}{(z;q^2)_\ell}. \tag{4.1}$$

Taking the $\ell\to\infty$ limit and setting $z=q$, this implies that

$$\sum_{\lambda\in\mathscr{C}_k} q^{|\lambda|} = \frac{(q^{k+1};q^{2k+2})_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty(q^{k+1};q^{2k+2})_\infty}{(q;q)_\infty},$$

in accordance with Conjecture 4.1 for $n=0$.

A final extremal case of the conjecture is $k=1$, i.e., $k_i=\delta_{i,a}$ for some fixed $a\in\{0,1,\dots,n\}$. For such a, define

$$\mathscr{C}_a^{(n)}(N) := \mathscr{C}_{0^a,1,0^{n-a}}(N) \quad\text{and}\quad \mathscr{C}_a^{(n)} := \bigcup_{N\geqslant 0}\mathscr{C}_a^{(n)}(N).$$

Then

$$\sum_{\lambda\in\mathscr{C}_a^{(n)}} q^{|\lambda|} = \frac{(q^{2a+2},q^{2n-2a+2},q^{2n+4};q^{2n+4})_\infty}{(q;q)_\infty}.$$

For $a=0$ this is [39, Theorem 12.1] and for arbitrary a this follows from the coloured partition theorem [24, Theorem 1.2] of Jing et al. In particular, the correspondence between $\mathscr{C}_a^{(n)}$ and the set of coloured partitions $C_N(M,r)$ defined by Jing, Misra and Savage is

$$M=2n+4,\quad r=\begin{cases} 2a+2 & \text{for } 0\leqslant a\leqslant \lfloor n/2\rfloor, \\ 2n-2a+2 & \text{for } \lceil n/2\rceil\leqslant a\leqslant n,\end{cases}$$

where in the second case the colour labelling of [24] needs to be reversed. Two further proofs of Conjecture 4.1 for level-one modules have been found to date. In [16], Dousse and Konan employed the theory of perfect crystals to prove a formula for the full character of any level-one standard module of $\mathrm{C}_n^{(1)}$. Upon principal specialisation their result implies the CMPP formula for level one. A third proof using functional equations was obtained by the second author in [46]. If

$$C_a^{(n)}(z,q) := \sum_{\lambda\in\mathscr{C}_a^{(n)}} z^{l(\lambda)}q^{|\lambda|},$$

it follows from [46] that

$$C_a^{(n)}(z,q) = \begin{cases} F_{2a+1,0}^{(n+1)}(z,q) & \text{for } 0\leqslant a\leqslant\lfloor n/2\rfloor, \\ F_{2n-2a+1,0}^{(n+1)}(z,q) & \text{for } \lceil n/2\rceil\leqslant a\leqslant n,\end{cases} \tag{4.2}$$

with $F_{a,\delta}^{(n)}(z,q)$ defined in (3.6). This implies the claimed product for $z=1$ thanks to Bressoud's even modulus analogue of the Andrews–Gordon identities proven in [9, 10].

Primc's complement of Conjecture 4.1 may be stated as follows.

Conjecture 4.2 (Primc [38, Conjecture 2.1])**.** *For k a nonnegative integer, n a positive integer and $k_0,\dots,k_n$ nonnegative integers such that $k_0+\cdots+k_n=k$,*

$$\sum_{\lambda\in\mathscr{D}_{k_0,\dots,k_n}} q^{|\lambda|} = \frac{(q^{2k+2n};q^{2k+2n})_\infty^n}{(q^2;q^2)_\infty(q;q)_\infty^{n-1}} \prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+1};q^{2k+2n}\big),$$

where $\lambda_i:=k_i+\cdots+k_n$ for $1\leqslant i\leqslant n$.

Primc does not actually state what the product on the right should be. His conjecture simply is that the generating function of partitions in $\mathscr{D}_{k_0,\dots,k_n}$ such that $k_0+\dots+k_n=k$ can be expressed as an infinite product with modulus $2k+2n$.

Since $\mathscr{D}_{k_0,k_1}$ is the set of ordinary partitions into even parts such that no part occurs more than $k_0+k_1=k$ times,

$$\sum_{\substack{\lambda\in\mathscr{D}_{k_0,k_1}\\ \lambda_1\leqslant 2\ell}} \Big(\frac{z}{q^2}\Big)^{l(\lambda)} q^{|\lambda|}=\frac{(z^{k+1};q^{2k+2})_\ell}{(z;q^2)_\ell}.$$

Taking the $\ell\to\infty$ limit and setting $z=q^2$ gives

$$\sum_{\lambda\in\mathscr{D}_{k_0,k_1}} q^{|\lambda|}=\frac{(q^{2k+2};q^{2k+2})_\infty}{(q^2;q^2)_\infty},$$

proving Conjecture 4.2 for $n=1$. Unlike Conjecture 4.1 however, the case of three-row frequency arrays (i.e., $n=2$) appears to still be open.

For $a\in\{0,1,\dots,n\}$, let

$$\mathscr{D}_a^{(n)}(N):=\mathscr{D}_{0^a,1,0^{n-a}}(N) \quad\text{and}\quad \mathscr{D}_a^{(n)}:=\bigcup_{N\geqslant 0}\mathscr{D}_a^{(n)}(N).$$

Then Conjecture 4.2 for $k=1$ is

$$\sum_{\lambda\in\mathscr{D}_a^{(n)}} q^{|\lambda|}=\frac{(q^{2a+1},q^{2n-2a+1},q^{2n+2};q^{2n+2})_\infty}{(q;q)_\infty}, \tag{4.3}$$

which again follows from the coloured partition theorem [24, Theorem 1.2] of Jing et al. The correspondence between $\mathscr{D}_a^{(n)}$ and the set of coloured partitions $C_N(M,r)$ considered in [24] is

$$M=2n+2,\quad r=\begin{cases} 2a+1 & \text{for } 0\leqslant a\leqslant\lfloor n/2\rfloor,\\ 2n-2a+1 & \text{for } \lceil n/2\rceil\leqslant a\leqslant n,\end{cases}$$

where in the second case the colour labelling of [24] needs to be reversed. Alternatively, if

$$D_a^{(n)}(z,q):=\sum_{\lambda\in\mathscr{D}_a^{(n)}} z^{l(\lambda)}q^{|\lambda|},$$

then [46]

$$D_a^{(n)}(z,q)=\begin{cases} F_{2a,0}^{(n)}(z,q) & \text{for } 0\leqslant a\leqslant\lfloor n/2\rfloor,\\ F_{2n-2a,0}^{(n)}(z,q) & \text{for } \lceil n/2\rceil\leqslant a\leqslant n.\end{cases} \tag{4.4}$$

Specialising $z=1$ and again appealing to Bressoud's Rogers–Ramanujan-type identities for even moduli [9, 10] implies (4.3).

By Proposition A.2 with $\lambda_i=k_i+\dots+k_n$ for $0\leqslant i\leqslant n$, Primc's conjecture is expressible in terms of the affine Lie algebra $\mathrm{D}_{n+1}^{(2)}$ in the following manner, resolving one of the open problems from [38].

Conjecture 4.3. *For nonnegative integers $k_0,\dots,k_n$, let $L(\Lambda)$ be the $\mathrm{D}_{n+1}^{(2)}$-standard module of highest weight $\Lambda=k_0\Lambda_0+k_1\Lambda_1+\dots+k_n\Lambda_n\in P_+^{2(k_0+\dots+k_n)}$. Then*

$$\sum_{\lambda\in\mathscr{D}_{k_0,\dots,k_n}} q^{|\lambda|}=\varphi_n(\chi_\Lambda).$$

In the following a number of functional equations satisfied by the coloured partitions of type $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ will be considered. To describe these we define

$$\mathcal{C}^{(n)}_{k_0\Lambda_0+\cdots+k_n\Lambda_n}(z)=\mathcal{C}^{(n)}_{k_0\Lambda_0+\cdots+k_n\Lambda_n}(z,q):=\sum_{\lambda\in\mathscr{C}_{k_0,\dots,k_n}} z^{l(\lambda)}q^{|\lambda|},$$
$$\mathcal{D}^{(n)}_{k_0\Lambda_0+\cdots+k_n\Lambda_n}(z)=\mathcal{D}^{(n)}_{k_0\Lambda_0+\cdots+k_n\Lambda_n}(z,q):=\sum_{\lambda\in\mathscr{D}_{k_0,\dots,k_n}} z^{l(\lambda)}q^{|\lambda|},$$

so that $C_a^{(n)}(z,q)=\mathcal{C}^{(n)}_{\Lambda_a}(z,q)$ and $D_a^{(n)}(z,q)=\mathcal{D}^{(n)}_{\Lambda_a}(z,q)$. Since the set of "$\mathrm{D}_2^{(2)}$-partitions" $\mathscr{D}_{k_0,k_1}$ only depends on the sum of k_1 and k_2, this also applies to $\mathcal{D}^{(1)}_{k_0\Lambda_0+k_1\Lambda_1}(z)$. By the $\mathbb{Z}_2$-symmetry of the $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$-partitions,

$$\mathcal{C}^{(n)}_{k_0\Lambda_0+k_1\Lambda_1+\cdots+k_n\Lambda_n}(z)=\mathcal{C}^{(n)}_{k_n\Lambda_0+\cdots+k_1\Lambda_{n-1}+k_0\Lambda_n}(z), \tag{4.5a}$$
$$\mathcal{D}^{(n)}_{k_0\Lambda_0+k_1\Lambda_1+\cdots+k_n\Lambda_n}(z)=\mathcal{D}^{(n)}_{k_n\Lambda_0+\cdots+k_1\Lambda_{n-1}+k_0\Lambda_n}(z), \tag{4.5b}$$

reflecting the diagram automorphisms of the corresponding Dynkin diagrams.

Proposition 4.4. *For a,k,n integers such that $0\leqslant a\leqslant k$,*

$$\mathcal{C}^{(n)}_{a\Lambda_0+(k-a)\Lambda_n}(z)=\sum_{i=0}^{a}\sum_{j=0}^{k-a}(zq)^{i+j}\,\mathcal{D}^{(n+1)}_{i\Lambda_0+(a-i)\Lambda_1+(k-a-j)\Lambda_n+j\Lambda_{n+1}}(zq), \tag{4.6a}$$
$$\mathcal{D}^{(n+1)}_{a\Lambda_0+(k-a)\Lambda_{n+1}}(z)=\mathcal{C}^{(n)}_{a\Lambda_0+(k-a)\Lambda_n}(zq), \tag{4.6b}$$

where $n\geqslant 1$ in (4.6a) *and $n\geqslant 0$ in* (4.6b).

For $n=0$ the functional equation (4.6b) is independent of the choice of a. Moreover, for $n>0$ the range of a may be restricted to $0\leqslant a\leqslant\lfloor k/2\rfloor$ due to the symmetry (4.5).

For the low-rank cases $\mathcal{C}^{(1)}_{\Lambda}(z)$ and $\mathcal{D}^{(2)}_{\Lambda}(z)$ additional equations holds. Since by (4.6b)

$$\mathcal{C}^{(1)}_{a\Lambda_0+(k-a)\Lambda_1}(z)=\mathcal{D}^{(2)}_{a\Lambda_0+(k-a)\Lambda_2}(z/q), \tag{4.7}$$

it suffices to consider functional equations for $\mathcal{D}^{(2)}_{\Lambda}(z)$.

Proposition 4.5. *For a,b,k nonnegative integers such that $a+b\leqslant k-1$,*

$$\begin{aligned}\mathcal{D}^{(2)}_{a\Lambda_0+(k-a-b)\Lambda_1+b\Lambda_2}(z)&-\mathcal{D}^{(2)}_{(a+1)\Lambda_0+(k-a-b-1)\Lambda_1+b\Lambda_2}(z)\\&=\sum_{i=0}^{a}\sum_{j=0}^{k-a}(zq)^{k+i-a+\min\{0,j-b\}}q^{i+j}\mathcal{D}^{(2)}_{i\Lambda_0+(k-i-j)\Lambda_1+j\Lambda_2}(zq^2).\end{aligned} \tag{4.8}$$

Not all of the equations in Proposition 4.5 are linearly independent. Assuming that $a+b\leqslant k-2$ and taking the difference between (4.8) and that same equation with b replaced by $b+1$ gives

$$\begin{aligned}\sum_{i,j=0}^{1}&(-1)^{i+j}\mathcal{D}^{(2)}_{(i+a)\Lambda_0+(k-i-j-a-b)\Lambda_1+(j+b)\Lambda_2}(z)\\&=-(1-zq)(zq)^{k-a-b-1}\sum_{i=0}^{a}\sum_{j=0}^{b}(zq^2)^{i+j}\mathcal{D}^{(2)}_{i\Lambda_0+(k-i-j)\Lambda_1+j\Lambda_2}(zq^2).\end{aligned}$$

By (4.5) this is invariant under the interchange of a and b, whereas (4.8) is not. If, symbolically, we write the equation (4.8) as $e_{a,b}$, this implies that $e_{a,b} - e_{a,b+1} = e_{b,a} - e_{b,a+1}$. Hence, of the $\binom{k+1}{2}$ equations of the proposition, there are only

$$\binom{k+1}{2} - \sum_{\substack{0\leqslant a<b\\ a+b\leqslant k-2}} 1 = \binom{k+1}{2} - \left\lfloor \frac{(k-1)^2}{4}\right\rfloor = \left\lfloor \frac{(k+2)^2}{4}\right\rfloor - 1$$

linearly independent equations. This should be further combined with the $n=1$ case of Proposition 4.4, which by (4.7) may be stated as

$$\mathcal{D}^{(2)}_{a\Lambda_0+(k-a)\Lambda_2}(z) = \sum_{i=0}^{a}\sum_{j=0}^{k-a}(zq^2)^{i+j}\,\mathcal{D}^{(2)}_{i\Lambda_0+(k-i-j)\Lambda_1+j\Lambda_2}(zq^2) \tag{4.9}$$

for $0\leqslant a\leqslant k$. As mentioned previously, this gives a further $\lfloor k/2\rfloor+1$ independent equations. However, if we take the sum of (4.8) for $b=k-a-1$ and (4.9) with $a\mapsto a+1$, we find

$$\begin{aligned}
\mathcal{D}^{(2)}_{a\Lambda_0+\Lambda_1+(k-a-1)\Lambda_2}(z) &= (1+zq)\sum_{i=0}^{a}\sum_{j=0}^{k-a-1}(zq^2)^{i+j}\mathcal{D}^{(2)}_{i\Lambda_0+(k-i-j)\Lambda_1+j\Lambda_2}(zq^2)\\
&\quad+\sum_{i=0}^{a}(zq^2)^{k+i-a}\mathcal{D}^{(2)}_{i\Lambda_0+(a-i)\Lambda_1+(k-a)\Lambda_2}(zq^2)\\
&\quad+\sum_{i=0}^{k-a-1}(zq^2)^{i+a+1}\,\mathcal{D}^{(2)}_{(a+1)\Lambda_0+(k-a-i-1)\Lambda_1+i\Lambda_2}(zq^2)
\end{aligned}$$

for $0\leqslant a\leqslant k-1$. Since by (4.5) this is is invariant under the substitution $a\mapsto k-a-1$, we have an additional $\lfloor k/2\rfloor$ dependencies. Equations (4.8) and (4.9) combined thus give

$$\left\lfloor \frac{(k+2)^2}{4}\right\rfloor - 1 + \left(\left\lfloor\frac{k}{2}\right\rfloor+1\right) - \left\lfloor\frac{k}{2}\right\rfloor = \left\lfloor \frac{(k+2)^2}{4}\right\rfloor$$

linearly independent equations. This is the exact same number as weights of the form $k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ such that $k_0+k_1+k_2=k$ and $k_2\leqslant k_0$, allowing us the conclude the following result.

Lemma 4.6. *Let k be a positive integer. Subject to the initial conditions $\mathcal{D}^{(2)}_{\Lambda}(0)=1$, the functional equations* (4.8) *and* (4.9) *combined with the symmetry relation* (4.5) *uniquely determine the set of generating functions $\{\mathcal{D}^{(2)}_{\Lambda}(z,q)\}_{\Lambda\in P^{2k}_{+}}$, where P^{2k}_{+} is the set of level-$2k$ dominant integral weights of $\mathrm{D}^{(2)}_3$.*

Proof of Proposition 4.4. Since the proof is very similar to that of Proposition 3.2, we will only give a minimal amount of detail.

A necessary condition for λ to be in $\mathscr{C}_{a,0^{n-1},k-a}$ is for the first five columns of its frequency array to be of the form as shown in the following (partial) frequency array on the left:

$$
\begin{array}{ccccc}
k-a & & j & & f_3^{(2n+1)} \\
& 0 & & f_2^{(2n)} & \\
0 & & 0 & & f_3^{(2n-1)} \\
& 0 & & f_2^{(2n-2)} & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
& 0 & & f_2^{(4)} & \\
0 & & 0 & & f_3^{(3)} \\
& 0 & & f_2^{(2)} & \\
a & & i & & f_3^{(1)}
\end{array}
\quad \mapsto \quad
\begin{array}{ccccc}
0 & & j & & f_3^{(2n+1)} \\
& k-a-j & & f_2^{(2n)} & \\
0 & & 0 & & f_3^{(2n-1)} \\
& 0 & & f_2^{(2n-2)} & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
& 0 & & f_2^{(4)} & \\
0 & & 0 & & f_3^{(3)} \\
& a-i & & f_2^{(2)} & \\
0 & & i & & f_3^{(1)}
\end{array}
$$

where, $i \in \{0, 1, \dots, a\}$ and $j \in \{0, 1, \dots, k-a\}$. The exact same set of admissible paths arises by replacing this by the (partial) frequency array shown on the right. Now eliminating the first column and relabelling $f_i^{(c)}$ (for $i+c$ even) as $f_{i-1}^{(c)}$, we end up with a frequency array of a $\mathrm{D}_{n+2}^{(2)}$-partition λ in $\mathscr{D}_{i,a-i,0,\dots,0,k-a-j,j}$. The contribution of the full set of arrays of this form to the generating function is

$$(zq)^{i+j}\mathcal{D}^{(n+1)}_{i\Lambda_0+(a-i)\Lambda_1+(k-a-j)\Lambda_n+j\Lambda_{n+1}}(zq).$$

Adding all the contributions from $i \in \{0, 1, \dots, a\}$ and $j \in \{0, 1, \dots, k-a\}$ results in the right-hand side of (4.6a). The $n = 1$ case of the above proof requires some additional justification since in the frequency array on the right of the above figure the two vertices in the second column labelled $a-i$ and $k-a-j$ coincide. The obvious interpretation would be to simply sum these two labels to give $k-i-j$. This, however, is a-independent and would therefore no longer require that $i \leqslant a$ or $j \leqslant k-a$. Hence it would lead to a larger range of admissible values for i and j. Given that we have summed the contributions of frequency arrays of the type shown on the right over $0 \leqslant i \leqslant a$ and $0 \leqslant j \leqslant k-a$ irrespective of the value of n, the proof and hence the functional equation (4.6a) holds for all $n \geqslant 1$.

The functional equation (4.6b) is trivial since the frequency array of an admissible $\mathrm{D}_{n+2}^{(2)}$-partition such that $k_0 = a$, $k_1 = \dots = k_n = 0$ and $k_{n+1} = k-a$ has only zeros in its first and third columns. Deleting the first column gives the frequency array of an $\mathrm{C}_n^{(1)}$-partition such that $k_0 = a$, $k_1 = \dots = k_{n-1} = 0$ and $k_n = k-a$. □

Proof of Proposition 4.5. We use $c := k-a-b$ in the proof to better fit some of the vertex labels used in the diagrams. The first four columns of the frequency arrays of partitions contributing to $\mathcal{D}^{(2)}_{a\Lambda_0+c\Lambda_1+b\Lambda_2}(z)$ and $\mathcal{D}^{(2)}_{(a+1)\Lambda_0+(c-1)\Lambda_1+b\Lambda_2}(z)$, respectively, take the form

$$
\begin{array}{cccc}
& b & & j \\
c & & l & \\
& a & & i
\end{array}
\qquad \text{and} \qquad
\begin{array}{cccc}
& b & & j \\
c-1 & & l & \\
& a+1 & & i
\end{array}
\tag{4.10}
$$

where the triple $(f_1^{(2)}, f_2^{(1)}, f_2^{(3)})$ has been replaced by (l, i, j). For admissibility, the array on the left requires that

$$i, j \geqslant 0, \quad i+l \leqslant k-b, \quad j+l \leqslant k-a, \quad 0 \leqslant l \leqslant c, \quad i+j+l \leqslant k,$$

and the array on the right that

$$i, j \geqslant 0, \quad i+l \leqslant k-b, \quad j+l \leqslant k-a-1, \quad 0 \leqslant l \leqslant c-1, \quad i+j+l \leqslant k.$$

Hence, in taking the difference between $\mathcal{D}^{(2)}_{a\Lambda_0+c\Lambda_1+b\Lambda_2}(z)$ and $\mathcal{D}^{(2)}_{(a+1)\Lambda_0+(c-1)\Lambda_1+b\Lambda_2}(z)$, the only arrays that contribute are those on the left of the above figure with

$$i,j\geqslant 0,\quad i+l\leqslant k-b,\quad j+l<k-a,\quad l=c,\quad i+j+l\leqslant k. \tag{4.11a}$$

or

$$i,j\geqslant 0,\quad i+l\leqslant k-b,\quad j+l=k-a,\quad 0\leqslant l\leqslant c,\quad i+j+l\leqslant k. \tag{4.11b}$$

The first set of inequalities may be simplified to

$$0\leqslant i\leqslant a,\quad 0\leqslant j<b,\quad l=c.$$

In terms of frequency arrays this may by symbolically written as

$$\sum_{i=0}^{a}\sum_{j=0}^{b-1}\begin{array}{cccc} & b & & j \\ c & & c & \\ & a & & i \end{array}$$

Now dropping the first two columns and replacing the c in the third column by $k-i-j$, so that the first two columns of the new array sum to k, this yields

$$\sum_{i=0}^{a}\sum_{j=0}^{b-1}\begin{array}{cc} & j \\ k-i-j & \\ & i \end{array}$$

Since, by the statement of the lemma, $a+b\leqslant k-1$ it follows that $k-i-j\geqslant 0$ as required. The contribution to the generating function of (4.11a) is thus

$$\sum_{i=0}^{a}\sum_{j=0}^{b-1}(zq)^{c+i+j}q^{i+j}\mathcal{D}^{(2)}_{i\Lambda_0+(k-i-j)\Lambda_1+j\Lambda_2}(zq^2).$$

Similarly, the inequalities (4.11b) may be simplified to

$$0\leqslant i\leqslant a,\quad b\leqslant j\leqslant k-a,\quad l=k-a-j.$$

In terms of frequency arrays this may by symbolically written as

$$\sum_{i=0}^{a}\sum_{j=b}^{k-a}\begin{array}{cccc} & b & & j \\ c & & k-a-j & \\ & a & & i \end{array}$$

Deleting the first two columns and replacing $k-a-j$ by $k-i-j$, so that once again the first two columns of the new array sum to k, this yields

$$\sum_{i=0}^{a}\sum_{j=b}^{k-a}\begin{array}{cc} & j \\ k-i-j & \\ & i \end{array}$$

Since, by the statement of the lemma, $a+b\leqslant k-1$ it follows that $k-i-j\geqslant 0$ as required. The contribution to the generating function of (4.11b) therefore is

$$\sum_{i=0}^{a}\sum_{j=b}^{k-a}(zq)^{(k-a-j)+i+j}q^{i+j}\mathcal{D}^{(2)}_{i\Lambda_0+(k-i-j)\Lambda_1+j\Lambda_2}(zq^2).$$

Adding up both contributions results in the right-hand side of (4.8). ☐

The analogue of Conjecture 1.4 for $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ is given by the following pair of identities.

Conjecture 4.7. *For k a nonnegative integer,*

$$\mathcal{C}_{k\Lambda_0}^{(n)}(z,q)=\sum_{\lambda\in\mathscr{C}_{k,0^n}} z^{l(\lambda)}q^{|\lambda|}=\sum_{\substack{\lambda\\ \lambda_1\leqslant k}}(zq)^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n}\big) \tag{4.12a}$$

and

$$\mathcal{D}_{k\Lambda_0}^{(n)}(z,q)=\sum_{\lambda\in\mathscr{D}_{k,0^n}} z^{l(\lambda)}q^{|\lambda|}=\sum_{\substack{\lambda\\ \lambda_1\leqslant k}}(zq^2)^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-2}\big), \tag{4.12b}$$

where in (4.12a) *it is assumed that $n\geqslant 0$ and in* (4.12b) *that $n\geqslant 1$.*

Proposition 4.8. *Equation* (4.12a) *holds for $n=0$,* (4.12b) *holds for $n=1$, and both equations hold for $k=1$. Moreover,* (4.12a) *holds for $z=1$.*

Proof. By the functional equation (4.6b) for $a=k$, it is enough to establish the claims pertaining to (4.12a).

We first will show that (4.12a) holds for $n=0$. By the $t=1$ case of [52, Theorem 1.2],

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant k}} m_{2\lambda}(x_1,\dots,x_\ell)=\sum_{\varepsilon_1,\dots,\varepsilon_\ell\in\{\pm1\}}\prod_{i=1}^{\ell}\frac{x_i^{k(1-\varepsilon_i)}}{1-x_i^{2\varepsilon_i}}=\prod_{i=1}^{\ell}\frac{1-x_i^{2k+2}}{1-x_i^2}.$$

Hence

$$\sum_{\substack{\lambda\in\mathscr{C}_k\\ \lambda_1\leqslant 2\ell-1}}\prod_{i=1}^{\ell}x_i^{2f_{2i-1}}=\sum_{\substack{\lambda\\ \lambda_1\leqslant k}} m_{2\lambda}(x_1,\dots,x_\ell),$$

where $f_i:=f_i^{(1)}$. Specialising $x_i=(zq^{2i-1})^{1/2}$ and using the homogeneity of the monomial symmetric functions yields

$$\sum_{\substack{\lambda\in\mathscr{C}_k\\ \lambda_1\leqslant 2\ell-1}} z^{l(\lambda)}q^{|\lambda|}=\sum_{\substack{\lambda\\ \lambda_1\leqslant k}}(zq)^{|\lambda|}m_{2\lambda}\big(1,q,\dots,q^{\ell-1}\big).$$

Since $P_\lambda(1)=m_\lambda$, this is a bounded version of (4.12a) for $n=0$.

To prove (4.12a) for $k=1$ we note that by Proposition 2.1 with $\delta=0$ we have

$$\sum_{r=0}^{\infty}(zq)^r P_{(2^r)}\big(1,q,q^2,\dots;q^{2n}\big)=F_{1,0}^{(n+1)}(z,q).$$

By (4.2) for $a=0$ this is equal to $C_0^{(n)}(z,q)=\mathcal{C}_{\Lambda_0}^{(n)}(z,q)$.

Finally, the $z=1$ case of (4.12a) follows by combining [21, Theorem 1.2] with the recent Primc–Trupčević proof of Conjecture 4.1 for $k_0=k$ and $k_2=\dots=k_n=0$. □

By (2.12), the $n=2$ instance of (4.12b) admits an alternative expression as a multisum. We conjecture that similar such multisums hold for $\mathcal{D}_{k\Lambda_1}^{(2)}(z,q)$ and $\mathcal{D}_{\Lambda_0+(k-1)\Lambda_1}^{(2)}(z,q)$.

Conjecture 4.9. *For k a nonnegative integer,*

$$\mathcal{D}^{(2)}_{k\Lambda_0}(z,q)=\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}}\prod_{i=1}^{k}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2+r_i+2s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}},$$

$$\mathcal{D}^{(2)}_{k\Lambda_1}(z,q)=\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}}\prod_{i=1}^{k}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}},$$

$$\mathcal{D}^{(2)}_{\Lambda_0+(k-1)\Lambda_1}(z,q)=\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}}\Omega^{s_1,\dots,s_k}_{r_1,\dots,r_k}(q)\prod_{i=1}^{k}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}},$$

where

$$\Omega^{s_1,\dots,s_k}_{r_1,\dots,r_k}(q):=\sum_{i=1}^{k-1}q^{r_i+2s_i}(1-q^{2s_{i+1}-2s_i})+q^{r_k+2s_k}$$

and $r_{k+1}=s_0:=0$.

Combining this with (4.7) and Conjectures 4.1 and 4.2 gives the following conjectural Andrews–Gordon-type identities:

$$\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}}\prod_{i=1}^{k}\frac{q^{(r_i+s_i)^2+s_i^2+s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}=\frac{(q,q^{k+1},q^{k+2};q^{k+2})_\infty}{(q;q^2)_\infty(q;q)_\infty},$$

$$\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}}\prod_{i=1}^{k}\frac{q^{(r_i+s_i)^2+s_i^2+r_i+2s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}=\frac{(q,q^2,q^{2k+2},q^{2k+3},q^{2k+4},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty(q;q)_\infty},$$

$$\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}}\prod_{i=1}^{k}\frac{q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}=\frac{(q^{k+1},q^{k+2},q^{k+2},q^{k+3},q^{2k+4},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty(q;q)_\infty}$$

and

$$\sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}}\Omega^{s_1,\dots,s_k}_{r_1,\dots,r_k}(q)\prod_{i=1}^{k}\frac{q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}$$
$$=\frac{(q^k,q^{k+1},q^{k+3},q^{k+4},q^{2k+4},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty(q;q)_\infty}.$$

Conjecture 4.9 for $k=2$ can be completed to a full set of weights, which is provable using the functional equations for $\mathcal{D}^{(2)}_{\Lambda}$.

Theorem 4.10. *We have*

$$\mathcal{D}^{(2)}_{2\Lambda_0}(z,q)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^{2}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2+r_i+2s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}},$$

$$\mathcal{D}^{(2)}_{2\Lambda_1}(z,q)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^{2}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}},$$

$$\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_1}(z,q)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^{2}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2+\delta_{i,2}(r_i+2s_i)}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}\Big(1+zq^{2+\sum_i(r_i+2s_i)}\Big),$$

$$\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_2}(z,q)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^{2}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2+r_i+2\delta_{i,2}s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}\Big(1+zq^{2+\sum_i(r_i+2s_i)}\Big),$$

where $r_3=s_0:=0$.

The expression for $\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_1}$ agrees with the one given in Conjecture 4.9. Indeed, splitting the above multisum for $\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_1}$ into two multisums in the obvious manner, then replacing $s_2\mapsto s_2-1$ in the second multisum, and finally recombining the two summations, yields

$$\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_1}(z,q)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^{2}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}\Big(q^{r_2+2s_2}+q^{r_1+2s_1}(1-q^{2s_2-2s_1})\Big).$$

In the same manner it may be shown that

$$\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_2}(z,q)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^{2}\frac{z^{r_i+s_i}q^{(r_i+s_i)^2+s_i^2+\delta_{i,1}r_1}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}\Big(q^{r_2+2s_2}+q^{r_1+2s_1}(1-q^{2s_2-2s_1})\Big).$$

Proof. For brevity, we denote

$$A(z)=\mathcal{D}^{(2)}_{2\Lambda_0}(z),\quad B(z)=\mathcal{D}^{(2)}_{2\Lambda_1}(z),\quad C(z)=\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_1}(z),\quad D(z)=\mathcal{D}^{(2)}_{\Lambda_0+\Lambda_2}(z),$$

where dependence on q has been suppressed. From the set of functional equations for $k=2$, we then choose the following four linearly independent equations:

$$\begin{aligned}A(z)&=B(zq^2)+zq^2C(zq^2)+z^2q^4A(zq^2),\\ D(z)&=B(zq^2)+2zq^2C(zq^2)+z^2q^4D(zq^2),\\ B(z)-C(z)&=z^2q^2B(zq^2)+z^2q^3C(zq^2)+z^2q^4A(zq^2),\\ C(z)-D(z)&=zqB(zq^2)+z^2q^3C(zq^2)+z^2q^4A(zq^2).\end{aligned}$$

The first two equations are (4.9) with $(a,k)=(2,2)$ and $(1,2)$ respectively, and the last two are (4.8) with $(a,b,k)=(0,0,2)$ and $(0,1,2)$. Here we have also applied the symmetry (4.5) to eliminate occurrences of the weights $\Lambda_1+\Lambda_2$ and $2\Lambda_2$. Because it will subsequently lead to a significant reduction in the number of terms, we subtract the first equation from the second and the fourth equation from the third, so that second and fourth equations are replaced by

$$\begin{aligned}D(z)-A(z)&=zq^2C(zq^2)+z^2q^4\big(D(zq^2)-A(zq^2)\big),\\ B(z)-2C(z)+D(z)&=-zq(1-zq)B(zq^2).\end{aligned}$$

Our aim is to show that the purported sums in Theorem (4.10) are the (unique) solutions to the above system of equations. However, instead of solving these equations in $\mathbb{Q}(q)[[z]]$ subject to the initial condition $f(0)=1$ for all $f\in\{A,B,C,D\}=:\mathcal{S}$, we will solve the w-deformed equations

$$0=A(z,w)-B(zq^2,wq^2)-zq^2C(zq^2,wq^2)-z^2q^4A(zq^2,wq^2), \tag{4.13a}$$

$$0=D(z,w)-A(z,w)-wq^2C(zq^2,wq^2)-z^2q^4\big(D(zq^2,wq^2)-A(zq^2,wq^2)\big), \tag{4.13b}$$

$$0=B(z,w)-\Big(1+\frac{w}{z}\Big)C(z,w)+\frac{w}{z}D(z,w)+wq\Big(1-\frac{z^2q}{w}\Big)B(zq^2,wq^2), \tag{4.13c}$$

$$0=C(z,w)-D(z,w)-zqB(zq^2,wq^2)-z^2q^3C(zq^2,wq^2)-zwq^4A(zq^2,wq^2), \tag{4.13d}$$

in $\mathbb{Q}(q)[[z,w]]$ subject to the initial conditions $f(0,0)=1$ for all $f\in\mathcal{S}$. If $f(z):=f(z,z)$ for $f\in\mathcal{S}$, then $\{f(z):f\in\mathcal{S}\}$ will satisfy the undeformed equations, obtained by setting $w=z$ in (4.13). Our claim is now that this system of equations is solved by

$$A(z,w)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^2\frac{z^{r_i}w^{s_i}q^{(r_i+s_i)^2+s_i^2+r_i+2s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}, \tag{4.14a}$$

$$B(z,w)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^2\frac{z^{r_i}w^{s_i}q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}, \tag{4.14b}$$

$$C(z,w)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^2\frac{z^{r_i}w^{s_i}q^{(r_i+s_i)^2+s_i^2+\delta_{i,2}(r_i+2s_i)}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}\Big(1+wq^{2+\sum_i(r_i+2s_i)}\Big), \tag{4.14c}$$

$$D(z,w)=\sum_{\substack{r_1,r_2\geqslant 0\\ s_1,s_2\geqslant 0}}\prod_{i=1}^2\frac{z^{r_i}w^{s_i}q^{(r_i+s_i)^2+s_i^2+r_i+2\delta_{i,2}s_i}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}\Big(1+wq^{2+\sum_i(r_i+2s_i)}\Big). \tag{4.14d}$$

Clearly, each of the four functions trivialises to 1 for $z=w=0$, as required.

To prove our claim, we adopt the computer-assisted procedure of [27] (see also [13]). For this it will be convenient to define

$$\begin{aligned}S_{k_1,k_2,\ell_1,\ell_2}&=S_{k_1,k_2,\ell_1,\ell_2}(z,w)\\&:=\sum_{\substack{m_1,m_2\geqslant 0\\ n_1,n_2\geqslant 0}}\frac{z^{M_1+M_2}w^{N_1+N_2}q^{(M_1+N_2)^2+(M_2+N_1)^2+N_1^2+N_2^2+k_1m_1+k_2m_2+2\ell_1n_1+2\ell_2n_2}}{(q;q)_{m_1}(q;q)_{m_2}(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}},\end{aligned}$$

where $k_1,k_2,\ell_1,\ell_2\in\mathbb{Z}$, $M_i=m_i+\cdots+m_2$, $N_i:=n_i+\cdots+n_2$ and where dependency on q is still being suppressed. There is some redundancy in this definition since

$$S_{k_1,k_2,\ell_1,\ell_2}(zq^m,wq^{2n})=S_{k_1+m,k_2+2m,\ell_1+n,\ell_2+2n}(z,w).$$

In terms of the function S, the claimed expressions (4.14) take the form

$$\begin{aligned}A(z,w)&=S_{1,2,1,2}(z,w), &\qquad C(z,w)&=S_{0,1,1,1}(z,w)+wq^2\,S_{1,3,2,3}(z,w),\\ B(z,w)&=S_{0,0,0,0}(z,w), &\qquad D(z,w)&=S_{1,2,1,1}(z,w)+wq^2\,S_{2,4,2,3}(z,w).\end{aligned}$$

We substitute this into (4.13) in which z, w have been replaced by $z/q, w/q^2$ in (4.13a), and z, w has been replaced by $z/q, w$ in (4.13b). The resulting four equations are

$$0 = S_{0,0,0,0} - S_{1,2,0,0} - zq\big(S_{1,3,1,1} + wq^2\, S_{2,5,2,3}\big) - z^2q^2\, S_{2,4,1,2}, \tag{4.15a}$$

$$\begin{aligned} 0 &= S_{0,0,1,1} + wq^2\, S_{1,2,2,3} - S_{0,0,1,2} - wq^2\big(S_{1,3,2,3} + wq^4\, S_{2,5,3,5}\big) \\ &\quad - z^2q^2\big(S_{2,4,2,3} + wq^4\, S_{3,6,3,5} - S_{2,4,2,4}\big), \end{aligned} \tag{4.15b}$$

$$\begin{aligned} 0 &= S_{0,0,0,0} - \Big(1 + \frac{w}{z}\Big)\big(S_{0,1,1,1} + wq^2\, S_{1,3,2,3}\big) \\ &\quad + \frac{w}{z}\big(S_{1,2,1,1} + wq^2\, S_{2,4,2,3}\big) + wq\Big(1 - \frac{z^2q}{w}\Big)S_{2,4,1,2}, \end{aligned} \tag{4.15c}$$

$$\begin{aligned} 0 &= S_{0,1,1,1} + wq^2\, S_{1,3,2,3} - S_{1,2,1,1} - wq^2\, S_{2,4,2,3} \\ &\quad - zq\, S_{2,4,1,2} - z^2q^3\big(S_{2,5,2,3} + wq^4\, S_{3,7,3,5}\big) - zwq^4\, S_{3,6,2,4}. \end{aligned} \tag{4.15d}$$

It is easily deduced that the function S satisfies the following four atomic relations:

$$\begin{aligned} R^{(1)}_{k_1,k_2,\ell_1,\ell_2} &:= S_{k_1,k_2,\ell_1,\ell_2} - S_{k_1+1,k_2,\ell_1,\ell_2} - zq^{k_1+1}\, S_{k_1+2,k_2+2,\ell_1,\ell_2+1} = 0, \\ R^{(2)}_{k_1,k_2,\ell_1,\ell_2} &:= S_{k_1,k_2,\ell_1,\ell_2} - S_{k_1,k_2+1,\ell_1,\ell_2} - z^2q^{k_2+2}\, S_{k_1+2,k_2+4,\ell_1+1,\ell_2+2} = 0, \\ R^{(3)}_{k_1,k_2,\ell_1,\ell_2} &:= S_{k_1,k_2,\ell_1,\ell_2} - S_{k_1,k_2,\ell_1+1,\ell_2} - wq^{2\ell_1+2}\, S_{k_1,k_2+2,\ell_1+2,\ell_2+2} = 0, \\ R^{(4)}_{k_1,k_2,\ell_1,\ell_2} &:= S_{k_1,k_2,\ell_1,\ell_2} - S_{k_1,k_2,\ell_1,\ell_2+1} - w^2q^{2\ell_2+4}\, S_{k_1+2,k_2+4,\ell_1+2,\ell_2+4} = 0 \end{aligned}$$

for all $k_1, k_2, \ell_1, \ell_2 \in \mathbb{Z}$. We now show that the equations (4.15a)–(4.15a) are in the linear span over $\mathbb{Q}(z, w, q)$ of a finite subset of $\{R^{(i)}_{k_1,k_2,\ell_1,\ell_2}\}_{k_1,k_2,\ell_1,\ell_2\in\mathbb{Z},i\in 1,2,3,4}$ First, (4.15a) is the same as

$$R^{(1)}_{0,1,0,0} - zqR^{(1)}_{1,3,1,1} + R^{(2)}_{0,0,0,0} + R^{(2)}_{1,1,0,0} + zqR^{(3)}_{2,3,0,1}.$$

Similarly, (4.15b) is

$$R^{(2)}_{0,0,1,1} - R^{(2)}_{0,0,1,2} + wq^2R^{(2)}_{1,2,2,3} + R^{(4)}_{0,1,1,1}.$$

The linear combination for (4.15c) is by far the most involved of the four cases:

$$\begin{aligned} &\frac{w^2}{z^3}R^{(1)}_{0,0,1,1} + R^{(1)}_{0,1,0,1} - \Big(1 + \frac{w^2}{z^3}\Big)R^{(1)}_{0,1,1,1} + \frac{w^2q}{z^2}R^{(1)}_{0,1,1,2} \\ &\quad - \frac{w}{z}\Big(1 + \frac{wq}{z}\Big)R^{(1)}_{0,2,1,2} - \frac{w}{z}\big(R^{(1)}_{1,1,0,1} - R^{(1)}_{1,1,1,1}\big) - \frac{w^2q^2}{z}R^{(1)}_{2,4,1,3} \\ &\quad + \Big(1 - \frac{w}{z^2q}\Big)R^{(2)}_{0,0,0,0} + \frac{w}{z^2q}R^{(2)}_{0,0,0,1} - \frac{w^2}{z^3}\big(R^{(2)}_{0,0,1,1} - R^{(2)}_{1,0,1,1}\big) \\ &\quad - \frac{w}{z}\Big(1 + \frac{wq}{z}\Big)R^{(2)}_{0,1,1,2} + \frac{w^2q}{z^2}\big(R^{(2)}_{1,1,1,2} + R^{(2)}_{2,2,0,2}\big) \\ &\quad + \Big(1 + \frac{w}{z}\Big)R^{(3)}_{1,1,0,1} - \frac{w}{z}R^{(3)}_{2,1,0,1} - \frac{w^2q}{z^2}R^{(3)}_{2,2,0,2} + \Big(1 + \frac{w^2}{z^3}\Big)zqR^{(3)}_{2,3,0,2} \\ &\quad + \frac{w^2q^2}{z}\big(R^{(3)}_{2,3,1,3} - R^{(3)}_{3,4,1,3}\big) - wq^2R^{(3)}_{3,3,0,2} \\ &\quad + \frac{w}{z^2q}R^{(4)}_{0,0,0,0} + \Big(1 - \frac{w}{z^2q}\Big)R^{(4)}_{0,1,0,0} - \frac{w}{z}\big(R^{(4)}_{0,1,1,1} - R^{(4)}_{1,2,1,1}\big). \end{aligned}$$

Finally, (4.15d) is

$$\begin{aligned} &R^{(1)}_{0,1,1,1} + R^{(1)}_{1,2,0,1} - R^{(1)}_{1,2,1,1} - z^2q^3R^{(1)}_{2,5,2,3} + R^{(2)}_{1,1,0,1} + zqR^{(2)}_{2,3,1,2} \\ &\quad - R^{(3)}_{1,1,0,1} + R^{(3)}_{2,2,0,1} + zq^2R^{(3)}_{3,4,0,2} + z^2q^3R^{(3)}_{3,5,1,3}. \end{aligned}$$

□

We conclude this section with a comment on the functional equations (4.13) and their solution (4.14). From this solution and the remark preceding Proposition 3.2 it follows that $A(z,0,q) = D(z,0,q) = \mathscr{A}^{(1)}_{2\Lambda_0}(z,q)$, $B(z,0,q) = \mathscr{A}^{(1)}_{2\Lambda_1}(z,q)$ and $C(z,0,q) = \mathscr{A}^{(1)}_{\Lambda_0+\Lambda_1}(z,q)$. Combinatorially this means that by setting $w=0$ those partitions in $\mathscr{D}_{k_0,k_1,k_2}$ that have a nonzero entry in the top row of their frequency array are eliminated. The resulting set of partitions is in one-to-one correspondence with $\mathscr{A}_{k_0,k_1}$. Accordingly, it is not hard to see that the $w=z$ case of (4.13) is equivalent to the Rogers–Selberg equations (3.8) for $k=2$. Interestingly, these same results with (z,q) replaced by (w,q^2) arise as a further special case since $A(0,w,q) = \mathscr{A}^{(1)}_{2\Lambda_0}(w,q^2)$, $B(0,w,q) = \mathscr{A}^{(1)}_{2\Lambda_1}(w,q^2)$ and $C(0,w,q) = D(0,w,q) = \mathscr{A}^{(1)}_{\Lambda_0+\Lambda_1}(w,q^2)$. The corresponding simplification of (4.13) is, up to the trivial $\mathscr{A}^{(1)}_{2\Lambda_0}(w,q^2) = \mathscr{A}^{(1)}_{2\Lambda_1}(wq^2,q^2)$, identical to the Rogers–Selberg equations with $(z,q) \mapsto (w,q^2)$. We do not yet know what the appropriate set of w-deformed equations is for $k \geqslant 3$ with the exception of the two equations

$$\mathcal{D}^{(2)}_{k\Lambda_0}(z,w) = \sum_{i=0}^{k} (zq^2)^i \, \mathcal{D}^{(2)}_{i\Lambda_0+(k-i)\Lambda_1}(zq^2,wq^2)$$

and

$$\mathcal{D}^{(2)}_{k\Lambda_1}(z,w) - \mathcal{D}^{(2)}_{\Lambda_0+(k-1)\Lambda_1}(z,w) = (zq)^k \sum_{i=0}^{k} \Big(\frac{wq}{z}\Big)^i \, \mathcal{D}^{(2)}_{i\Lambda_0+(k-i)\Lambda_1}(zq^2,wq^2).$$

For $\mathcal{D}^{(2)}_{(k-a)\Lambda_0+a\Lambda_1}(z,w) \in \mathbb{Q}(q)[[z,w]]$ these equations have the property

$$\begin{aligned}\mathcal{D}^{(2)}_{a\Lambda_0+(k-a)\Lambda_1}(z,0,q) &= \mathcal{A}^{(1)}_{a\Lambda_0+(k-a)\Lambda_1}(z,q)\\ \mathcal{D}^{(2)}_{a\Lambda_0+(k-a)\Lambda_1}(0,w,q) &= \mathcal{A}^{(1)}_{a\Lambda_0+(k-a)\Lambda_1}(w,q^2).\end{aligned}$$

A reasonable guess is thus that

$$\begin{aligned}\mathcal{D}^{(2)}_{k\Lambda_0}(z/q,w/q^2,q) &= \mathcal{D}^{(2)}_{k\Lambda_1}(z,w,q)\\ &= \sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}} \prod_{i=1}^{k} \frac{z^{r_i} w^{s_i} q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}}\\ \mathcal{D}^{(2)}_{\Lambda_0+(k-1)\Lambda_1}(z,w,q) &= \sum_{\substack{r_1,\dots,r_k\geqslant 0\\ s_1,\dots,s_k\geqslant 0}} \Omega^{s_1,\dots,s_k}_{r_1,\dots,r_k}(q) \prod_{i=1}^{k} \frac{z^{r_i} w^{s_i} q^{(r_i+s_i)^2+s_i^2}}{(q;q)_{r_i-r_{i+1}}(q^2;q^2)_{s_i-s_{i-1}}},\end{aligned}$$

where $r_{k+1} = s_0 := 0$. Indeed, for $w=0$ this yields the multisums on the right of (1.7) for $a=0$, $a=k$ and, since $\Omega^{0,\dots,0}_{r_1,\dots,r_k}(q) = q^{r_k}$, $a=k-1$. Similarly, for $z=0$ it gives (1.7) with $(z,q)\mapsto(w,q^2)$ for $a=0$, $a=k$ and, since $\Omega^{s_1,\dots,s_k}_{s_1,\dots,s_k}(q) = q^{2s_1}$, $a=k-1$. The above series thus have the structure of two interwoven copies of the Andrews–Gordon multisums. It is an open problem to find the w,z-generalisations of the Andrews–Gordon multisums (1.7) for $1 \leqslant a \leqslant k-2$. Of course, the correct such sum should equate to

$$\frac{(q^{a+1},q^{a+2},q^{2k-a+2},q^{2k-a+3},q^{2k+4},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty(q;q)_\infty}$$

when $z=w=1$.

5. Towards completing Conjectures 1.4 and 4.7

Recall the definition of the bivariate generating function for coloured partitions of type $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$:

$$\mathcal{G}_\Lambda^{(n)}(z,q) := \sum_{\lambda\in\mathscr{G}_{k_0,\dots,k_n}} z^{l(\lambda)} q^{|\lambda|},$$

where $\Lambda = k_0\Lambda_0 + \cdots + k_n\Lambda_n$ and $(\mathcal{G},\mathscr{G})$ is one of $(\mathcal{A},\mathscr{A})$, $(\mathcal{C},\mathscr{C})$ or $(\mathcal{D},\mathscr{D})$. For the weights $\Lambda = k\Lambda_0$ and $\Lambda = k\Lambda_n$ explicit formulas for these generating functions in terms of Hall–Littlewood symmetric functions are proposed in Conjectures (1.4) and (4.7). No such expressions in terms of Hall–Littlewood functions seem to exists for other weights. By [21, Lemma 2.1], for $k \geqslant 0$, $n \geqslant 1$ and $t := q^n$,

$$\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\leqslant k}} (zq)^{|\lambda|} P_{2\lambda}(1,q,q^2,\dots;t)\\
&\quad = \sum \prod_{i=1}^{2k} \Biggl\{ \frac{(zq)^{\frac{1}{2}\mu_i^{(0)}}}{(t;t)_{\mu_i^{(0)}-\mu_{i+1}^{(0)}}} \prod_{a=1}^{n} q^{\mu_i^{(a)}} t^{\binom{\mu_i^{(a-1)}-\mu_i^{(a)}}{2}} \begin{bmatrix} \mu_i^{(a-1)}-\mu_{i+1}^{(a)} \\ \mu_i^{(a-1)}-\mu_i^{(a)} \end{bmatrix}_t \Biggr\}\\
&\quad =: \mathrm{HL}_{k,n}(z,q),
\end{aligned}$$

where the sum on the right is over sequences of partitions $0 = \mu^{(n)} \subseteq \cdots \subseteq \mu^{(1)} \subseteq \mu^{(0)}$ such that all parts of $(\mu^{(0)})'$ are even and $l(\mu^{(0)}) \leqslant 2k$. Here $\mu \subseteq \lambda$ is shorthand for partition-inclusion, that is, $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$, and, by abuse of notation, 0 denotes the unique partition of 0. The condition on the partition $\mu^{(0)}$ implies that $\mu_{2i-1}^{(0)} = \mu_{2i}^{(0)}$ for all $1 \leqslant i \leqslant k$, so that

$$\prod_{i=1}^{2k} \frac{(zq)^{\frac{1}{2}\mu_i^{(0)}}}{(t;t)_{\mu_i^{(0)}-\mu_{i+1}^{(0)}}} = \prod_{i=1}^{k} \frac{(zq)^{\mu_{2i-1}^{(0)}}}{(t;t)_{\mu_{2i-1}^{(0)}-\mu_{2i+1}^{(0)}}}.$$

Conjectures 1.4 and 4.7 can thus be written in the following alternative form.

Conjecture 5.1. *For k a nonnegative integer and n a positive integer,*

$$\begin{aligned}
\mathcal{A}_{k\Lambda_n}^{(n)}(z,q) &= \mathrm{HL}_{k,2n-1}(z,q),\\
\mathcal{A}_{k\Lambda_0}^{(n)}(z,q) &= \mathrm{HL}_{k,2n-1}(zq,q),\\
\mathcal{C}_{k\Lambda_0}^{(n)}(z,q) &= \mathrm{HL}_{k,2n}(z,q),\\
\mathcal{D}_{k\Lambda_0}^{(n)}(z,q) &= \mathrm{HL}_{k,2n-2}(zq,q),
\end{aligned}$$

where the final equation requires $n \geqslant 2$.

By the symmetry (4.5), Λ_0 may be replaced by Λ_n in the last two results.

Let $S_{k,n}$ denote the set of all sequences $0 = \mu^{(n)} \subseteq \cdots \subseteq \mu^{(1)} \subseteq \mu^{(0)}$ of partitions such that $(\mu^{(0)})'$ is even and $l(\mu^{(0)}) \leqslant 2k$. For $\boldsymbol{\mu} \in S_{k,n}$ and $t := q^n$, define

$$\mathrm{HL}_{k,n;\boldsymbol{\mu}}(z,q) := \prod_{i=1}^{2k} \Biggl\{ \frac{(zq)^{\frac{1}{2}\mu_i^{(0)}}}{(t;t)_{\mu_i^{(0)}-\mu_{i+1}^{(0)}}} \prod_{a=1}^{n} q^{\mu_i^{(a)}} t^{\binom{\mu_i^{(a-1)}-\mu_i^{(a)}}{2}} \begin{bmatrix} \mu_i^{(a-1)}-\mu_{i+1}^{(a)} \\ \mu_i^{(a-1)}-\mu_i^{(a)} \end{bmatrix}_t \Biggr\}.$$

Conjecture 5.2. *For k, n positive integers,*

$$\sum_{\boldsymbol{\mu}\in S_{1,n}} q^{n\mu_1^{(0)}-n\mu_1^{(1)}} \,\mathrm{HL}_{1,n;\boldsymbol{\mu}}(z/q,q) = \begin{cases} \mathcal{A}_{\Lambda_1}^{(n/2+1/2)}(z,q), & \textit{for odd } n,\\ \mathcal{D}_{\Lambda_1}^{(n/2+1)}(z,q), & \textit{for even } n \end{cases} \tag{5.1a}$$

and

$$\sum_{\boldsymbol{\mu}\in S_{k,2}} q^{\sum_{i=1}^{2k}\mu_i^{(0)}-2\mu_1^{(1)}}\,\mathrm{HL}_{1,2;\boldsymbol{\mu}}(z/q,q)=\mathcal{D}^{(2)}_{(k-1)\Lambda_0+\Lambda_1}(z,q). \tag{5.1b}$$

Since $\mu^{(0)}_{2i-1}=\mu^{(0)}_{2i}$ the two conjectures are consistent. Equation (5.1a) for $n=1$ is (3.7a) for $n=1$. For $n=2$ it may be proved as follows.

Proof. Taking $\mu^{(0)}=(r_1,r_1)$ and $\mu^{(2)}=(r_2,r_3)$, we must show that

$$\mathcal{D}^{(2)}_{\Lambda_1}(z,q)=\sum_{r_1,r_2,r_3\geqslant 0}\frac{z^{r_1}q^{(r_1-r_2)^2+(r_1-r_3)^2+r_2^2+r_3^2-r_2+r_3}}{(q^2;q^2)_{r_1-r_2}(q^2;q^2)_{r_2-r_3}(q^2;q^2)_{r_3}}$$

Since $\mathcal{D}^{(2)}_{\Lambda_1}(z,q)=D^{(2)}_1(z,q)$, if follows from (4.4) with $a=1$ and $n=2$ that

$$\mathcal{D}^{(2)}_{\Lambda_1}(z,q)=\sum_{r_1,r_2\geqslant 0}\frac{z^{r_1}q^{r_1^2+r_2^2}}{(q;q)_{r_1-r_2}(q^2;q^2)_{r_2}}.$$

If we can prove that these two multisums are the same we are done. Equating coefficients of $z^{r_1}q^{-r_1^2}$, we are to show that

$$\sum_{r_2\geqslant 0}\frac{q^{r_2^2}}{(q;q)_{r_1-r_2}(q^2;q^2)_{r_2}}=\sum_{r_2,r_3\geqslant 0}\frac{q^{(r_1-r_2-r_3)^2+(r_2-r_3)^2-r_2+r_3}}{(q^2;q^2)_{r_1-r_2}(q^2;q^2)_{r_2-r_3}(q^2;q^2)_{r_3}}$$

for all nonnegative integers r_1. To this end, we multiply the above by z^{r_1} and sum over r_1. By then making the substitution $r_1\mapsto r_1+r_2$ on the left and $(r_1,r_2)\mapsto(r_1+r_2+r_3,r_2+r_3)$ on the right, this boils down to showing that

$$\sum_{r_1\geqslant 0}\frac{z^{r_1}}{(q;q)_{r_1}}\cdot\sum_{r_2\geqslant 0}\frac{z^{r_2}q^{r_2^2}}{(q^2;q^2)_{r_2}}=\sum_{r_1,r_3\geqslant 0}\frac{z^{r_1+r_3}q^{(r_1-r_3)^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_3}}\cdot\sum_{r_2\geqslant 0}\frac{(z/q)^{r_2}q^{r_2^2}}{(q^2;q^2)_{r_2}}$$

for $|z|<1$. The sums over r_1 and r_2 on the left are can be carried out by [19, Equation (II.2)]

$$\sum_{n\geqslant 0}\frac{z^n}{(q;q)_n}=\frac{1}{(z;q)_\infty},\quad |z|<1,$$

and [19, Equation (II.1)]

$$\sum_{n\geqslant 0}\frac{z^n q^{\binom{n}{2}}}{(q;q)_n}=(-z;q)_\infty \tag{5.2}$$

respectively. On the right the sum over r_2 can also be performed by (5.2). Moreover, we can use this same summation to sum over either r_1 or r_3. Choosing r_3, we are left with

$$\sum_{r_1\geqslant 0}\frac{(-q/z;q^2)_{r_1}}{(q^2;q^2)_{r_1}}\,z^{2r_1}=\frac{(-zq;q^2)_\infty}{(z^2;q^2)_\infty},$$

which is a special case of the q-binomial theorem [19, Equation (II.3)]

$$\sum_{n\geqslant 0}\frac{(a;q)_n}{(q;q)_n}\,z^n=\frac{(az;q)_\infty}{(z;q)_\infty},\quad |z|<1. \qquad\square$$

Appendix A. Proof of the non-standard specialisations

In this appendix we prove the non-standard specialisations given in (1.5) and (2.5), which were first stated without proof in [21]. The results presented here can be viewed as an addendum to the recent survey [11] of the Lepowsky and Wakimoto product formulas. We should remark that there exists a second non-standard specialisation for the affine Lie algebra $\mathrm{D}_{n+1}^{(2)}$ that will not be considered below, see [42, Theorem 5.14]. This specialisation differs from (2.2) in that

$$\big(\mathrm{e}(-\alpha_0),\dots,\mathrm{e}(-\alpha_n)\big)\mapsto(q,q^2,\dots,q^2,-1).$$

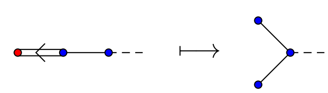
Let $\mathfrak{g}=\mathfrak{g}(A)$ be an arbitrary affine Lie algebra with generalised Cartan matrix A of size $(n+1)\times(n+1)$, and let $L(\Lambda)$ be a standard module of $\mathfrak{g}$ with normalised character $\chi_\Lambda\in\mathbb{Z}[[\mathrm{e}(-\alpha_0),\dots,\mathrm{e}(-\alpha_n)]]$, defined in the exact same manner as was done in Section 2.1 for $\mathfrak{g}$ one of $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ or $\mathrm{D}_{n+1}^{(2)}$. Denote the usual principal specialisation by ϕ_n, that is, $\phi_n:\mathbb{Z}[[\mathrm{e}(-\alpha_0),\dots,\mathrm{e}(-\alpha_n)]]\to\mathbb{Z}[[q]]$ is given by $\phi_n(\mathrm{e}(-\alpha_i))\to q$ for all $0\leqslant i\leqslant n$. Then [32]

$$\phi_n(\chi_\Lambda)=\prod\bigg(\frac{1-q^{\langle\Lambda+\rho,\alpha\rangle}}{1-q^{\langle\rho,\alpha\rangle}}\bigg)^{\mathrm{mult}(\alpha)}, \tag{A.1}$$

where the product runs over the positive roots α of the dual root system of $\mathfrak{g}$, i.e., the positive coroots of $\mathfrak{g}$. To prove (A.1), one first applies ϕ_n to the Weyl–Kac formula (2.1) (which holds for all $\mathfrak{g}(A)$). To then turn the sum into a product requires the denominator or Macdonald identity

$$\sum_{w\in W}\mathrm{sgn}(w)\,\mathrm{e}(w(\rho)-\rho)=\prod_{\alpha>0}(1-\mathrm{e}(-\alpha))^{\mathrm{mult}(\alpha)}$$

applied to the case of the dual affine Lie algebra $\mathfrak{g}({}^tA)$. In contrast, the specialisations of $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ proven below require the Macdonald identities for $\mathrm{B}_n^{(1)}$ and $\mathrm{D}_n^{(1)}$ respectively. In other words, instead of dualising by reversing the arrows of the Dynkin diagram of $\mathfrak{g}$, arrows between blue vertices are reversed but in the case of a red-blue pair we have



This maps $\mathrm{A}_{2n}^{(2)}$ to $\mathrm{B}_n^{(1)}$ (for $n\geqslant 3$), $\mathrm{D}_{n+1}^{(2)}$ to $\mathrm{D}_n^{(1)}$ (for $n\geqslant 4$), and $\mathrm{C}_n^{(1)}$ to $\mathrm{D}_{n+1}^{(2)}$ (for $n\geqslant 2$). For small values of n the appropriate degenerations of $\mathrm{B}_n^{(1)}$ and $\mathrm{D}_n^{(1)}$ need to be used instead.

A.1. The $\mathrm{A}_{2n}^{(2)}$ case.

Proposition A.1. *For $k\geqslant 0$ an integer or half-integer and $n\geqslant 1$ an integer, parametrise $\Lambda\in P_+^{2k}$ as in (2.3), where $(\lambda_1,\dots,\lambda_n)$ is a partition such that $\lambda_1\leqslant\lfloor k\rfloor$. Then*

$$\begin{aligned}\varphi_n(\chi_\Lambda)=\frac{(q^{2k+2n+1};q^{2k+2n+1})_\infty^n}{(q;q)_\infty^n}&\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};q^{2k+2n+1}\big)\\ \times\prod_{1\leqslant i<j\leqslant n}&\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};q^{2k+2n+1}\big)\end{aligned} \tag{A.2a}$$

if k is an integer, and

$$\varphi_n(\chi_\Lambda)=0 \tag{A.2b}$$

if k is a half-integer.

This should be compared with

$$\begin{aligned}\phi_n(\chi_\Lambda)&=\frac{(q^{2n+1};q^{4n+2})_\infty(q^{2k+2n+1};q^{2k+2n+1})_\infty^n}{(q;q^2)_\infty(q;q)_\infty^n}\\&\quad\times\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};q^{2k+2n+1}\big)\theta\big(q^{2k-2\lambda_i+2i-1};q^{4k+4n+2}\big)\\&\quad\times\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};q^{2k+2n+1}\big),\end{aligned}$$

as follows from (A.1). Writing $\chi_\Lambda^{\mathfrak{g}}$ instead of χ_Λ, we also remark that in the non-vanishing or integral-k case

$$\frac{1}{(q;q^2)_\infty}\,\varphi_n\Big(\chi_\Lambda^{\mathrm{A}_{2n}^{(2)}}\Big)=\phi_n\Big(\chi_{\Lambda'}^{\mathrm{A}_{2n-1}^{(2)}}\Big),$$

where

$$\Lambda'=(2k+1-\lambda_1-\lambda_2)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n\in P_+^{2k+1}$$

is a weight of $\mathrm{A}_{2n-1}^{(2)}$ and $1/(q;q^2)_\infty=\phi_n\Big(\chi_{\Lambda_0}^{\mathrm{A}_{2n-1}^{(2)}}\Big)$.

Proof. For $x=(x_1,\dots,x_n)$, let

$$\Delta_{\mathrm{B}}(x):=\prod_{i=1}^n(1-x_i)\prod_{1\leqslant i<j\leqslant n}(1-x_i/x_j)(1-x_ix_j)$$

be the Vandermonde product for the root system B_n. As mentioned above, at the heart of the proof of (A.2a) and (A.2b) is the $\mathrm{B}_n^{(1)}$ Macdonald identity [34]

$$\begin{aligned}&\sum_{\substack{r\in\mathbb{Z}^n\\|r|\text{ even}}}\Delta_{\mathrm{B}}(xq^r)\prod_{i=1}^n q^{(2n-1)\binom{r_i}{2}+(i-1)r_i}x_i^{(2n-1)r_i}\\&\qquad=(q;q)_\infty^n\prod_{i=1}^n\theta(x_i;q)\prod_{1\leqslant i<j\leqslant n}\theta(x_i/x_j,x_ix_j;q)=:\Pi_{\mathrm{B}}(x,q),\end{aligned}$$

which in the above explicit form holds for all $n\geqslant 1$. By the substitution $(r_1,x_1)\mapsto(r_1+1,x_1/q)$ and the use of the quasi-periodicity relation $\theta(a;q)=-a\theta(aq;q)$, this yields the exact same identity as above except that the condition on the parity of $|r|:=r_1+\cdots+r_n$ has switched from even to odd and the right-hand side has picked up a minus sign. Taking the sum respectively difference of the even and odd cases implies

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(xq^r)\prod_{i=1}^n(-\sigma)^{r_i}q^{(2n-1)\binom{r_i}{2}+(i-1)r_i}x_i^{(2n-1)r_i}=2\,\Pi_{\mathrm{B};\sigma}(x,q),$$

where $\sigma\in\{-1,1\}$ and $\Pi_{\mathrm{B};1}(x,q)=\Pi_{\mathrm{B}}(x,q)$, $\Pi_{\mathrm{B};-1}(x,q)=0$. By the B_n Vandermonde determinant

$$\Delta_{\mathrm{B}}(x)=\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-n}-x_i^{n-j+1}\big)\prod_{i=1}^n x_i^{n-i}$$

followed by multilinearity, this may also be written as

$$\det_{1\leqslant i,j\leqslant n}\bigg(\sum_{r\in\mathbb{Z}}(-\sigma)^r q^{(2n-1)\binom{r}{2}+(n-1)r}x_i^{(2n-1)r+n-i}\Big((x_iq^r)^{j-n}-(x_iq^r)^{n-j+1}\Big)\bigg)=2\,\Pi_{\mathrm{B};\sigma}(x,q).$$

Writing the sum over r as $\sum_r(a_r-b_r)$, this can be replaced by $\sum_r(a_r-b_{-r})$. Then once again using multilinearity, we obtain

$$\sum_{r\in\mathbb{Z}^n}\det_{1\leqslant i,j\leqslant n}\Big(x_i^{(2n-1)r_j+j-n}-x_i^{-(2n-1)r_j+n-j+1}\Big)\prod_{i=1}^n(-\sigma)^{r_i}q^{(2n-1)\binom{r_i}{2}+(i-1)r_i}x_i^{n-i}$$
$$=2\,\Pi_{\mathrm{B};\sigma}(x,q).$$

We are now ready to prove (A.2a) and (A.2b). First we recall the rewriting of the Weyl–Kac character formula (2.1) in the case of $\mathrm{A}_{2n}^{(2)}$ given in [8, Lemma 2.2]:

$$\chi_\Lambda=\frac{1}{(q;q)_\infty^n\prod_{i=1}^n\theta(x_i;q)\theta(x_i^2/q;q^2)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_i/x_j,x_ix_j/q;q)}$$
$$\times\sum_{r\in\mathbb{Z}^n}\det_{1\leqslant i,j\leqslant n}\bigg(q^{\kappa\binom{r_i}{2}}x_i^{\kappa r_i+\lambda_i+n}\Big((x_iq^{r_i})^{j-n-1-\lambda_j}-(x_iq^{r_i-1})^{n-j+1+\lambda_j}\Big)\bigg),$$

where $\kappa:=2k+2n+1$, $q:=\mathrm{e}(-\delta)$ and $x_i:=q\,\mathrm{e}(\alpha_0+\dots+\alpha_{i-1})$ for $1\leqslant i\leqslant n$. The marks for $\mathrm{A}_{2n}^{(2)}$ are the comarks read in reverse order, i.e., $a_i=a_{n-i}^\vee$. Hence $\delta=2\alpha_0+\dots+2\alpha_{n-1}+\alpha_n$, so that $\varphi_n(q)=q^{2n-1}$ and $\varphi_n(x_i)=-q^{2n-i}$. This implies

$$\varphi_n(\chi_\Lambda)=\frac{1}{2(q;q)_\infty^n}\sum_{r\in\mathbb{Z}^n}\bigg(\prod_{i=1}^n(-\sigma)^{r_i}p^{(2n-1)\binom{r_i}{2}+(2n-i)r_i}y_i^{n-i}$$
$$\times\det_{1\leqslant i,j\leqslant n}\Big(y_j^{-(2n-1)r_i+i-n}-y_j^{(2n-1)r_i+n-i+1}\Big)\bigg),$$

where $p:=q^\kappa$, $y_i:=q^{\lambda_i+n-i+1}$ and $\sigma=1$ if κ is odd and $\sigma=-1$ if κ is even, i.e., $\sigma=1$ if k is an integer and $\sigma=-1$ if k is a half-integer. Replacing $r_i\mapsto -r_i$ and interchanging i and j in the determinant yields

$$\varphi_n(\chi_\Lambda)=\frac{1}{(q;q)_\infty^n}\,\Pi_{\mathrm{B};\sigma}(p,y).$$

For $\sigma=-1$ this implies the vanishing of $\varphi_n(\chi_\Lambda)$ and for $\sigma=1$ this yields the claimed product form since, for $p=q^\kappa$ and $x_i=q^{\lambda_i+n-i+1}$,

$$\Pi_{\mathrm{B};1}(x,p)=\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};q\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j},q^{\lambda_i+\lambda_j+2n-i-j+2}\big).\qquad\square$$

A.2. **The $\mathrm{D}_{n+1}^{(2)}$ case.** A half-partition $(\lambda_1,\dots,\lambda_n)$ is a weakly decreasing sequence such that all $\lambda_i-1/2\in\mathbb{N}_0$ for all i.

Proposition A.2. *For $k\geqslant 0$ an integer or half-integer and $n\geqslant 2$ an integer, parametrise $\Lambda\in P_+^{2k}$ as in (2.3), where $(\lambda_1,\dots,\lambda_n)$ is a partition or half-partition such that $\lambda_1\leqslant k$. Then*

$$\varphi_n(\chi_\Lambda)=\frac{(q^{2k+2n};q^{2k+2n})_\infty^n}{(q^2;q^2)_\infty(q;q)_\infty^{n-1}}\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+1};q^{2k+2n}\big)\tag{A.3a}$$

if k is an integer and λ is a partition, and

$$\varphi_n(\chi_\Lambda)=0\tag{A.3b}$$

if k is a half-integer or λ is a half-partition.

This is to be compared with the principal specialisation

$$\begin{aligned}\phi_n(\chi_\Lambda)&=\frac{(q^{2k+2n};q^{2k+2n})_\infty^n}{(q;q^2)_\infty(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{2\lambda_i+2n-2i+1};q^{2k+2n}\big)\\&\quad\times\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+1};q^{2k+2n}\big)\end{aligned}$$

which holds for any level and λ a partition or half-partition.

Proof. The key steps of the proof are identical to the $\mathrm{A}_{2n}^{(2)}$ case with the exception of how we deal with the half-partitions. Starting point is the $\mathrm{D}_n^{(1)}$ Macdonald identity [34]

$$\begin{aligned}&\sum_{\substack{r\in\mathbb{Z}^n\\|r|\text{ even}}}\Delta_{\mathrm{D}}(xq^r)\prod_{i=1}^n q^{2(n-1)\binom{r_i}{2}+(i-1)r_i}x_i^{2(n-1)r_i}\\&\qquad=(q;q)_\infty^n\prod_{1\leqslant i<j\leqslant n}\theta(x_i/x_j,x_ix_j;q)=:\Pi_{\mathrm{D}}(x,q),\end{aligned}$$

where $n\geqslant 2$ and, for $x=(x_1,\dots,x_n)$,

$$\Delta_{\mathrm{D}}(x):=\prod_{1\leqslant i<j\leqslant n}(1-x_i/x_j)(1-x_ix_j).$$

As before, we carry out the substitution $(r_1,x_1)\mapsto(r_1+1,x_1/q)$ and use quasi-periodicity. This time the exact same identity as above arises but with the condition on the parity of $|r|$ changed to odd. Taking the sum respectively difference of the even and odd cases implies

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}(xq^r)\prod_{i=1}^n\sigma^{r_i}q^{2(n-1)\binom{r_i}{2}+(i-1)r_i}x_i^{2(n-1)r_i}=2\,\Pi_{\mathrm{D};\sigma}(x,q),$$

where $\sigma\in\{-1,1\}$ and $\Pi_{\mathrm{D};1}(x,q)=\Pi_{\mathrm{D}}(x,q)$, $\Pi_{\mathrm{D};-1}(x,q)=0$. To subsequently be able to handle the half-partition case, we enhance the above identity to

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}^{(\tau)}(xq^r)\prod_{i=1}^n\sigma^{r_i}q^{2(n-1)\binom{r_i}{2}+(i-1)r_i}x_i^{2(n-1)r_i}=2\,\Pi_{\mathrm{D};\sigma,\tau}(x,q),$$

where $\tau\in\{-1,1\}$,

$$\Delta_{\mathrm{D}}^{(\tau)}(x)=\begin{cases}\Delta_{\mathrm{D}}(x)&\text{if }\tau=1,\\0&\text{if }\tau=-1,\end{cases}\quad\text{and}\quad\Pi_{\mathrm{D};\sigma,\tau}(x,q)=\begin{cases}\Pi_{\mathrm{D};\sigma}(x,q)&\text{if }\tau=1,\\0&\text{if }\tau=-1.\end{cases}$$

In other words, for $\tau=-1$ we simply get $0=0$. Applying the D_n-Vandermonde determinant

$$\Delta_{\mathrm{D}}^{(\tau)}(x)=\frac{1}{2}\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-n}+\tau x_i^{n-j}\big)\prod_{i=1}^n x_i^{n-i}$$

and then carrying out the same sequence of steps as in the $\mathrm{A}_{2n}^{(2)}$ case, we obtain

$$\begin{aligned}&\sum_{r\in\mathbb{Z}^n}\det_{1\leqslant i,j\leqslant n}\Big(x_i^{2(n-1)r_j+j-n}+\tau x_i^{-2(n-1)r_j+n-j}\Big)\prod_{i=1}^n\sigma^{r_i}q^{2(n-1)\binom{r_i}{2}+(i-1)r_i}x_i^{n-i}\\&\qquad=4\,\Pi_{\mathrm{D};\sigma,\tau}(x,q).\end{aligned}\tag{A.4}$$

Next we recall the rewriting of the Weyl–Kac character formula (2.1) in the case of $\mathrm{D}_{n+1}^{(2)}$ as given in [8, Lemma 2.4]:

$$\chi_\Lambda = \frac{1}{(q^2;q^2)_\infty^{n-1}(q;q)_\infty \prod_{i=1}^n \theta(x_i;q)\prod_{1\leqslant i<j\leqslant n} x_j\theta(x_i/x_j,x_ix_j;q^2)}$$

$$\times \sum_{r\in\mathbb{Z}^n} \det_{1\leqslant i,j\leqslant n}\Big(q^{\kappa r_i^2-(2n+2\lambda_i-1)r_i}x_i^{\kappa r_i}\Big((x_iq^{2r_i})^{j-1+\lambda_i-\lambda_j}-(x_iq^{2r_i})^{2n-j+\lambda_i+\lambda_j}\Big)\Big),$$

where $\kappa:=2k+2n$, $q:=\mathrm{e}(-\delta)$ and $x_i:=\mathrm{e}(-\alpha_i-\dots-\alpha_n)$ for $1\leqslant i\leqslant n$. Since the marks for $\mathrm{D}_{n+1}^{(2)}$ are the comarks for $\mathrm{C}_n^{(1)}$, $\delta=\alpha_0+\alpha_1+\dots+\alpha_n$. Hence $\varphi_n(q)=q^{n-1}$ and $\varphi_n(x_i)=-q^{n-i}$, leading to

$$\varphi_n(\chi_\Lambda)=\frac{1}{4(q;q)_\infty^{n-1}(q^2;q^2)_\infty}\sum_{r\in\mathbb{Z}^n}\Bigg(\prod_{i=1}^n \sigma^{r_i}p^{2(n-1)\binom{-r_i}{2}-(i-1)r_i}y_i^{n-i}$$

$$\times \det_{1\leqslant i,j\leqslant n}\Big(y_j^{-2(n-1)r_i+i-n}+\tau y_j^{2(n-1)r_i+n-i}\Big)\Bigg),$$

where $p:=q^\kappa$, $y_i:=q^{\lambda_i+n-i+1/2}$, and $\sigma=1$ if κ is even (i.e., k is an integer), $\sigma=-1$ if κ is odd (i.e., k is a half-integer), $\tau=1$ if λ is a partition and $\tau=-1$ if λ is a half-partition. Replacing $r_i\mapsto -r_i$ and interchanging i and j in the determinant, it follows from (A.4) that

$$\varphi_n(\chi_\Lambda)=\frac{\Pi_{\mathrm{D};\sigma,\tau}(y,p)}{(q;q)_\infty^{n-1}(q^2;q^2)_\infty}.$$

For $\sigma=\tau=1$ this yields the product-form (A.3a) and for $\sigma=-1$ or $\tau=-1$ it implies (A.3b). □

References

[1] G. E. Andrews, *Sieves in the theory of partitions*, Amer. J. Math. **94** (1972), 1214–1230.

[2] G. E. Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. USA **71** (1974), 4082–4085.

[3] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading, Massachusetts, 1976.

[4] G. E. Andrews, *Multiple series Rogers–Ramanujan type identities*, Pacific J. Math. **114** (1984), 267–283.

[5] G. E. Andrews, *Parity in partition identities*, Ramanujan J. **23** (2010) 45–90.

[6] W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **50** (1949), 1–10.

[7] I. Baranović, M. Primc and G. Trupčević, *Bases of Feigin–Stoyanovsky's type subspaces for* $\mathrm{C}_\ell^{(1)}$, Ramanujan J. **45** (2018), 265–289.

[8] N. Bartlett and S. O. Warnaar, *Hall–Littlewood polynomials and characters of affine Lie algebras*, Adv. Math. **285** (2015), 1066–1105.

[9] D. M. Bressoud, *A generalization of the Rogers–Ramanujan identities for all moduli*, J. Combin. Theory Ser. A **27** (1979), 64–68.

[10] D. M. Bressoud, *Analytic and combinatorial generalizations of the Rogers–Ramanujan identities*, Memoirs Amer. Math. Soc. **24** (1980), 1–54.

[11] M. Butorac, S. Kožić, A. Meurman and M. Primc, *Lepowsky's and Wakimoto's product formulas for the affine Lie algebras* $C_l^{(1)}$, arXiv:2403.05456.

[12] S. Capparelli, A. Meurman, A. Primc, M. Primc, *New partition identities from* $C_\ell^{(1)}$*-modules*, Glas. Mat. Ser. III **57(77)** (2022), 161–184.

[13] S. Chern, *Linked partition ideals, directed graphs and q-multi-summations*, Electron. J. Combin. **27** (2020), Paper No. 3.33, 29pp.

[14] S. Chern, Z. Li, D. Stanton, T. Xue and A. J. Yee, *The Ariki–Koike algebras and Rogers–Ramanujan type partitions*, arXiv:2209.07713.

[15] S. Corteel and T. A. Welsh, *The* A_2 *Rogers–Ramanujan identities revisited*, Ann. Comb. **23** (2019), 683–694.

[16] J. Dousse and I. Konan, *Characters of level* 1 *standard modules of* $C_n^{(1)}$ *as generating functions for generalised partitions*, arXiv:2212.12728.
[17] E. Feigin, *The PBW filtration*, Represent. Theory **13** (2009), 165–181.
[18] J. Fulman, *The Rogers–Ramanujan identities, the finite general linear groups, and the Hall–Littlewood polynomials*, Proc. Amer. Math. Soc. **128** (2000), 17–25.
[19] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, second edition, Encyclopedia of Mathematics and its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.
[20] B. Gordon, *A combinatorial generalization of the Rogers–Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.
[21] M. J. Griffin, K. Ono and S. O. Warnaar, *A framework of Rogers–Ramanujan identities and their arithmetic properties*, Duke Math. J. **165** (2016), 1475–1527.
[22] Y. Huang and J. Ruofan, *Generating series for torsion-free bundles over singular curves: rationality, duality and modularity*, arXiv:2312.12528.
[23] M. Ishikawa, F. Jouhet and J. Zeng, *A generalization of Kawanaka's identity for Hall–Littlewood polynomials and applications*, J. Algebraic Combin. **23** (2006), 395–412.
[24] N. Jing, K. C. Misra, C. D. Savage, *On multi-color partitions and the generalized Rogers–Ramanujan identities*, Commun. Contemp. Math. **3** (2001), 533–548.
[25] F. Jouhet and J. Zeng, *New identities for Hall–Littlewood polynomials and applications*, Ramanujan J. **10** (2005), 89–112.
[26] V. G. Kac, *Infinite-dimensional Lie Algebras*, 3rd Edition, Cambridge University Press, Cambridge, 1990.
[27] S. Kanade and M. C. Russell, *Completing the* A_2 *Andrews–Schilling–Warnaar identities*, Int. Math. Res. Not. IMRN **(2023)**, 17100–17155.
[28] S. Kim and A. J. Yee, *The Rogers–Ramanujan–Gordon identities, the generalized Göllnitz–Gordon identities, and parity questions*, J. Combin. Theory Ser. A **120** (2013), 1038–1056.
[29] A. N. Kirillov, *New combinatorial formula for modified Hall–Littlewood polynomials*, in *q-Series from a Contemporary Perspective*, pp. 283–333, Contemp. Math. 254, Amer. Math. Soc., Providence, RI, 2000.
[30] M. Lassalle and M. Schlosser, *Inversion of the Pieri formula for Macdonald polynomials* Adv. Math. **202** (2006), 289–325.
[31] J. Lepowsky and S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. in Math. **29** (1978), 15–59.
[32] J. Lepowsky, *Affine Lie algebras and combinatorial identities* in *Lie algebras and related topics*, pp. 130–156, Lecture Notes in Mathematics No. 933, Springer, Berlin–New York, 1982.
[33] J. Lepowsky and R. L. Wilson, *The structure of standard modules. II. The case* $A_1^{(1)}$*, principal gradation*, Inv. Math. **79** (1985), 417–442.
[34] I. G. Macdonald, *Affine root systems and Dedekind's* η*-function*, Invent. Math. **15** (1972), 91–143.
[35] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, New York, 1995.
[36] P. A. MacMahon, *Combinatory analysis, Vol. I & II*, Dover Publications, Inc., Mineola, NY, 2004.
[37] A. Meurman and M. Primc, *Annihilating fields of standard modules of* $\mathfrak{sl}(2,\mathbb{C})^\sim$ *and combinatorial identities*, Mem. Amer. Math. Soc. **137** (1999), no. 652, viii+89 pp.
[38] M. Primc, *New partition identities for odd w odd*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **28(558)** (2024), 49–56.
[39] M. Primc and T. Šikić, *Combinatorial bases of basic modules for affine Lie algebras* $C_n^{(1)}$, J. Math. Phys. **57** (2016), 1–19.
[40] M. Primc and T. Šikić, *Leading terms of relations for standard modules of* $C_n^{(1)}$, Ramanujan J. **48** (2019), 509–543.
[41] M. Primc and G. Trupčević, *Linear independence for* $C_\ell^{(1)}$ *by using* $C_{2\ell}^{(1)}$, arXiv:2403.06881.
[42] E. M. Rains and S. O. Warnaar, *Bounded Littlewood identities*, Mem. Amer. Math. Soc. **270** (2021), no. 1317, vii+115 pp.
[43] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.
[44] L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. (2) **16** (1917), 315–336.
[45] L. J. Rogers and S. Ramanujan, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Phil. Soc. **19** (1919), 211–216.
[46] M. C. Russell, *Companions to the Andrews–Gordon and Andrews–Bressoud identities, and recent conjectures of Capparelli, Meurman, Primc, and Primc*, arXiv:2306.16251.
[47] A. Schilling and S. O. Warnaar, *A higher level Bailey lemma: proof and application*, Ramanujan J. **2** (1998), 327–349.

[48] I. J. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1917), 302–321.

[49] A. Selberg, *Über einige arithmetische Identitäten*, Avh. Norske Vidensk.-Akad. Oslo Mat.-Naturvidensk. Klasse: Avh. **8** (1936), 1–23.

[50] A. V. Sills, *An Invitation to the Rogers–Ramanujan Identities*, CRC Press, Boca Raton, Florida, 2018.

[51] L. J. Slater, *A new proof of Rogers's transformations of infinite series*, Proc. London Math. Soc. (2) **53** (1951), 460–475.

[52] J. R. Stembridge, *Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities*, Trans. Amer. Math. Soc. **319** (1990), 469–498.

[53] S. Tsuchioka, *A proof of the second Rogers–Ramanujan identity via Kleshchev multipartitions*, Proc. Japan Acad. Ser. A Math. Sci. **99** (2023), 23–26.

[54] S. O. Warnaar, *50 Years of Bailey's lemma*, in *Algebraic Combinatorics and Applications*, pp. 333–347, A. Betten et al. eds., Springer, Berlin, 2001.

[55] S. O. Warnaar, *Rogers–Szegő polynomials and Hall–Littlewood symmetric functions*, J. Algebra **303** (2006), 810–830.

[56] S. O. Warnaar, *Hall–Littlewood functions and the A_2 Rogers–Ramanujan identities*, Adv. Math. **200** (2006), 403–434.

[57] S. O. Warnaar, *The A_2 Andrews–Gordon identities and cylindric partitions*, Trans. Amer. Math. Soc., Series B **10** (2023), 715–765.

[58] S. O. Warnaar and W. Zudilin, *Dedekind's η-function and Rogers–Ramanujan identities*, Bull. Lond. Math. Soc. **44** (2012), 1–11.