Schubert and Macdonald Polynomials, a parallel

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Abstract

Schubert and (non-symmetric) Macdonald polynomials are two linear bases of the ring of polynomials which can be characterized by vanishing conditions. We show that both families satisfy similar branching rules related to the multiplication by a single variable. These rules are sufficient to recover a great part of the theory of Schubert and Macdonald polynomials.

1 Introduction

Let n be a positive integer, $\mathbf{x} = \{x_1, \ldots, x_n\}$. Schubert polynomials $\{Y_v : v \in \mathbb{N}^n\}$ and Macdonald polynomials $\{M_v : v \in \mathbb{N}^n\}$ are two linear bases of the ring of polynomials in $\mathbf{x}$, which are triangular (with respect to two different orders) in the basis of monomials. In the case of Schubert polynomials, one takes coefficients in an infinite set of indeterminates $\mathbf{y} = \{y_1, y_2, \ldots\}$. In the case of Macdonald polynomials, coefficients are rational functions in two parameters t, q.

These polynomials can be defined by vanishing conditions. Writing $|v| := v_1 + \cdots + v_n$, then one requires, both for Schubert and Macdonald polynomials, the vanishing in $d-1$ specific points, where $d = \binom{n+|v|}{n}$ is the dimension of the space of polynomials in $\mathbf{x}$ of degree $\leq |v|$. It just remains to fix a normalization condition to determine these polynomials uniquely.

The interpolation points are chosen in such a way that there is an easy relation between the polynomials Y_v and Y_{vs_i} (resp. M_v and M_{vs_i}), where s_i is a simple transposition. In fact, these relations reduce to a simple computation in the ring of polynomials in x_i, x_{i+1} as a free module over symmetric polynomials in x_i, x_{i+1}. As such, this space is a two-dimensional space with basis $\{1, x_i\}$. The relations are given by Newton's divided differences in the first case, and a deformation of them in the case of Macdonald polynomials (the generators of the Hecke algebra).

There is an extra "affine" operation, which sends M_v onto $M_{v\tau}$, with $v\tau = [v_2, \ldots, v_n, v_1{+}1]$. Together with $v \to vs_i$, $i = 1, \ldots, n{-}1$, the affine operation suffices to generate all Macdonald polynomials starting from $M_{0,\ldots,0} = 1$.

In the case of Schubert polynomials, one needs more starting points. They are the polynomials Y_v, with v *dominant*, i.e. such that $v_1 \geq v_2 \geq \cdots \geq v_n$. We also say that v is a *partition* and write $v \in \mathfrak{Part}$. In that case Y_v is a product of linear factors $x_i - y_j$, which can be read instantly by representing v by a diagram of boxes in the plane.

Having a distinguished linear basis, one has to recover the *multiplicative structure*. In the Schubert world, it is *Monk's formula*, describing the product by any x_i, which answers this problem. We give a similar formula for Macdonald polynomials. Coefficients are no more ± 1, but products of factors of the type $t^i q^j - 1$.

Given v, one can choose an i such that the multiplication by x_i furnishes a *transition formula* which allows to decompose the polynomial Y_v (resp M_v) into "smaller" polynomials. Iterating, one gets a canonical decomposition of Y_v into "shifted monomials" (meaning products of $x_i - y_j$), and a canonical decomposition of M_v into "shifted monomials" which are now products of $x_i q^j - t^k$.

As in every interpolation problem, it is useful to know a point $\aleph$ where the polynomials take explicit and all different values. In the case of Schubert polynomials, this point is just the origin $[0, \ldots, 0]$, but one can also use $[1, t, \ldots, t^{n-1}]$, or specialize instead the y_i's : $y_i \to t^{i-1}$. In the case of Macdonald polynomials, one takes $\aleph = [ut^{n-1}, \ldots, ut, u]$, with $u \neq 1$.

We mention at the end an interesting domain, introduced by Feigin, Jimbo, Miwa and Mukhin [5], which ought to be further developed, that is, the specialization of Macdonald polynomials under a *wheel condition* $t^\alpha q^\beta = 1$, with $\alpha, \beta \in \mathbb{Z}$.

2 Interpolation in the case of a single variable

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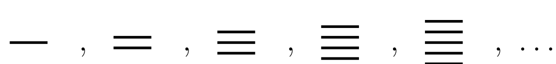


everybody continues with

Galileo, recording the positions of a falling stone at regular intervals of time, confronted to a little more difficult task :

$$I,\ IV,\ IX,\ XVI,\ XXV,\ XXXVI,\ \ldots$$

but was fortunately guided by the metaphysical principle that if it is not the increment of space which is constant, then it must be the increment of velocity[1].

Before being able to formulate any algebraic law concerning gravitation, Newton had to address the question of transforming discrete sets of data, say the positions of a planet at different times, into algebraic functions. Contrary to the case of a stone solved by Galileo, comets are not likely to appear at regularly spaced times, and to handle their seemingly erratic apparitions, Newton found the solution of normalizing differences of positions by the interval of time to which they correspond. These operations are called *Newton's divided differences*, we shall see more about them later.

Thanks to them, Newton was able to write an interpolation formula for the position $f(t)$ of a comet at times $t_0, t_1, \ldots$:

$$f(t) = f(t_0) + f^{\partial}(t - t_0) + f^{\partial\partial}(t - t_0)(t - t_1) + \cdots$$

where the coefficients f^{∂}, $f^{\partial\partial}$ are the successive divided differences of the positions at time $t_0, t_1, t_2, \ldots$.

Of course, one can characterize Newton's polynomials 1, $(t - t_0)$, $(t - t_0)(t - t_1), \ldots$ by their respective degrees and vanishing properties. Writing the specializations of Newton's formula at times t_0, $t_1, \ldots$ allows then to recover the recursive definition of f^{∂}, $f^{\partial\partial}, \ldots$.

[1]Quando, dunque, osservo che una pietra, che discende dall'alto a partire dalla quiete, acquista via nuovi incrementi di velocità, perché non dovrei credere che tali aumenti avvengano secondo la pi semplice e pi ovvia proporzione? Ora, se consideriamo attentamente la cosa, non troveremo nessun aumento o incremento più semplice di quello che aumenta sempre nel medesimo modo. Il che facilmente intenderemo considerando la stretta connessione tra tempo e moto: come infatti la equabilità e uniformità del moto si definisce e si concepisce sulla base della eguaglianza dei tempi e degli spazi (infatti chiamiamo equabile il moto, allorché in tempi eguali vengono percorsi spazi eguali), così, mediante una medesima suddivisione uniforme del tempo, possiamo concepire che gli incrementi di velocit avvengano con [altrettanta] semplicità; [lo possiamo] in quanto stabiliamo in astratto che risulti uniformemente e, nel medesimo modo, continuamente accelerato, quel moto che in tempi eguali, comunque presi, acquista eguali aumenti di velocità.

In fact, all classical interpolation formulas in one variable rely on the fact that if one knows the value of a polynomial $P(t)$ in a, then $(P(t)-P(a))(t-a)^{-1}$ (divided difference !) is a polynomial of smaller degree.

The situation is not at all the same in the case of several variables, because it is not evident how to reduce the degree of a polynomial, knowing its values in some points.

Nevertheless, we shall give two families of polynomials in several variables, Schubert and Macdonald polynomials, which behave almost as simply as the polynomials of Newton, and can be defined by vanishing properties.

Schubert polynomials have originally been defined as representatives of Schubert varieties [15], but, due to their close relation with the Ehresmann-Bruhat order on the symmetric group [16, 17], it is clear that they can also be characterized by vanishing properties. It is more surprising, and due to Sahi, Knop and Okounkov [24, 25, 11, 9, 10, 20, 21], that Macdonald polynomials can also be defined by vanishing conditions.

We shall study both families of polynomials by using interpolation methods only, i.e. by computing specializations (*Grothendieck polynomials* [16] could have been treated in the like manner).

3 Schubert polynomials

Given n, let $\mathfrak{Pol}(\mathbf{x},\mathbf{y})$ (resp. $\mathfrak{Pol}_d(\mathbf{x},\mathbf{y})$) be the space of polynomials in $\mathbf{x}=\{x_1,\ldots,x_n\}$ with polynomial coefficients in $\mathbf{y}=\{y_1,y_2,\ldots,y_\infty\}$ (resp. of total degree $\leq d$ in $\mathbf{x}$).

We need to use two different indexings, either by permutations or by codes, for the polynomials that we want to describe.

Given σ in the symmetric group $\mathfrak{S}_N$, its *code* $\mathfrak{c}(\sigma)$ is the vector v of components $v_i := \#\{j : j>i \,\&\, \sigma_i>\sigma_j\}$. One identifies σ and $[\sigma, N{+}1, N{+}2,\ldots]$; this corresponds to concatenating 0's to the code of σ. We write $\sigma=\langle v\rangle$ when $\mathfrak{c}(\sigma)=v$ (up to terminal zeros), and $\mathbf{y}^\sigma=\{y_{\sigma_1},y_{\sigma_2},\ldots\}$.

Definition 1 *Given $v\in\mathbb{N}^n$, the Schubert polynomial $Y_v(\mathbf{x})$, also denoted $X_\sigma(\mathbf{x})$ with $\sigma=\langle v\rangle$, is the only polynomial in $\mathfrak{Pol}_{|v|}(\mathbf{x},\mathbf{y})$ such that*

$$Y_v(\mathbf{y}^{\langle u\rangle}) = 0\,,\ u\neq v,\ |u|\leq|v| \tag{1}$$

$$Y_v(\mathbf{y}^{\langle v\rangle}) = \Cap(v) := \prod_{i<j,\,\sigma_i>\sigma_j}(y_{\sigma_i}-y_{\sigma_j}) \tag{2}$$

The space $\mathfrak{Pol}_{|v|}(\mathbf{x},\mathbf{y})$ has dimension exactly the number of conditions that we have just imposed on the putative $Y_v(\mathbf{x})$. The existence (unicity is clear) of this polynomial will follow from the recursive construction that (1, 2) imply.

Polynomials which are products of linear factors $x_i - y_j$ are easy to specialize. A "pigeon-hole" analysis gives the following lemma.

Lemma 2 *Let $v \in \mathbb{N}^n$ be dominant. Then*

$$\prod_{i=1}^{n}\prod_{j=1}^{v_i}(x_i - y_j)$$

satisfies (1, 2).

We can now reason by induction on the number of indices i such that $v_i < v_{i+1}$ to treat the general case.

Given a polynomial $f(x_1,\ldots,x_n)$, and $i : 1 \le i \le n{-}1$, denote f^{s_i} the image of f under the exchange of x_i, x_{i+1}. Let $f \to f\partial_i := (f - f^{s_i})(x_i - x_{i+1})^{-1}$ be the i-th *Newton's divided difference* (denoted on the right).

Lemma 3 *Let $v \in \mathbb{N}^n$, $\sigma = \langle v\rangle$, i be such that $v_i > v_{i+1}$. Suppose that Y_v satisfies (1, 2). Then*

$$f := X_\sigma(\mathbf{x})\,\partial_i = \big(X_\sigma(\mathbf{x}) - X_\sigma(\mathbf{x}^{s_i})\big)\,(x_i - x_{i+1})^{-1}$$

also satisfies (1, 2) for the index $v' = [v_1,\ldots,v_{i-1},v_{i+1},v_i{-}1,v_{i+2},\ldots,v_n]$.

Proof. The polynomial f vanish in all $\mathbf{x} = y^{\langle u\rangle}$, $|u| < |v|$, except for $u = v'$, because $X_\sigma(\mathbf{x})$ as well as $X_\sigma(\mathbf{x}^{s_i})$ vanish in these points. Moreover, $\big(X_\sigma(\mathbf{y}^\sigma) - X_\sigma(\mathbf{y}^{\sigma s_i})\big)\,(y_{\sigma_i} - y_{\sigma_{i+1}})^{-1}$ is indeed equal to $⋒(\sigma s_i)$, the two permutations σ and σs_i having the same inversions, except the inversion σ_i, σ_{i+1}. Q.E.D.

In conclusion, conditions (1, 2) define a linear basis of $\mathfrak{Pol}(\mathbf{x},\mathbf{y})$, Lemma 3 showing that the Schubert polynomials can be generated by divided differences from the dominant Schubert polynomials.

As we already said, we shall recover the multiplicative structure of $\mathfrak{Pol}(\mathbf{x},\mathbf{y})$ by describing the effect of multiplying the Schubert basis by $x_1,\ldots,x_n$.

Definition 4 *$v \in \mathbb{N}^n$ is a* successor *of u if $|v| = |u| + 1$ & $Y_u(\mathbf{y}^{\langle v\rangle}) \neq 0$. Correspondingly, for two permutations ζ, σ, ζ is a* successor *of σ iff $\ell(\zeta) = \ell(\sigma) + 1$ and $X_\sigma(\mathbf{y}^\zeta) \neq 0$.*

Theorem 5 *ζ is a successor of σ iff $\zeta\sigma^{-1}$ is a transposition (a,b), and $\ell(\zeta) = \ell(\sigma)+1$. In that case,*

$$X_\sigma(\mathbf{y}^\zeta) = \Cap(\mathfrak{c}(\zeta))\,(y_{\zeta_b} - y_{\zeta_a})^{-1}\,.$$

Proof. If $u = \mathfrak{c}(\sigma)$ is dominant, then it is immediate to write the specializations of Y_u and check the proposition in that case. Let us therefore suppose that there exists $i:\ u_i < u_{i+1}$, and let $\eta = \langle u_1,\ldots,u_{i-1},u_{i+1}{+}1,u_i,u_{i+2},\ldots,u_n\rangle$. Since for any ζ,

$$\big(X_\eta(\mathbf{y}^\zeta) - X_\eta(\mathbf{y}^{\zeta s_i})\big)\,(y_{\zeta_i} - y_{\zeta_{i+1}})^{-1} = X_\sigma(\mathbf{y}^\zeta)\,,$$

then ζ can be a successor of σ only if $\zeta=\eta$, or if ζs_i is a successor of η. In the first case,

$$X_\sigma(\mathbf{y}^\eta) = X_\eta(\mathbf{y}^\eta)(y_{\eta_i} - y_{\eta_{i+1}})^{-1} = \Cap(\mathfrak{c}(\eta))\,,$$

while in the second,

$$\frac{-X_\eta(\mathbf{y}^{\zeta s_i})}{y_{\zeta_i} - y_{\zeta_{i+1}}} = \frac{\Cap(\mathfrak{c}(\zeta s_i))}{(y_{\zeta_{i+1}} - y_{\zeta_i})(y_{\zeta_b} - y_{\zeta_a})} = \frac{\Cap(\mathfrak{c}(\zeta))}{y_{\zeta_b} - y_{\zeta_a}}\,,$$

and this proves the proposition. Q.E.D.

Corollary 6 (Monk formula [12]) *Given $v\in\mathbb{N}^n$, $\sigma=\langle v\rangle$, $i\in\{1,\ldots,n\}$, then*

$$(x_i - y_{\sigma_i})X_\sigma(\mathbf{x}) = \sum_{j>i} X_{\sigma\tau_{i,j}} - \sum_{j<i} X_{\sigma\tau_{i,j}}\,, \tag{3}$$

summed over all transpositions $\tau_{i,j}$ such that $\ell(\sigma\tau_{i,j}) = \ell(\sigma)+1$.

Proof. The polynomial $(x_i - y_{\sigma_i})X_\sigma(\mathbf{x})$ belongs to the linear span of $Y_w:\ |w| = |v|+1$, because it is of degree $|v|+1$ and vanishes in all $\mathbf{y}^{\langle w\rangle}:\ |w|\le|v|$. Writing it $\sum c_\zeta X_\zeta(\mathbf{x})$, and testing all the specializations $\mathbf{y}^\zeta$, one finds that the permutations appearing in the sum are exactly the successors of σ such that $y_{\zeta_i}\ne y_{\sigma_i}$. Q.E.D.

Let us put the *right lexicographic order* on monomials : $v\ge u$ iff either $|v|>|u|$ or $\Big(|v| = |u|\ \&\ \exists k:\ v_k > u_k, v_{k+1} = u_{k+1},\ldots,v_n = u_n\Big)$. The recursive definition of Schubert polynomials provided by Lemma 3 entails that $Y_v = x^v +$ lower terms with respect to this order.

Given $v \in \mathbb{N}^n$, let $k \leq n$ be such that $v_k > 0$, $v_{k+1} = 0 = \cdots = v_n$, and let v' be obtained from v by changing v_k into v_k-1 and $\sigma = \langle v' \rangle$. Then Monk formula rewrites in that case as

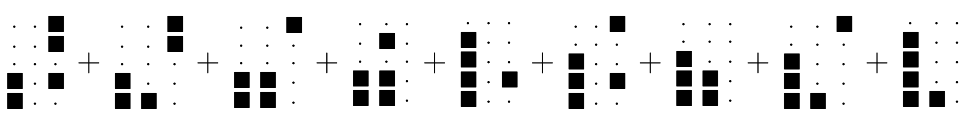
$$Y_v = (x_k - y_{\sigma_k}) Y_{v'} + \sum\nolimits_u Y_u \ , \tag{4}$$

summed over all u such that $|u| = |v|$ and $\langle u \rangle \sigma^{-1}$ is a transposition τ_{ik} with $i < k$. Let us remark that $u < v$, and therefore, the above equation, called *transition formula* [15], provides a *positive* recursive definition of Schubert polynomials, and decomposes them into sums of "shifted monomials" $\prod (x_i - y_j)$.

For example, starting with $v = [2, 0, 3]$, $\langle v' \rangle = \sigma = [3, 1, 5, 2, 4]$, one has the following sequence of transitions :

$$\begin{array}{rcl} Y_{203} &=& (x_3 - y_5) Y_{202} + Y_{230} + Y_{401} \ , \\ Y_{230} &=& (x_2 - y_4) Y_{220} + Y_{320} \ , \\ Y_{401} &=& (x_3 - y_2) Y_{400} + Y_{410} \ , \\ \cdots & & \cdots \end{array}$$

that one terminates when attaining dominant indices. In final, writing each shifted monomial as a diagram of black squares in the Cartesian plane ($x_i - y_j$ gives a box in column i, row j), the polynomial Y_{203} reads



the first diagram, for example, coding $(x_1 - y_1)(x_1 - y_2)(x_3 - y_2)(x_3 - y_4)(x_3 - y_5)$.

Fomin and Kirillov [6] give configurations from which one reads a different decomposition of Schubert polynomials into shifted monomials.

4 Some properties of Schubert polynomials

Their definition by vanishing conditions show that Schubert polynomials are compatible with the embedding $\mathfrak{Pol}(x_1, \ldots, x_n) \hookrightarrow \mathfrak{Pol}(x_1, \ldots, x_{n+1})$:
$Y_v = Y_{v,0}$. As a consequence, one can index Schubert polynomials by vectors $v \in \mathbb{N}^\infty\}$ having only a finite number of components different from 0, these

polynomials constituting a basis of the ring $\mathfrak{Pol}(x_1, x_2, \ldots)$ of polynomials in an infinite number of variables x_i, with coefficients in $\mathbf{y}$.

The vanishing conditions also show a symmetry between $\mathbf{x}$ and $\mathbf{y}$: $X_\sigma(\mathbf{x}, \mathbf{y}) = (-1)^{\ell(\sigma)} X_{\sigma^{-1}}(\mathbf{y}, \mathbf{x})$. The polynomials $X_\sigma(\mathbf{x}, \mathbf{0})$, as well as the polynomials $X_\sigma(\mathbf{0}, \mathbf{y})$ are linearly independent. The polynomials $X_\sigma(\mathbf{x}, 1/(1-q))$ (i.e. specializing $y_i = q^{i-1}$) can be used in problems of q-interpolation [23].

Any simple transposition s_i can be written $\partial_i(x_{i+1} - x_i) + 1$. More generally, given a permutation σ and any decomposition $\sigma = s_i \cdots s_j$, then

$$\sigma = \big(\partial_i(x_{i+1} - x_i) + 1\big) \cdots \big(\partial_j(x_{j+1} - x_j) + 1\big) . \tag{5}$$

Reordering this expression, one can show [13, Prop 9.6.2] that

$$\sigma = \sum_\zeta \partial_\zeta X_\zeta(\mathbf{x}^\sigma, \mathbf{x}) ,$$

defining $\partial_\zeta = \partial_i \cdots \partial_j$ with the help of any *reduced* decomposition $s_i \cdots s_j = \zeta$. Thus, Schubert polynomials occur in the expansion of permutations in terms of divided differences (and also in the expansion of divided differences in terms of permutations [13, Prop 10.2.5]).

Clearly, the permutation ζ obtained by expanding (5) are smaller than σ in the Bruhat order (since they have a decomposition which is subword of $s_i \cdots s_j$). This implies that $X_\zeta(\mathbf{x}^\sigma, \mathbf{x}) = 0$ for all ζ which are not smaller than σ. We have already met this criterium when $\ell(\sigma) = \ell(\zeta) + 1$, the comparison with respect to the Bruhat order corresponding in that case to the requirement in Theorem 5 that $\zeta\sigma^{-1}$ be a transposition. Each Schubert polynomial X_ζ, apart from $X_{123\ldots} = 1$, vanishes for an infinite number of specializations $X_\zeta(\mathbf{x}^\sigma, \mathbf{x})$. For example, $X_{13245\ldots} = x_1 + x_2 - y_1 - y_2$ vanishes for all σ having a reduced decomposition which does not contain s_2 (equivalently, such that $\{\sigma_1, \sigma_2\} = \{1, 2\}$).

Lemma 3 shows that $Y_v : v \in \mathbb{N}^n$ is symmetrical in x_i, x_{i+1} if $v_i \leq v_{i+1}$. Therefore, if v is *anti-dominant* (i.e $v_1 \leq v_2 \leq \cdots \leq v_n$), then Y_v is a symmetric function, which is called a *Grassmannian Schubert polynomial*. It is obtained from Y_λ, $\lambda = [v_n + n - 1, \ldots, v_2 + 1, v_1]$, by a sequence of divided differences corresponding to the "maximal permutation" $\omega := [n, \ldots, 1]$. Indeed, let $\partial_\omega = \partial_1(\partial_2\partial_1) \cdots (\partial_{n-1} \cdots \partial_1)$ be such a product. Then, thanks to (3), $Y_\lambda \partial_\omega = Y_v$, divided differences acting just by reordering and decreasing indices (or by annihilation). On the other hand, ∂_ω is an operator which commutes with multiplication with symmetric functions, and decreases degrees by $n(n-1)/2$. From this, it is easy to conclude that ∂_ω is equal to

$\mathfrak{Pol}(\mathbf{x}) \ni f \to \sum_{\sigma\in\mathfrak{S}_n}(f/\Delta)^\sigma$, where $\Delta = \prod_{1\le i<j\le n}(x_i - x_j)$. In the case where $f = f_1(x_1)\dots f_n(x_n)$, then $\sum(-1)^{\ell(\sigma)} f^\sigma$ can be written as the determinant $\det(f_i(x_j))$. In particular,

$$Y_v = \det(x_i^{\langle\lambda_j\rangle})/\Delta\,,$$

where $x^{\langle k\rangle}$ stands for the "modified power" $(x-y_1)\dots(x-y_k)$.

The special case of Grassmannian Schubert polynomials when $y_i = q^{i-1}$ appear frequently in the literature under the name *q-factorial Schur functions.*

5 Macdonald polynomials

As in the case of Schubert polynomials, our fundamental objects will be indexed by elements of $\mathbb{N}^n$, but this time we shall not decode $v \in \mathbb{N}^n$ as a permutation. This is the affine symmetric group that we have to use now.

It is convenient to consider $v \in \mathbb{N}^n$ as the n first components of an infinite vector v such that $v_{i+rn} = v_i + r$, $r \in \mathbb{Z}$. Similarly, we shall use an infinite set of indeterminates $x_i : i \in \mathbb{Z}$, such that $x_{i+rn} = q^r x_i$. Now, apart from the simple transpositions s_i, $0 < i < n$ (which transpose x_{i+rn} and x_{i+1+rn}, resp. v_{i+rn} and v_{i+1+rn}, for all r at the same time), we also have a translation τ $\tau :\, x_i \to x_{i+1}$, $v_i \to v_{i+1}$, and its inverse $\bar\tau = \tau^{-1}$, that one can also write

$$\begin{aligned} [x_1,\dots,x_{n-1},x_n] &\xrightarrow{\tau} [x_2,\dots,x_n,qx_1]\,,\\ [v_1,\dots,v_{n-1},v_n] &\xrightarrow{\tau} [v_2,\dots,v_n,v_1+1]\,. \end{aligned}$$

Let moreover $s_0 := \tau s_1 \bar\tau = \bar\tau s_{n-1}\tau$, and $\Phi := \bar\tau\,(x_n - 1)$.

Given $v \in \mathbb{N}^n$, one superscriptizes its components with the numbers $0, 1, \dots, n-1$, reading by increasing values, from right to left (this is one of the many ways to *standardize* a word). This allows to associate to v a new vector $\langle v\rangle := [q^{v_1}t^a, \dots, q^{v_n}t^b]$, where $a,\dots,b$ are the superscripts. For example, for $v = [5,0,8,5]$, the superscripts are $[2,0,3,1]$, and $\langle v\rangle = [q^5t^2, q^0t^0, q^8t^3, q^5t^1]$.

Given $u, v \in \mathbb{N}^n$, let

$$\eth(u,v) := \frac{1}{1-t}\widehat{\prod}_i \frac{1-t}{\langle v\rangle_i\langle u\rangle_i^{-1} - 1}\ , \qquad (6)$$

product over all i such that $\langle v\rangle_i \ne \langle u\rangle_i$.

Given $u, v \in \mathbb{N}^n$ such that v is a permutation of $w = u\tau$, let vv, ww be the images of v, w after putting superscripts. Let

$$\mathbb{U}(u,v) := \prod \frac{(\gamma - t)(t\gamma - 1)}{(\gamma - t)^2} \tag{7}$$

products over all pairs $i < j$ such that $v_i < v_j$ and v_i^a, v_j^b is a subword of vv and not of ww, with $\gamma := q^{v_j - v_i} t^{b-a}$.

For example, $u = [4,5,0,8]$ gives $ww = [5^2, 0^0, 8^3, 5^1]$, $v = [0,5,5,8]$ gives $vv = [0^0, 5^2, 5^1, 8^3]$. Only the two subwords $[0^0, 5^2]$, $[5^1, 8^3]$ contribute to

$$\mathbb{U}([4,5,0,8],[0,5,5,8]) = \frac{(tq^5 - 1)(t^3q^5 - 1)(q^3t - 1)(q^3t^3 - 1)}{(q^5t^2 - 1)^2 (q^3t^2 - 1)^2} .$$

We extend the definition of $\mathbb{U}(u,v)$ by requiring that $\mathbb{U}$ be invariant under the action of τ: $\mathbb{U}(u\tau^k, v\tau^k) = \mathbb{U}(u,v)\ \forall k \geq 0$. Thus $\mathbb{U}([4,5,0,8],[0,5,5,8]) = \mathbb{U}([5,0,8,5],[5,5,8,1]) = \mathbb{U}([0,8,5,6],[5,8,1,6]) = \mathbb{U}([8,5,6,1],[8,1,6,6])$.

The lexicographic order is no more convenient. We have to extend the natural order on partitions. Given $v \in \mathbb{N}^n$, denote $\lambda(v)$ the partition obtained by reordering v decreasingly. Then we set

$$u < v \text{ iff} |u| < |v| \text{ or } \Big(|u| = |v| \ \&\ \lambda(u) < \lambda(v)\Big) \text{ or } \Big(\lambda(u) = \lambda(v) \ \&\ u <_{\mathfrak{S}} v\Big),$$

$<_{\mathfrak{S}}$ being the Bruhat order on the permutations of an element of $\mathbb{N}^n$. For example $[4,0,0] > [0,0,4] > [2,2,0] > [2,0,2] > [1,2,1] > [3,0,0]$ is a chain.

The *leading term* of a polynomial is the restriction of the polynomial to its maximal elements with respect to this order.

Definition 7 *For any $v \in \mathbb{N}^n$, M_v is the only polynomial such that $M_v(\langle u \rangle) = 0$, $\forall u : |u| \leq |v|, u \neq v$, and such that the leading term of M_v is $x^v q^{-\sum_i \binom{v_i}{2}}$.*

The number of conditions is $\binom{|v|+n}{n}$, i.e. the dimension of the space of polynomials of degree $\leq |v|$. The existence of such a family of polynomials will follow from the recursions $v \to vs_i$, and $v \to v\tau$ that the vanishing conditions impose.

Okounkov [20, 22] considers more general interpolation points (essentially, he replaces the vectors $\langle u \rangle$ by $[\langle u \rangle_1 + c\langle u \rangle_1^{-1}, \ldots, \langle u \rangle_n + c\langle u \rangle_n^{-1}]$, where c is an extra parameter).

Lemmas 8,9 and Proposition 10 below are due to Sahi and Knop. We repeat their proof for completeness.

The first lemma is straightforward, being a statement in a space of dimension 2.

Lemma 8 *Let f be a polynomial in x_1, x_2, and $a, b \in \{t^i q^j,\, i, j \in \mathbb{Z}\}$, $a \neq b$, be such that $f(b,a) = 0$.*

Let T be the operator, commuting with multiplication with symmetric functions in x_1, x_2, such that $1T = t$, $x_2 T = x_1$. Then $g = f(T + (t-1)(ba^{-1} - 1)^{-1}$ is such that $g(a,b) = 0$, $g(b,a) = (ta-b)(a-b)^{-1} f(a,b)$.

The function g can be written

$$g(x_1, x_2) = \left(\frac{(t-1)x_1}{x_1 - x_2} + \frac{t-1}{b-a} \right) f(x_1, x_2) + \frac{x_1 - t x_2}{x_1 - x_2} f(x_2, x_1)\,. \qquad (8)$$

Suppose that x, y, $x \neq y$ are such that $f(y,x) = 0$. Then the preceding formula entails

$$g(x,y) = \frac{(t-1)(yx^{-1} - ba^{-1})}{(ba^{-1} - 1)(yx^{-1} - 1)} f(x,y) \quad \& \quad g(y,x) = \frac{yx^{-1} - t}{yx^{-1} - 1} f(x,y) \quad (9)$$

Remark. The operators $T = T_1, T_2, \ldots, T_{n-1}$, corresponding to the successive pairs of indeterminates $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$, generate a faithful representation of the Hecke algebra of $\mathfrak{S}_n$. They satisfy the *braid relations*

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \& \quad T_i T_j = T_j T_i\,,\ |i-j| \neq 1\,,$$

together with the *Hecke relation*

$$(T_i - t)(T_i + 1) = 0\,.$$

I have defined with M.P. Schützenberger [16, 17] more general representations of the Hecke algebra acting on polynomials by deformation of divided differences.

Lemma 9 *Let $v \in \mathbb{N}^n$, $f \in \mathfrak{Pol}(x)$ be such that $f(\langle u \rangle) = 0$ for all $u :\ |u| \leq |v|, u \neq v$. Then $g := f\,\Phi$ is such that $g(\langle w \rangle) = 0$ for all $w :\ |w| \leq |v|+1, w \neq v\tau$, and $g(\langle v\tau \rangle) = f(\langle v \rangle)\big(t\langle v \rangle_1 - 1\big)$.*

Proof. If w is such that $w_n \neq 0$, then $w = u\tau$, with $u = w\bar{\tau}$. The vanishing of f in $\langle u \rangle$ insures the vanishing of g in $\langle w \rangle$. If $w_n = 0$, then $\langle w \rangle_n = 1$, and the linear factor $x_n - 1$ specializes to 0. Q.E.D.

Proposition 10 *Let $v \in \mathbb{N}^n$. Suppose that a polynomial M_v satisfy the conditions of Definition 7. Then*

$$M_{vs_i} = M_v \left(T_i + \frac{t-1}{\langle v \rangle_{i+1} \langle v \rangle_i^{-1} - 1} \right) \ , \ \textit{if} \quad v_i < v_{i+1} \, , \tag{10}$$

and

$$M_{v\tau} = M_v \, \Phi \tag{11}$$

also satisfy (7) for the indices vs_i and $v\tau$ respectively.

Proof. The required vanishing properties, knowing those of M_v, are a consequence of the two lemmas 8,9. The coefficient of x^{vs_i} in M_{vs_i} is the same as the one of x^v in M_v. The monomial $x^v q^{-\sum \binom{v_i}{2}}$ gives the term $q^{-v_1} x^{v\tau} q^{-\sum \binom{v_i}{2}} = x^{v\tau} q^{-\sum \binom{(v\tau)_i}{2}}$, and therefore the normalization conditions are also respected. Q.E.D.

One should note that one can relax the condition $v_i < v_{i+1}$ in the proposition. In other words, the action of the Hecke algebra generates any M_u from M_v when u is a permutation of v. However, one has to take into account that

$$\left(T_i + \frac{t-1}{\frac{\langle v \rangle_{i+1}}{\langle v \rangle_i} - 1} \right) \left(T_i + \frac{t-1}{\frac{\langle v \rangle_i}{\langle v \rangle_{i+1}} - 1} \right) = \frac{\left(t \frac{\langle v \rangle_{i+1}}{\langle v \rangle_i} - 1 \right) \left(\frac{\langle v \rangle_{i+1}}{\langle v \rangle_i} - t \right)}{\left(\frac{\langle v \rangle_{i+1}}{\langle v \rangle_i} - 1 \right)^2} \tag{12}$$

and Equation 10, when $v_i > v_{i+1}$, gives the polynomial M_{vs_i} multiplied by this last constant.

Equations 10,12 describe the action of a generator T_i on the Macdonald basis. More generally, the action of a "maximal cycle" $T_{n-1} \cdots T_1$ is given by the following proposition.

Proposition 11 *Let $w \in \mathbb{N}^n$. For any $v \in \mathbb{N}^n$, let*

$$\eth(w, v) = \widehat{\prod}_{i=1\ldots n-1} \frac{1-t}{\langle v \rangle_{i+1} \langle w \rangle_i^{-1} - 1} \, ,$$

product over all i such that $\langle v \rangle_{i+1} \neq \langle w \rangle_i$.

Then, for any M_v occurring in the expansion of $M_w T_{n-1} \cdots T_1$, its coefficient is equal to

$$\text{Ↄ}(w,v)\ \Cup\left(w\tau^{n-1}, v\tau^n\right).$$

Proof. The formula results from the iteration of the action of T_i, decomposing it into the cases :

$$M_v T_i = cM_{vs_i} + (1-t)\left(\frac{\langle v\rangle_{i+1}}{\langle v\rangle_i} - 1\right)^{-1} M_v\ ,\ v_i \neq v_{i+1}\,,$$

$$M_v T_i = tM_v\ ,\ v_i = v_{i+1}\,,$$

where $c = 1$ if $v_i < v_{i+1}$, and otherwise c is the constant in (12). Since v is a permutation of w, one can rewrite $(1-t)\left(\frac{\langle v\rangle_{i+1}}{\langle v\rangle_i} - 1\right)^{-1}$ as a factor of $\text{Ↄ}(w,v)$. The definition (7) of $\Cup(u,v)$ involved the comparison of $u\tau = w$ and v. Here, $w\tau^{-1}$ does not necessarily exist, this is why we have to write $\Cup(w\tau^{n-1}, v\tau^n)$, though comparing w and v. Q.E.D.

For example,

$$M_{303}T_2 = M_{330} + (1-t)(tq^3-1)^{-1}M_{303}\,,$$

$$M_{303}T_2T_1 = tM_{330} + \frac{1-t}{tq^3-1}\frac{(t^3q^3-1)(t^2q^3-t)}{(t^2q^3-1)^2}M_{033} + \frac{(1-t)^2}{(tq^3-1)(t^{-2}q^{-3}-1)}M_{303}\,,$$

and $\text{Ↄ}(303,330) = t\,\widehat{\tfrac{1}{0}}$, $\text{Ↄ}(303,033) = \widehat{\tfrac{1}{0}}\,\frac{1-t}{tq^3-1}$, $\text{Ↄ}(303,303) = \frac{1-t}{t^{-2}q^{-3}-1}\,\frac{1-t}{tq^3-1}$.

Repeated application of Lemmas 8,9 furnishes the specialization $M_v(\langle v\rangle)$. One can show that it is a polynomial, and not only a rational fraction. For example, writing ij for $t^iq^j - 1$, then $M_{641}(\langle 6,4,1\rangle)$ is equal to the product of all the contents of the boxes below :

$$\begin{array}{|c|c|c|}
\hline
26 & 14 & 01 \\
\hline
t15 & t03 & \\
\cline{1-2}
t14 & t02 & \\
\cline{1-2}
t13 & t01 & \\
\cline{1-2}
t^202 & & \\
\cline{1-1}
t^201 & & \\
\cline{1-1}
\end{array}$$

and we leave it to the reader to formulate the general rule.

6 Multiplication by an indeterminate

As with the Schubert basis, to recover the multiplicative structure of the ring of polynomials, we essentially need to describe the multiplication by $x_1, \ldots, x_n$ in the Macdonald basis. However, in the present case, multiplication by $x_1 + \cdots + x_n$ will suffice.

Given $u \in \mathbb{N}^n$, we use the same approach as in the Schubert case, and say that v is a *successor* of u iff $|v| = |u|+1$ and $M_u(\langle v \rangle) \neq 0$.

Lemma 12 *The terms appearing with a non-zero coefficient in the expansion*

$$\big(x_1 + \cdots + x_n - |\langle u \rangle|\big)\, M_u = \sum_v c_u^v M_v$$

are exactly the successors of u.

Proof. The LHS is a polynomial of degree $|u| + 1$, which vanishes in every point $\langle w \rangle : |w| \leq |u|$, because the linear factor provides the extra vanishing in $\langle u \rangle$. The RHS belongs therefore to the span of $M_v : |v| = |u| + 1$. Since the linear factor does not vanish for any such $\langle v \rangle$, the RHS is a sum over the successors of u. Q.E.D.

Of course, as we have used for Schubert polynomials, specializing the LHS in every successor in turn furnishes the coefficients c_u^v :

$$(x_1 + \cdots + x_n - |\langle u \rangle|)\; M_u = \sum_v \frac{\big(|\langle v \rangle| - |\langle u \rangle|\big)\, M_u\left(\langle v \rangle\right)}{M_v\left(\langle v \rangle\right)}\; M_v\,. \tag{13}$$

This is however not very informative, one does not want to test all the specializations of $M_u(\langle v \rangle)$. For example, for $u = [5, 0, 2]$, the successors are much fewer than the compositions of 8 in three parts, being

$$[5,0,3], [0,2,6], [5,1,2], [6,0,2], [2,0,6], [0,6,2], [5,2,1]$$

This list can be structured by connecting its elements by the simple transpositions s_0 $- -$, s_1 —–, s_2 $=\!=$.

$$\begin{tikzcd}
503 \arrow[r, dash, dashed] & 206 \arrow[dr, dash] & & & \\
& & 026 \arrow[r, dash, double] \arrow[dl, dash, dashed] & 062 \arrow[r, dash] & 602 \\
512 \arrow[r, dash, double] & 521 & & &
\end{tikzcd}$$

Theorem 13 *Let $u, v \in \mathbb{N}^n$, $|v| = |u| + 1$. Then v is a successor of u iff there exists a subword $s_i \cdots s_j$ of one of the words $s_{n-1} \cdots s_1$, $s_{n-2} \cdots s_1 s_0, \ldots,$ $s_0 s_{n-1} \cdots s_2$, such that $v = u\tau s_i \cdots s_j$. In that case,*

$$\frac{M_u(\langle v \rangle)}{M_v(\langle v \rangle)} = t^{1-n}\, \eth(u,v) \Cup (u,v)\,. \tag{14}$$

Proof. The two sides of (14) are invariant under the action of τ. One remarks that $M_u(\langle v \rangle) = 0$ if $u_n \neq 0$, $v_n = 0$, because $x_n - 1$ is a factor of $M_u = M_{u\bar{\tau}}\Phi$. Therefore, using the maximal possible power of $\bar{\tau}$, one is reduced to compute $M_u(\langle v \rangle)$ in the case $u_n = 0$.

Since all the successors of u occur in the product $(x_1 + \cdots + x_n - |\langle u \rangle|)\, M_u$ and since the linear factor commutes with $T_1, \ldots, T_{n-1}$, one also knows how to relate the successors of all the permutations of u. We have only to check that the constants appearing in (10,12) correspond to the variation of $\eth(u,v) \Cup (u,v)$ under the action of $T_1, \ldots, T_{n-1}$, to prove the theorem by induction on $|u|$. Q.E.D.

For example, the successors of $[5,0,2]$ seen above are determined by those of $[5,0,2]\,\tau^{-1} = [1,5,0]$:

$$\begin{tikzcd}
250 \arrow[r, no head] & 520 \arrow[dr, no head, double] & & & \\
& & 502 \arrow[r, no head, dashed] & 106 \arrow[r, no head, double] & 160 \\
151 \arrow[r, no head, dashed] & 052 \arrow[ur, no head] & & &
\end{tikzcd}$$

Using T_2, for example, one deduces $M_{520}(\langle 503 \rangle) M_{503}(\langle 503 \rangle)^{-1}$ and $M_{520}(\langle 530 \rangle) M_{530}(\langle 530 \rangle)^{-1}$ from the value $M_{502}(\langle 503 \rangle) M_{503}(\langle 503 \rangle)^{-1}$.

The linear factor $x_1 + \cdots + x_n - |\langle u \rangle|$ is not compatible with the translation of indices. However, the sum of all the linear factors $x_i \langle u \rangle_i^{-1} - 1$ is such, since $\langle u\tau \rangle_n = q \langle u \rangle_1$. Therefore one has

$$M_u \left(\frac{x_1}{\langle u \rangle_1} + \cdots + \frac{x_n}{\langle u \rangle_n} - n \right) \Phi = M_w \left(\frac{x_1}{\langle w \rangle_1} + \cdots + \frac{x_n}{\langle w \rangle_n} - n \right), \tag{15}$$

with $w = u\tau = [u_2, \ldots, u_n,\, u_1{+}1]$.

Using this new linear factor, (13) rewrites as

$$M_u\left(\frac{x_1}{\langle u\rangle_1}+\cdots+\frac{x_n}{\langle u\rangle_n}-n\right)=\sum_v \frac{M_u(\langle v\rangle)}{M_v(\langle v\rangle)}\big(|\langle v\rangle|-|\langle u\rangle|\big)\,M_v\,, \tag{16}$$

summed over all the successors v of u.

Let us now turn our attention towards the single elements $x_i\langle u\rangle_i^{-1}-1$. They also vanish in $\langle u\rangle$, and therefore give the following analogs of Monk's formula (3).

Proposition 14 *Let $u\in\mathbb{N}^n$, $i\in\{1,\ldots,n\}$. Then*

$$M_u\left(\frac{x_i}{\langle u\rangle_i}-1\right)=\sum_v \frac{M_u(\langle v\rangle)}{M_v(\langle v\rangle)}\left(\frac{\langle v\rangle_i}{\langle u\rangle_i}-1\right)M_v\,, \tag{17}$$

summed over all the successors v of u such that $\langle v\rangle_i\neq\langle u\rangle_i$.

In the case $i=1$, the constants in (17), expressed with (14), differ from those appearing in (11) only by the linear factor $\langle v\rangle_i\langle u\rangle_i^{-1}-1$. In that case, (17) is equivalent to the action of the Ξ_1 operator of Knop [10, Th.3.6], that we reformulate as :

Corollary 15 *For any $u\in\mathbb{N}^n$, one has*

$$M_u\left(\frac{x_1}{\langle u\rangle_1}-1\right)t^{n-1}=M_{u\tau}\,T_{n-1}\cdots T_1\,. \tag{18}$$

More generally, one knows how to expand a product $M_u(x_i\langle u\rangle_i^{-1}-1)(x_j\langle u\rangle_j^{-1}-1)\cdots(x_k\langle u\rangle_k^{-1}-1)$, $i,j,\ldots,k$ all different. For example, for $n=4$, one has

$$M_u\left(\frac{x_1}{\langle u\rangle_1}-1\right)\left(\frac{x_3}{\langle u\rangle_3}-1\right)t^3=M_uT_2\,\Phi^2\,T_2T_3T_1-t^2(t-1)M_uT_2\,\Phi\,T_3\,.$$

Higher degree cases would require products of operators similar to those in [14, Lemma 7.4]. However, coefficients will no more factorize into simple factors t^iq^j-1 as in (17).

As for Schubert polynomials, choosing an appropriate i provides a recursion on the Macdonald basis, that we shall call a *transition*.

Theorem 16 *Given $v \in \mathbb{N}^n$, $v \neq 0^n$, let i be the leftmost position of the maximum of $(v_1, \ldots, v_n)$. Let $u = [v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1}, \ldots, v_n]$ and $\langle u \rangle_i = q^a t^b$. Then*

$$M_v = (x_i q^{-a} - t^b) M_u - \sum_w \frac{M_u(\langle w \rangle)}{M_w(\langle w \rangle)} \left(\frac{\langle w \rangle_i}{\langle u \rangle_i} - 1 \right) M_w \,, \tag{19}$$

summed over the successors of u such that $\langle w \rangle_i \neq \langle u \rangle_i$, and $w \neq v$. Those successors are such that $w < v$.

Proof. By translation, one can suppose that $i = 1$. Let $v = [\alpha, \ldots]$, $u = [a, \ldots]$, with $a = \alpha - 1$; let β be the number of components of u equal to α. Then $b = n-1-\beta$. The only linear factors $\langle v \rangle_i \langle u \rangle_i^{-1}$, apart from $\langle v \rangle_1 \langle u \rangle_1^{-1} = t^\beta q - 1$, which do not vanish are those corresponding to the components of u equal to α. They are equal to $q^\alpha t^i / (q^\alpha t^{i+1} - 1)$, $i = n-2, \ldots, n-1+\beta$. On the other hand, $\Cup(v, u\tau) = 1$. In final, $M_v(\langle v \rangle)/M_u(\langle v \rangle) = t^b(t^\beta q - 1)$, and therefore, the coefficient of M_v in the expansion of $(x_1 q^{-a} - t^b) M_u$ is equal to 1. Moreover, the other successors of u are $< v$. Q.E.D.

Notice that the exponent b is equal to

$$n - 1 - \#(j : j > i, v_j = a{+}1) - \#(j : j < i, v_j = a) \,. \tag{20}$$

One can iterate the transition formula. This gives a canonical decomposition of any Macdonald polynomial into sums of products of "shifted monomials" $\prod (x_i q^{-a} - t^b)$, the specialization $t = 0$ of these monomials being of degree $|v|$.

For example, writing ij for a factor $t^i q^j - 1$, starting with $v = [2, 0, 2]$, one has $u = [1, 0, 2]$, $\langle u \rangle = [tq, 1, t^2 q^2]$ and the following sequence of transitions :

$$\begin{aligned}
M_{202} &= \left(x_1 q^{-1} - t\right) M_{102} + \frac{10}{22} M_{022} \,, \\
M_{022} &= \left(x_2 q^{-1} - t\right) M_{012} + tq \frac{10 \cdot 10}{11 \cdot 21} M_{121} + \frac{10 \cdot 31}{21 \cdot 21} M_{112} \,, \\
M_{121} &= \left(x_2 q^{-1} - t\right) M_{111} + \frac{10}{21} M_{112} \,, \\
M_{112} &= \left(x_3 q^{-1} - 1\right) M_{111} \,,
\end{aligned}$$

leading to polynomials of degree 3 that one assumes to be known by induction on the degree.

To reduce the size of the output, let us represent each factor $x_j/q^{i-1} - t^k$ by a black square in the Cartesian plane (row i, column j) (k is determined by i, j, according to (20)). Then the final outcome of the transitions for M_{202} is

$$\begin{smallmatrix}\cdot&\cdot&\cdot\\ \cdot&\blacksquare&\cdot\\ \blacksquare&\blacksquare&\blacksquare\end{smallmatrix}\ \frac{10\cdot 10}{11\cdot 22} + \begin{smallmatrix}\cdot&\cdot&\cdot\\ \cdot&\cdot&\blacksquare\\ \blacksquare&\blacksquare&\blacksquare\end{smallmatrix}\ \frac{10\cdot 10}{11\cdot 22} + \begin{smallmatrix}\cdot&\cdot&\cdot\\ \blacksquare&\cdot&\cdot\\ \blacksquare&\blacksquare&\blacksquare\end{smallmatrix}\ \frac{10}{11} + \begin{smallmatrix}\cdot&\cdot&\cdot\\ \blacksquare&\cdot&\blacksquare\\ \cdot&\blacksquare&\blacksquare\end{smallmatrix}\ \frac{10}{11} + \begin{smallmatrix}\cdot&\cdot&\cdot\\ \cdot&\blacksquare&\blacksquare\\ \cdot&\blacksquare&\blacksquare\end{smallmatrix}\ \frac{10}{22} + \begin{smallmatrix}\cdot&\cdot&\cdot\\ \blacksquare&\cdot&\blacksquare\\ \blacksquare&\cdot&\blacksquare\end{smallmatrix}$$

with leading term $\begin{smallmatrix}\cdot&\cdot&\cdot\\ \blacksquare&\cdot&\blacksquare\\ \blacksquare&\cdot&\blacksquare\end{smallmatrix} = (x_1 q^{-1} - t)(x_1 - t)(x_3 q^{-1} - t)(x_3 - 1)$.

Haglund, Haiman, Loehr [7] give a combinatorial formula for the component of degree $|v|$ of M_v, which involves, in general, another enumeration than the one by transition.

We do not see how to relate the combinatorial expressions of Okounkov [21, 22] for symmetric Macdonald polynomials to the present one. Both use shifted monomials which are not linearly independent, and different algorithms can produce different expressions.

7 Principal specialization

The *homogeneous non-symmetric Macdonald polynomials* E_v [2, 19] are the terms of degree $|v|$ of M_v. Their specialization in $x_1 = t^{n-1}, \ldots, x_n = t^0$ is called the *principal specialization*. But all M_v , except when $v = [0, \ldots, 0]$ vanish in $\langle 0, \ldots, 0\rangle = [t^{n-1}, \ldots, 0]$, we have to find another specialization for them.

Let us introduce another indeterminate z and define the *principal specialization* to be the ring morphism $x_1\theta_z = zt^{n-1}, \ldots,\ x_n\theta_z = z$. One has the following commutative diagrams (that one has only to test on the polynomials $1, x_{i+1}$:

$$\begin{tikzcd} \bullet \arrow[r, "T_i"] \arrow[d, "\theta_z"'] & \bullet \arrow[d, "\theta_z"] \\ \bullet \arrow[r, "t"] & \bullet \end{tikzcd} \quad , \quad \begin{tikzcd} \bullet \arrow[r, "T_i+(t-1)/(\alpha-1)"] \arrow[d, "\theta_z"'] & \bullet \arrow[d, "\theta_z"] \\ \bullet \arrow[r, "(t\alpha-1)/(\alpha-1)"] & \bullet \end{tikzcd} \ . \tag{21}$$

The compatibility of θ_z is less straightforward. For example, for $n = 3$, a polynomial of degree 1 gives the diagram

$$\begin{tikzcd} ax_1+bx_2+cx_3+d \arrow[r, "\Phi"] \arrow[d, "\theta_z"'] & (ax_3q^{-1}=bx_1+cx_2+d)(x_3-1) \arrow[d, "\theta_z"] \\ azt^2+bzt+cz+d \arrow[r, "?"] & (azq^{-1}+bzt^2+czt+d)(z-1) \end{tikzcd} .$$

Fortunately, we have found expressions involving Φ which are easy to specialize. Indeed, $M_v\,(x_1\langle v\rangle_1^{-1}-1)t^{n-1} = M_v\Phi T_{n-1}\cdots T_1$. Therefore

$$M_v\theta_z\left(\frac{t^{n-1}z}{\langle v\rangle_1}-1\right) = M_{v\tau}\,\theta_z\,. \tag{22}$$

Iterating, one sees that $M_v\theta_z$ is a product of factors of the type $(t\alpha-1)(\alpha-1)^{-1}$ or of the type $(t^{n-1}z\langle v\rangle_1^{-1}-1)$. The first factors are associated to the inversions of v (as an infinite vector), the second ones record the affine steps in the recursive definition of M_v. More precisely, let

$$⋒(v) := \prod_{i=1}^{n}\prod_{v_i>v_j}^{j=i+1\ldots\infty}\frac{t\langle v\rangle_i\langle v\rangle_j^{-1}-1}{\langle v\rangle_i\langle v\rangle_j^{-1}-1}\,,$$

and

$$\varphi_z(v) := \prod_{i=1}^{n}\left(\frac{zt^{n-1}q^1}{\langle v\rangle_i}-1\right)\cdots\left(\frac{zt^{n-1}q^{v_i}}{\langle v\rangle_i}-1\right)\,.$$

Then (21), (22) imply the following description of the principal specialization.

Proposition 17 *Let $v\in\mathbb{N}^n$. Then*

$$M_v(zt^{n-1},\ldots,z) = \varphi_z(v)\ ⋒(v)\,. \tag{23}$$

In particular,

$$M_v(0,\ldots,0) = (-1)^{|v|}\ ⋒(v)\,. \tag{24}$$

One notices that the homogeneous polynomials E_v are obtained from the same recursions than the M_v, except for a factor x_n, instead of x_n-1, in an affine step $E_v\to E_{v\tau}$. Therefore, the specialization $E_v(zt^{n-1},\ldots,z)$ is equal to the coefficient of $z^{|v|}$ in $M_v(zt^{n-1},\ldots,z)$, and this gives the following value of the principal specialization of E_v.

Corollary 18 *Let $v \in \mathbb{N}^n$ and λ be the decreasing reordering of v. Then*

$$E_v(zt^{n-1}, \ldots, z) = z^{|v|} t^{\sum(i-1)\lambda_i} q^{-\sum \binom{v_i}{2}} \Cap(v)\,. \tag{25}$$

Given a polynomial f, filter it by the degree in $x_1, \ldots, x_n$: $f = f_d + \cdots + f_0$. Then, thanks to (24), (18), the operator $f \to f_d(t^{n-1}, \ldots, 1) f_0^{-1} f$, on polynomials with constant term, has eigenfunctions the Macdonald polynomials M_v, with eigenvalues $t^{\sum(i-1)\lambda_i} q^{-\sum \binom{v_i}{2}}$. Apart from the change $q \to 1/q$, this operator induces the "nabla operator" of Garsia et al [1] on symmetric homogeneous Macdonald polynomials.

Remark. The numerator and denominator of $\Cap(v)$ have common factors. One can reduce the denominator to the contribution of the inversions v_i, v_j : $v_i > v_j$, $j > i, i \leq n$ such that there is no $k : i < k < j$, $v_k = v_j$. This new denominator $D(v)$ is a factor of the normalizing factor that Knop [9, Th. 5.1] uses to remove the denominator of M_v. In fact, the analysis of Knop of the case where v has equal components can be refined to show that $q^{\sum v_i(v_i-1)/2} D(v) M_v$ is integral. For example , $\Cap([0,2,4]) = (t^2q-1)(t^3q-1)(t^3q^2-1)(t^3q^3-1)/D([0,2,4])$, $D([0,2,4]) = (tq-1)^2(t^2q^2-1)(t^2q^3-1)$, and $q^7 D([0,2,4]) M_{0,2,4}$ is integral.

8 Some Properties

We have seen that a subset of Schubert polynomials constitute a basis of the ring of symmetric polynomials. This basis is the image of the dominant polynomials Y_λ, $\lambda \in \mathfrak{Part}$, under ∂_ω. *Symmetric Macdonald polynomials P_λ* can be defined by the same vanishing conditions than the M_v :

$$P_\lambda(\langle \mu \rangle) = 0 \quad \forall \mu \in \mathfrak{Part},\ |\mu| \leq |\lambda|,\ \mu \neq \lambda\ , \tag{26}$$

together with some normalization conditions.

However, $\{P_\lambda\}$ is not a subfamily of $\{M_v\}$. One needs to use a symmetrization operator, that we call the *Euler-Poincaré characteristic* [4, 14], because of its use in the study of flag manifolds: $\square_\omega := \Delta_t\, \partial_\omega$, where Δ_t is the t-Vandermonde $\prod_{1 \leq i < j \leq n} (tx_i - x_j)$.

When $n = 2$, $\square_{21} = (tx_1 - x_2)\partial_1$ is equal to $T_1 + 1$. Therefore, $\square_i := (tx_i - x_{i+1})\partial_i$, $i = 1, \ldots, n{-}1$ generate the Hecke algebra, but satisfy the

Yang-Baxter relations instead of the braid relations :

$$\left(\square_i + \frac{t-\alpha}{\alpha-1}\right)\left(\square_{i+1} + \frac{t-\alpha\beta}{\alpha\beta-1}\right)\left(\square_i + \frac{t-\beta}{\beta-1}\right)$$
$$= \left(\square_{i+1} + \frac{t-\beta}{\beta-1}\right)\left(\square_i + \frac{t-\alpha\beta}{\alpha\beta-1}\right)\left(\square_{i+1} + \frac{t-\alpha}{\alpha-1}\right),$$

for any α, β such that $\alpha, \beta, \alpha\beta \neq 1$. In particular (cf. [4]),

$$\begin{aligned}\square_\omega &= \square_1 \left(\square_2 - t\frac{[1]}{[2]}\right)\square_1 \cdots \left(\square_{n-1} - t\frac{[n-2]}{[n-1]}\right)\cdots\left(\square_1 - t\frac{[0]}{[1]}\right)\\ &= \left(T_1 + \frac{1}{[1]}\right)\left(T_2 + \frac{1}{[2]}\right)\left(T_1 + \frac{1}{[1]}\right)\cdots\left(T_{n-1} + \frac{1}{[n+1]}\right)\cdots\left(T_1 + \frac{1}{[1]}\right), \end{aligned} \tag{27}$$

where $[i]$ is the t-integer $(t^i - 1)(t-1)^{-1}$.

Given $\lambda \in \mathfrak{Part}$, let $\nu = [\lambda_n, \ldots, \lambda_1]$. Then $M_\nu \square_\omega$ is symmetrical, and, thanks to (27), belongs to the linear span of $\{M_v,\ v \text{ permutation of } \lambda\}$. Therefore, $M_\nu \square_\omega$ satisfies the required vanishing conditions. In other words, $\square_\omega$ projects the non-symmetric Macdonald polynomials onto the symmetric ones, up to normalization. Notice that the image of a dominant monomial x^λ is the *Hall-Littlewood polynomial* (still, up to normalization). In that case, one interprets $\square_\omega$ as a product of raising operators [18, III.2].

Both Schubert and Macdonald bases acquire special properties when one specializes the parameters ($\mathbf{y}$ in the Schubert case, t, q in the Macdonald case).

The specialization $q = 1$ for Macdonald polynomials is related to the specialization $y_i = t^{i-1}$ of Schubert polynomials. I shall just give one property in that respect.

Proposition 19 *Let v be dominant, k be such that $v_k \neq 0$, $v_{k+1} = 0 = \cdots = v_n$. Let the partition conjugate to $[v_1, \ldots, v_k]$ be $1^{\mu_1}2^{\mu_2}\cdots$. Then*

$$M_v\big|_{q=1} = (Y_{0^{n-1},1})^{\mu_1}\,(Y_{0^{n-2},1,1})^{\mu_2}\,(Y_{0^{n-3},1,1,1})^{\mu_3}\cdots,$$

specializing $y_1 = 1, y_2 = t, y_3 = t^2, \ldots$ inside the Schubert polynomials.

Proof. One can suppose by induction the statement to be true for $u = [v_1{-}1, \ldots, v_k{-}1, 0, \ldots, 0]$. In particular $M_u|_{q=1} = f$ is a symmetric function.

Therefore, $M_{[0,\ldots,0,v_1,\ldots,v_k]}\big|_{q=1} = f\,(x_{n-k+1}-1)\cdots(x_n-1)$, and $M_v\big|_{q=1}$ is, up to a t-factorial, equal to

$$\begin{aligned} f\,(x_{n-k+1}-1)\cdots(x_n-1)\,\Box_\omega\big|_{q=1} &= (x_{n-k+1}-1)\cdots(x_n-1)\,\Box_\omega\big|_{q=1} f\,, \\ &= M_{0^{n-k}1^k}\Box_\omega\big|_{q=1} f\,. \end{aligned}$$

Because the constants used in the recursion (10) specialize into the inverses of t-integers appearing in (27), then $M_{0^{n-k}1^k}\Box_\omega|_{q=1}$ is proportional to $M_{1^k0^{n-k}}|_{q=1}$. But the specialized Schubert polynomial $Y_{0^{n-k}1^k}$ satisfy the same vanishing conditions than the polynomial $M_{1^k0^{n-k}}|_{q=1}$; moreover the coefficient of $x_1\cdots x_k$ is 1 in both polynomials. Q.E.D.

Instead of $\Box_\omega$, one can use the operator $\nabla_\omega := \partial_\omega\,\widetilde{\Delta}_t$, where $\widetilde{\Delta}_t := \prod_{1\le i<j\le n}(tx_j - x_i)$. This operator factorizes in the same way as $\Box_\omega$. For example, writing $\nabla_i := \partial_i(tx_{i+1}-x_i) = T_i - t$, then for $n=3$,

$$\begin{aligned} \nabla_\omega = \partial_{321}(tx_2-x_1)(tx_3-x_1)(tx_3-x_2) &= \nabla_1\left(\nabla_2 + \frac{t}{1+t}\right)\nabla_1 \\ &= \left(T_1 + \frac{t-1}{t^{-1}-1}\right)\left(T_2 + \frac{t-1}{t^{-2}-1}\right)\left(T_1 + \frac{t-1}{t^{-1}-1}\right). \end{aligned} \quad (28)$$

Let us finish on an example of the use of ∇_ω.

Proposition 20 *Let $k\in\mathbb{N}$, $\rho=[n{-}1,\ldots,0]$, $\rho\cdots\rho$ be the concatanation of k copies of ρ. The $M_{\rho\cdots\rho}$ specializes, for $q=t^{-1-k}$, into the product*

$$(-1)^{kn(n-1)/2}\,\widetilde{\Delta}_t(x_1,\ldots,x_n)\cdots\widetilde{\Delta}_t(x_{(k-1)n+1},\ldots,x_{kn})\,.$$

Proof. Let $v=[0,\ldots,n{-}1,\ldots,0,\ldots,n{-}1]$. The vector $<v>$ is the concatanation of k vectors of the type $[z, zt^kq, z(t^kq)^2,\ldots,z(t^kq)^{n-1}]$. Such vector specializes, for $q=t^{-1-k}$, into $[z, zt^{-1},\ldots,zt^{1-n}]$. Recognizing the constants appearing in (28), one concludes that the operator transforming M_v into $M_{\rho\cdots\rho}$ is a direct product of operators ∇_ω on each block of variables $x_1,\ldots,x_n$, …, $(x_{(k-1)n+1},\ldots,x_{kn})$.

For degree reasons, controlling the coefficient of $x^{\rho\cdots\rho}$, one sees that $M_{\rho\cdots\rho}$ coincides with the written product of t-Vandermonde determinants. Q.E.D.

Notice that ∇_i is not invertible; indeed, $\nabla_i^2 = -(t-1)\nabla_i$. We have obtained M_{1010} from M_{0101}, but we cannot recover M_{0101} from M_{1010}, after specializing q. As a matter of fact, the action of the Hecke algebra on, say M_{210210}, generate a 5-dimensional space which is a t-deformation of the *Specht representation* of $\mathfrak{S}_6$ indexed by the partition $[2,2,2]$. The polynomials $M_{210210}, M_{212010}, M_{221010}, M_{212100}, M_{221100}$ constitute a basis of this space. Physicists [3, 8] use other bases.

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