PARTIAL THETA FUNCTIONS

S. OLE WARNAAR

1. Introduction

The Jacobi theta function

$$\theta_3(u;q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nu) = \sum_{n=-\infty}^{\infty} x^n q^{n^2} \tag{1.1}$$

(where $u, q \in \mathbb{C}$, $0 < |q| < 1$ and $x := \exp(2\mathrm{i}u)$) and its close cousins θ_1, θ_2 and θ_4 form an important class of transcendental functions [142]. They are quasi doubly-periodic entire functions and the key building blocks of elliptic functions. The theta functions satisfy a large number of identities, chief among them the Jacobi triple product identity [8, 59]

$$\theta_3(u;q) = \prod_{k=1}^{\infty}(1+xq^{2k-1})(1+x^{-1}q^{2k-1})(1-q^{2k}). \tag{1.2}$$

Ramanujan contributed extensively to the theory of theta functions, often favouring the symmetric form obtained by setting $q^2 = ab$ and $x^2 = a/b$ in (1.1).

In the Lost Notebook Ramanujan stated numerous identities for functions that closely resemble ordinary theta functions. These functions, which were given the name 'partial theta functions' by Andrews [10], take the form

$$\Theta_{\mathrm{p}}(x;q) := \sum_{n=0}^{\infty} x^n q^{n^2} \tag{1.3}$$

or

$$\prod_{k=1}^{n}(1+xq^k)(1+x^{-1}q^k). \tag{1.4}$$

In (1.3) the sum over $\mathbb{Z}$ defining an ordinary theta function is replaced by a sum over the 'positive cone' $\{n \in \mathbb{Z} : n \geqslant 0\}$. In (1.4) the infinite product expression for ordinary theta functions is replaced by a finite product. By the q-binomial theorem [8, 59], such a product can be expanded as

$$(1+x)\prod_{k=1}^{n}(1+xq^k)(1+x^{-1}q^k) = \sum_{k=-n}^{n+1} x^k q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}, \tag{1.5}$$

where, for integers $0 \leqslant k \leqslant n$, $\begin{bmatrix} n \\ k \end{bmatrix} := \prod_{i=1}^{k}(1-q^{i+n-k})/(1-q^i)$ is a q-binomial coefficient. Up to a factor $\prod_{k\geqslant 1}(1-q^k)$, in the large-$n$ limit the left-hand side is a theta function in product form and the right-hand side a theta function in sum form.

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Before stating two representative examples of Ramanujan's identities for partial theta functions, we introduce some standard q-series notation. Ramanujan himself did not use this, and would typically just write the first few terms in a sum or a product in explicit form. Let n be a nonnegative integer. Then the q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ are defined by

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}) \quad \text{and} \quad (a;q)_\infty = \prod_{k\geqslant 1}(1-aq^{k-1}).$$

We also use the condensed notation

$$(a_1,\dots,a_k;q)_n = (a_1;q)_n\cdots(a_k;q)_n \quad \text{and} \quad (a_1,\dots,a_k;q)_\infty = (a_1;q)_\infty\cdots(a_k;q)_\infty,$$

so that the partial theta function (1.4) can be written as $(-xq,-q/x;q)_n$.

With the above notation we can state the following two examples of identities for partial theta functions from the Lost Notebook [122, p. 37]:[1]

$$\sum_{n=0}^{\infty}\frac{q^n}{(aq,q/a;q)_n} = (1-a)\sum_{n=0}^{\infty}(-1)^n a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}) + \frac{a}{(aq,q/a;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n} q^{\binom{n+1}{2}} \tag{1.6}$$

and [122, p. 29]:

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n}}{(aq^2,q^2/a;q^2)_n} = (1-a)\sum_{n=0}^{\infty}a^n q^{\binom{n+1}{2}} + \frac{a(q;q^2)_\infty}{(aq^2,q^2/a;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{3n} q^{n(3n+2)}(1+aq^{2n+1}). \tag{1.7}$$

Further identities of this type, as well as some simpler partial theta function identities from the Lost Notebook, will be discussed in Section 3.

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2. The early history of partial theta functions

Before presenting a more detailed discussion of Ramanujan's identities for partial theta functions, we briefly review some of the early history of these functions.

The first occurrence of partial theta functions appears to be in two papers by Eisenstein [50, 51] published in Crelle in 1844. In these papers Eisenstein gave a continued fraction expansion for $\Theta_{\mathrm{p}}(x;q)$ from which he deduced the simple fact that $\Theta_{\mathrm{p}}(r;1/k)$ is irrational for $k\in\mathbb{Z}\setminus\{-1,0,1\}$ and $r\in\mathbb{Q}^*$. Two years later Heine [63] recovered Eisenstein's continued fraction as a special case of a more general continued fraction expansion for the ${}_2\phi_1$ basic hypergeometric function. In 1915 Eisenstein's irrationality result for Θ_{p} was sharpened in a joint paper by Bernstein and Szász [25] and a follow-up work by Szász [115]. Shortly thereafter Tschakaloff [136, 137] studied the partial theta function

[1] Ramanujan stated (1.6) and (1.7) with (a,q) replaced by $(-a,x)$.

$\sum_{n\geqslant 0} x^n q^{\binom{n}{2}} = \Theta_{\mathrm{p}}(q^{-1/2}x; q^{1/2})$ and established not just irrationality but also linear independence results. He also noted the functional equation $\Theta_{\mathrm{p}}(x;q) - xq\,\Theta_{\mathrm{p}}(xq^2;q) = 1$. It is the 1 on the right, rather than the 0 in the case of ordinary theta functions, that obstructs quasi-periodicity along annuli. Since Tschakaloff's work on the arithmetics of partial theta functions, in certain parts of the literature partial theta functions are referred to as 'Tschakaloff functions' or 'Tschakaloff series'.

In an unrelated early development, in 1904 Hardy [61] studied the zeros of entire functions of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for positive real a_n. As one of his key examples he proved that all of the roots of the partial theta function $\Theta_{\mathrm{p}}(x;q)$ for $1/q^2 \geqslant 9$ are real and negative, with exactly one root in the interval $\bigl(-(1/q^2)^n, -(1/q^2)^{n-1}\bigr)$. Several years later, and inspired by Hardy's paper as well as earlier work of Laguerre [102] on limits of real-rooted polynomials, Petrovitch [117] was led to consider real-rooted entire functions f all of whose finite sections $a_0 + a_1x + \cdots + a_nx^n$ are real-rooted. Once again $\Theta_{\mathrm{p}}(x;q)$ served as a key example. In 1923 Hutchinson [68] pushed this line of study further, proving that $a_n^2/(a_{n-1}a_{n+1}) \geqslant 4$ for all positive integers n if and only if

(i) all zeros of f are real, simple and negative;
(ii) all zeros of $a_mx^m + \cdots + a_nx^n$ are real and negative (with the possible exception of $x = 0$) for all pairs of nonnegative integers $m < n$.[2]

Since $a_n^2/(a_{n-1}a_{n+1}) = 1/q^2$ for $f(x) = \Theta_p(x;q)$, Hutchinson's result improved Hardy's lower bound on $1/q^2$ for the roots of the partial theta function to be real and negative from 9 to 4.[3] Hutchinson also remarked that approximately and asymptotically the roots of $\Theta_p(x;q)$ for $|q| < 1$ are given by the sequence $(-q^{1-2n})_{n\geqslant 1}$.

Not much other work related to partial theta functions seems to predate that of Ramanujan, although special cases of (1.3) certainly did arise earlier in the q-series literature. For example, the specialisation $x = -q$ of (1.3) corresponds to Rogers' false theta series [124]

$$\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} = \biggl(\sum_{n\geqslant 0} - \sum_{n<0} \biggr) q^{2n(2n+1)}. \tag{2.1}$$

For more on such series in the work of Rogers and Ramanujan we refer the interested reader to [127].

3. Partial theta functions and the Lost Notebook

There are essentially two general approaches for proving partial theta function identities such as (1.6) and (1.7). Both are reviewed in full detail in Chapter 6 of Part II of the edited version of the Lost Notebook by Andrews and Berndt [15]. As was characteristic of his style, there are no hints in the Lost Notebook as to how Ramanujan discovered his identities and whether or not he had proofs.

The first approach relies upon the following master identity.

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[2] Hutchinson incorrectly stated 'real, simple and negative', a condition which either would require that $a_n^2/(a_{n-1}a_{n+1}) > 4$ [48, p. 215] or $n > m + 2$ [148].

[3] Hutchinson's theorem applied to $\Theta_{\mathrm{p}}(x;q)$ appears in Pólya and Szegő's classic text 'Problems and Theorems in Analysis II' [120, Problem 176, p. 66] without reference to Hutchinson.

Theorem 3.1. *For arbitrary a,b,c,d such that $|a|<1, |c/b|<1$ and $|q|<1$,*

$$\sum_{n=0}^{\infty}\frac{(c,d;q)_n q^n}{(aq,bq;q)_n}=(1-a)\sum_{n=0}^{\infty}\frac{(1/b;q)_{n+1}(cd/ab;q)_n a^n}{(c/b,d/b;q)_{n+1}} + \frac{(c,d;q)_\infty}{b(aq,bq;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq/c;q)_n}{(d/b;q)_{n+1}}\Big(\frac{c}{b}\Big)^n. \tag{3.1}$$

This theorem was first obtained by Andrews [10] using a range of summation and transformation formulas for basic hypergeometric series, including the q-binomial theorem, one of Heine's transformations for ${}_2\phi_1$ series and Sears' transformation for ${}_4\phi_3$ series.[4] Several years later Agarwal [1] (see also [2]) observed that, up to a Heine transformation of one of the terms, (3.1) is a specialisation of a three-term relation for ${}_3\phi_2$ series due to Sears, see e.g., [59, Exercise 3.6]. It is not inconceivable that Ramanujan knew (3.1) or one of the closely related identities obtained by transforming one or more of the infinite series. Modulo Heine's transformation [59, Eq. (III.2)] Ramanujan did record the $c=d=0$ case of (3.1) in the Lost Notebook [122, p. 40] (see also [9, p. 98]):

$$\sum_{n=0}^{\infty}\frac{q^n}{(aq,bq;q)_n}=(1-b^{-1})\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(aq;q)_n}\Big(\frac{a}{b}\Big)^n + \frac{1}{b(aq,bq;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}\Big(\frac{a}{b}\Big)^n, \tag{3.2}$$

a result proved combinatorially by Kim [75] and Levande [104]. Equation (1.6) from the introduction follows from (3.2) by specialising $b=1/a$ and then transforming the first term on the right using the Rogers–Fine identity [52, 124]. Ramanujan certainly knew of the latter; the Third Notebook [21, Chapter 16, Entry 7] contains an identity from which the Rogers–Fine identity easily follows, and by the time he returned to India he was of course well familiar with the work of Rogers. The second example from the introduction follows in a similar fashion from (3.1). This time one must replace $q\mapsto q^2$ followed by $(b,c,d)\mapsto(1/a,q,0)$. Transforming both series on the right using the Rogers–Fine identity then yields (1.7).

Almost all other partial theta function identities of the type (1.6) found in Lost Notebook can be obtained from (3.1). For some, such as [122, p. 4]

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^n}{(aq,q/a;q)_n}=(1-a)\sum_{n=0}^{\infty}a^n q^{\binom{n+1}{2}}+\frac{a(q;q^2)_\infty}{(aq,q/a;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n}q^{n^2+n}, \tag{3.3}$$

the Rogers–Fine identity alone does not suffice for writing the two series on the right in desired form, and further results from the theory of basic hypergeometric functions are required, see [10, 15]. The exception is Ramanujan's [122, p. 12]

$$\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_n q^n}{(aq,q/a;q)_n}=(1-a)\sum_{n=0}^{\infty}a^n q^{n^2+n}+\frac{a}{(aq,q/a;q)_\infty}\sum_{n=0}^{\infty}a^{3n}q^{n(3n+2)}(1-aq^{2n+1}), \tag{3.4}$$

which does not follow from (3.1). This is where the second general method, based on Bailey pairs, comes to the rescue.

[4] For a proof of the $c=0$ case of (3.1) which does not require the Sears transformation, see [130].

Two sequences $\alpha=(\alpha_n)_{n\geqslant 0}$ and $\beta=(\beta_n)_{n\geqslant 0}$ are said to form a Bailey pair relative to a if [12, 19, 139]

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{3.5}$$

Bailey pairs are related to partial theta functions by the following result [140].[5]

Theorem 3.2. *Let (α,β) be a Bailey pair relative to q, and extend the α-sequence to all integers n by $\alpha_{-n-1}:=\alpha_n$ for $n\geqslant 0$. Then*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\beta_n(q;q)_{2n}q^n}{(aq,q/a;q)_n}&=(1-a)\sum_{n=0}^{\infty}\alpha_n(-1)^n a^n q^{-\binom{n}{2}}\frac{1-q}{1-q^{2n+1}}\\&\quad+\frac{1}{(q,aq,q/a;q)_\infty}\sum_{r=1}^{\infty}(-1)^{r+1}a^r q^{\binom{r}{2}}\biggl(\sum_{n=-\infty}^{\infty}\alpha_n q^{(1-r)n}\,\frac{1-q}{1-q^{2n+1}}\biggr),\end{aligned} \tag{3.6}$$

provided $a\neq 0$ and all series converge.

A Bailey pair which readily implies the second Rogers–Ramanujan identity [124] is

$$\mathrm{B}(3):\quad \alpha_n=(-1)^n q^{n(n+1)+\binom{n}{2}}\,\frac{1-q^{2n+1}}{1-q},\qquad \beta_n=\frac{1}{(q;q)_n},$$

where B(3) refers to the Rogers–Slater labelling. It is easy to check that the expression for α_n is invariant under the substitution $n\mapsto -n-1$ so that the above α-sequence applies to all integers n. Substituting B(3) into (3.6) and using the Jacobi triple product identity to simplify the second sum over α_n to $(q;q)_\infty(-1)^{\lfloor r/3\rfloor}q^{-(r-1)(r-2)/6}$ for $r\not\equiv 0 \pmod 3$ and zero otherwise, (3.4) follows. The earlier examples also follow from (3.6). In the case of (3.3) the required Bailey pair is E(3) [128], and in the case of (1.6) and (1.7) two Bailey pairs not part of Slater's original list are needed, see [140] for details.

One advantage of using Bailey pairs to prove Ramanujan's identities for partial theta functions is that it suggests how to generalise each of his results. As independently discovered by Andrews [12] and Paule [116], Bailey pairs can be iterated to give new Bailey pairs, leading to what is known as the Bailey chain. This implies that many of Ramanujan's identities for partial theta functions extend to infinite families of such identities. For example, (3.4) is the modulus-3 case of the following partial theta identity for all odd moduli [140]:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(q;q)_{2n}q^n}{(aq,q/a;q)_n}\sum_{0\leqslant n_{k-1}\leqslant\cdots\leqslant n_1\leqslant n}\frac{q^{n_1^2+\cdots+n_{k-1}^2+n_1+\cdots+n_{k-1}}}{(q;q)_{n-n_1}(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-2}-n_{k-1}}(q;q)_{n_{k-1}}}\\&\quad=(1-a)\sum_{n=0}^{\infty}a^n q^{kn(n+1)}+\sum_{i=1}^{\kappa-1}(-1)^{i+1}a^i q^{\binom{i}{2}}\frac{(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q,aq,q/a;q)_\infty}\sum_{n=0}^{\infty}a^{\kappa n}q^{kn(\kappa n+2i)},\end{aligned} \tag{3.7}$$

where k is a positive integer and $\kappa:=2k+1$ the odd modulus.

It was observed by Andrews [10] (see also [79, 80, 106, 140]) that by an appropriate residue calculus identities for partial theta functions of the form (1.6) yield new identities for the same class of functions. For example, (3.3) implies [140]

$$\sum_{n=0}^{\infty}(-1)^n a^n q^{n^2+n}=\sum_{n=0}^{\infty}\frac{(q,aq;q^2)_n(aq)^n}{(-aq;q)_{2n+1}}. \tag{3.8}$$

[5] The a in (3.6) should not be confused with the a in (3.5), which has been fixed to the value q.

Curiously, Ramanujan stated the $a=1$ case of this identity in the Lost Notebook [122, p. 13], but appears to have missed the full form despite its connection with (3.3) and the occurrence of other very similar identities, such as [122, p. 28]

$$\sum_{n=0}^{\infty} a^n q^{\binom{n+1}{2}} = \sum_{n=0}^{\infty} \frac{(-q;q)_{n-1} a^n q^{\binom{n+1}{2}}}{(-aq^2;q^2)_n}. \tag{3.9}$$

Interestingly, (3.8) is relevant to another problem first considered by Ramanujan, that of determining the asymptotic expansion of partial theta functions. In the Second Notebook (see [22, p. 547]) he noted that the false theta series (2.1) admits the asymptotic expansion

$$2\sum_{n=0}^{\infty}(-1)^n q^{n^2+n} \sim 1+t+t^2+2t^3+5t^4+17t^5+\cdots \tag{3.10}$$

where $q=(1-t)/(1+t)\to 1^-$ (i.e., $t\to 0^+$). Berndt and Kim [23] have shown, more generally, that the partial theta function (1.3) for $0<q<1$ and $x=-q^b$ ($b\in\mathbb{R}$) admits the asymptotic expansion

$$2\sum_{n=0}^{\infty}(-1)^n q^{n^2+bn} \sim \sum_{n=0}^{\infty} a_n t^n. \tag{3.11}$$

They then used (3.8) to show that for b a positive integer the coefficients a_n are again integral. They also conjectured that for sufficiently large n all a_n have the same sign. This was settled by Bringmann and Folsom [34] using methods from the theory of modular forms. Specifically, for n sufficiently large, $a_n>0$ if $b\equiv 1,2 \pmod 4$ and $a_n<0$ if $b\equiv 0,3 \pmod 4$. In the case of Ramanujan's original asymptotic expansion (3.10), the series on the right has been shown by Stanley [133] to be the generating function of fixed-point-free alternating involutions in $\mathfrak{S}_{2n}$. For example, there are 5 fixed-point-free alternating involutions in $\mathfrak{S}_8$, given in cycle notation by (12)(34)(56)(78), (12)(38)(46)(57), (16)(24)(35)(78), (16)(24)(38)(57) and (18)(24)(36)(57). For more on the asymptotics of partial theta function, see [33, 56, 67, 71, 76, 81, 109, 111].

Viewed as a function of x, the partial theta function (1.3) is entire. By a special case of Hadamard's factorisation theorem, if f is a rank-0 entire function with simple zeros at $x_1, x_2, \dots$ and $f(0)=1$, then $f(x)$ admits the Hadamard factorisation $f(x)=\prod_{n\geqslant 1}(1-x/x_n)$. On page 26 of the Lost Notebook Ramanujan [122] gave the following remarkable formula for the Hadamard factorisation of the partial theta function (1.3):

$$\Theta_{\mathrm{p}}(x;q) = \prod_{n=1}^{\infty}\Big(1+xq^{2n-1}\big(1+y_1(n)+y_2(n)+\cdots\big)\Big), \tag{3.12}$$

where

$$y_1(n)=\frac{\sum_{i=n}^{\infty}(-1)^i q^{i^2+i}}{\sum_{i=0}^{\infty}(-1)^i(2i+1)q^{i^2+i}}, \qquad y_2(n)=y_1(n)\,\frac{\sum_{i=n}^{\infty}(-1)^i(i+1)q^{i^2+i}}{\sum_{i=0}^{\infty}(-1)^i(2i+1)q^{i^2+i}}.$$

This is significantly deeper than Hutchinson's observation about the zeros as discussed in the previous section. In [13] Andrews proved (3.12) by showing that

$$x_n^{-1} = -q^{2n-1}\big(1+y_1(n)+y_2(n)+O(q^{3n(n+1)})\big)$$

provided that $0<q<2^{-n-7}/3$.[6]

[6] The bound on q, which is not believed to be sharp, is taken from [16, Theorems 18.1.3 and 18.1.4].

The final type of results for partial theta functions that may be found in the Lost Notebook are a number of expansions of ordinary theta functions in terms of partial theta functions, not necessarily of the same nome q. One illustrative example is [122, p. 33]

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_{2n}}\,(aq,q/a;q)_n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{3n^2}. \tag{3.13}$$

Although elegant, results such as these are much less deep than his other formulas for partial theta functions. The q-binomial theorem allows for the partial theta function in the summand to be expanded as a Laurent polynomial in a, much like (1.5). Interchanging sums, and using a special case of the q-Chu–Vandermonde sum known as the Durfee rectangle identity, (3.13) follows. For a comprehensive treatment of (3.13) and related results, see [11] and [132].

4. On the general theory of partial theta functions

Unlike ordinary theta functions, a general theory of partial theta is still far from complete, and many properties of partial theta functions are not yet fully-understood. Below we survey a number of recent developments.

4.1. **The generalised triple product identity.** One of the central results in the theory of theta functions, the Jacobi triple product identity, does not have an analog for partial theta functions, and (1.3) does not admit a simple product form. Instead, there is a pair of closely related triple-product-like identities for the sum and product of two partial theta functions [140]:

$$1+\sum_{n=1}^{\infty}(-1)^n(a^n+b^n)q^{\binom{n}{2}} = (q,a,b;q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q;q)_{2n}q^n}{(q,a,b,ab;q)_n} \tag{4.1}$$

and [18]

$$\biggl(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\biggr)\biggl(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\biggr) = (q,a,b;q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q;q)_{2n}q^n}{(q,a,b,ab/q;q)_n}. \tag{4.2}$$

It is the first of these identities, which simplifies to the Jacobi triple product identity for $ab=q$, that underpins (3.6). For additional proofs, variants or generalisations of (4.1) and (4.2), see [**?**, 20, 108, 125, 135, 138, 141].

4.2. **The zeros and spectrum of partial theta functions.** Thanks to its representation as a product, we have full control of the zeros of a theta function. For a partial theta function, however, the structure of its zeros is much more complicated, and sensitively depends on the nome q. We already discussed the Hardy, Hutchinson and Andrews–Ramanujan results on the zeros of $\Theta_{\mathrm{p}}(x;q)$. To put these results in broader context, an entire function f belongs to the Laguerre–Pólya class if all of its roots are real, f has rank 1 (i.e., the sum $\sum_{n\geqslant 1} x_n^{-2}$ of nonzero roots, counted with multiplicity, is bounded), and f admits the Hadamard factorisation

$$f(x) = x^m\,\mathrm{e}^{a+bx+cx^2} \prod_{n\geqslant 1} \mathrm{e}^{x/x_n}\Bigl(1-\frac{x}{x_n}\Bigr),$$

where $a,b,c\in\mathbb{R}$ (with $c\leqslant 0$) and m is a nonnegative integer. The significance of this class of entire functions stems from the following theorem, established by Laguerre [101] and Pólya [118].

(i) If $(P_n)_{n\geqslant 1}$ is a sequence of complex polynomials all of whose roots are real and which converges uniformly inside a disk of positive radius, then this sequence converges uniformly on compact subsets in $\mathbb{C}$ to an entire function in the Laguerre–Pólya class.
(ii) If f belongs to the Laguerre–Pólya class then there exists a sequence of complex polynomials, all of whose roots are real, which converges uniformly to f on compact subsets of $\mathbb{C}$.

As a more restrictive notion, a function f is of type I in the Laguerre–Pólya class if it admits the representation

$$f(x)=x^m\,\mathrm{e}^{a+bx}\prod_{n\geqslant 1}\Big(1-\frac{x}{x_n}\Big),$$

where $b\geqslant 0$, $\sum_{n\geqslant 1}x_n^{-1}<\infty$ and $x_n<0$. An entire functions f of type I in the Laguerre–Pólya class is the uniform limit on compact subsets of $\mathbb{C}$ of a sequence of polynomials with negative real zeros and positive real coefficients [119]. From Hutchinson's theorem it follows that all zeros of $\Theta_{\mathrm{p}}(x;q)$ are real and negative for $1/q^2\geqslant 4$, a bound which was improved to $(37/20)^2=3.4255$ by Craven and Csordas [45]. Katkova, Lobova and Vishnyakova [72] then proved the existence of a constant $q_\infty=3.233636\dots$ such that $\Theta_{\mathrm{p}}(x;q)$ for $q>0$ has only negative real zeros—and hence is of type I in the Laguerre–Pólya class—if and only if $1/q^2\geqslant q_\infty$. See also [29]. Moreover, as shown by Nguyen and Vishnyakova [114], the partial theta function is extremal in the sense that any entire function of the form $f(x)=\sum_{n\geqslant 0}a_nx^n$ $(a_n>0)$ for which the sequence $a_n^2/(a_{n-1}a_{n+1})$ is decreasing with limit $\lim_{n\to\infty}a_n^2/(a_{n-1}a_{n+1})=b\geqslant q_\infty$ is in the Laguerre–Pólya class. In a similar vein, Katkova and Vishnyakova [73] have proven the existence of a constant, s_∞, such that all roots of $\Theta_{\mathrm{p}}(x;q)$ have negative real parts if $1/q<s_\infty$.

In a long series of papers [83–98] Kostov has made a detailed study of the zeros and spectrum of the partial theta function

$$\theta_{\mathrm{p}}(x;q):=\sum_{n=0}^{\infty}x^nq^{\binom{n+1}{2}}=\Theta_{\mathrm{p}}\big(xq^{1/2};q^{1/2}\big).$$

Here q for $|q|<1$ is a spectral value if $\theta_{\mathrm{p}}(x;q)$ has multiple zeros as a function of x. Kostov has shown that the only spectral value in the (closed) disk of radius 0.31 is $\tilde{q}_1:=1/q_\infty=0.309249\dots$. Moreover, for $0<q<1$ the partial theta function $\theta_{\mathrm{p}}(x;q)$ has infinitely many negative real zeros, and there exists a strictly increasing (real) sequence $(\tilde{q}_i)_{i\geqslant 1}$ (with $\tilde{q}_1$ as above) converging to 1 such that $\theta_{\mathrm{p}}(x;\tilde{q}_i)$ has a single double zero, which is the right-most of the real zeros. For all $0<q<1$ not contained in the sequence of spectral values, $\theta_{\mathrm{p}}(x;q)$ has no multiple zeros. Finally, for $q\in(\tilde{q}_i,\tilde{q}_{i+1}]$ there are exactly i pairs of complex conjugate zeros counted with multiplicity. Kostov and Shapiro [100] have also used the partial theta function $\theta_{\mathrm{p}}(x;q)$ to settle the Hardy–Petrovitch–Hutchinson problem. Let Δ_n denote the set of all polynomials $a_0+a_1x+\dots+a_nx^n$ with positive real a_i such that all of its sections $a_0+a_1x+\dots+a_ix^i$ for $1\leqslant i\leqslant n$ are real-rooted. The Hardy–Petrovitch–Hutchinson problem then asks for the lower bounds

$$m_n:=\inf_{f\in\Delta_{n+1}}\frac{a_n^2}{a_{n-1}a_{n+1}}.$$

Petrovitch [117] showed that $m_1=4$, $m_2=27/8$ and $m_3\approx 3.264$. Kostov and Shapiro proved that the m_n are algebraic, $m_n>m_{n+1}$ and $\lim_{n\to\infty}m_n=q_\infty$. Furthermore, they constructed a sequence of polynomials $\big(p_n(x)\big)_{n\geqslant 1}=(1+x,1+x+x^2/4,1+x+x^2/4+x^3/54,\dots)$ such that (i) $p_n(x)-p_{n-1}(x)=A_nx^n$, (ii) $p_{n+1}(x)$ realises the lower bound m_n, viz. $m_n=A_n^2/(A_{n-1}A_{n+1})$, and (iii) a scaled version of $p_{n+1}(1/x)$ converges to $\theta_{\mathrm{p}}(x;1/q_\infty)$.

Similar ratios of coefficients arise in the theory of Padé approximants of entire functions of the form $f(x) = \sum_{n\geqslant 0} a_n x^n$ ($a_n \neq 0$ but not necessarily real), with smooth coefficients, i.e., coefficients a_n such that $a_n^2/(a_{n-1}a_{n+1})$ has a finite limit as $n \to \infty$. Once again, partial theta functions are the simplest examples of such functions, since $a_n^2/(a_{n-1}a_{n+1})$ is constant. We refer the reader to the work of Lubinsky and Saff [107] for more on Padé approximants of partial theta functions and how they relate to the zero distribution of the Rogers–Szegő polynomials.

Sokal [129] considered a different (albeit related) kind of zero for partial theta functions. He viewed $\theta_{\mathrm{p}}(x;q)$ as a formal power series in q and defined the unique formal power series $-\xi_0(q)/q$, termed 'leading root', by the equation $\theta_{\mathrm{p}}(-\xi_0(q)/q;q) = 0$. He showed that $\xi_0(q) = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + \cdots =: a_0 + a_1 q + \cdots$ has strictly positive coefficients and that the ratio a_{k+1}/a_k converges to Katkova–Lobova–Vishnyakova constant q_∞ in the large-k limit. Sokal also posed the conjecture, which is still open, that the sequence $(c_n)_{n\geqslant 1}$, defined by $\xi_0(q) = \prod_{n\geqslant 1}(1-q)^{-c_n}$, is strictly positive, strictly increasing and strictly convex. Subsequently, Prellberg [121] gave a simple combinatorial interpretation of the coefficients m_k of the leading root in terms of rooted trees. Flores and González-Meneses [53] have further shown that these same coefficients, as well as the coefficients $m_{k,i} := [q^k]\big(\xi_0(q)\big)^i$ of the ith power of the leading root, arise naturally from counting braids in the Artin–Tits monoid

$$\mathrm{A}_\infty = \left\langle a_1, a_2, a_3, \dots \,\middle|\, \begin{matrix} a_i a_j = a_j a_i, & |i-j| \geqslant 2 \\ a_i a_j a_i = a_j a_i a_j, & |i-j| = 1 \end{matrix} \right\rangle$$

on infinitely many strands. Assuming the standard lex order $a_1 < a_2 < \cdots$, each element or braid of A_∞ has a unique maximal lex representative. Flores and González-Meneses proved that $m_{k,i}$ is the number of elements of length k in A_∞ whose maximal lex representative starts with one of the letters $a_1, \dots, a_i$. For example, the braids of length 4 starting with a_1 are

$$a_1^4,\ a_1^3 a_2,\ a_1^2 a_2^2,\ a_1^2 a_2 a_3,\ a_1 a_2^2 a_1,\ a_1 a_2^3,\ a_1 a_2^2 a_3,\ a_1 a_2 a_3^2,\ a_1 a_2 a_3 a_4,$$

so that $m_{4,1} = a_4 = 9$. As a consequence of this result, q_∞ admits the interpretation as the growth rate of A_∞.

4.3. **Modularity.** Unlike theta functions, partial theta functions are not modular forms. They provide, however, examples of the more recent concept of a quantum modular form as introduced by Zagier in [149]. Let Γ be a congruence subgroup of the modular group $\mathrm{SL}(2,\mathbb{Z})$ and $\mathcal{Q} \subset \mathbb{Q}$ an infinite 'quantum set'. Then $f : \mathcal{Q} \to \mathbb{C}$ is called a quantum modular form of weight k on Γ if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

$$f(z) - (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

extends to an open subset of $\mathbb{R}$ and has appropriate continuity or analyticity properties. If to each point of $\mathcal{Q}$ one can attach a formal power series over $\mathbb{C}$ subject to additional modularity properties, f is said to be a strong quantum modular form. Folsom, Ono and Rhoades [55] have shown that the partial theta series

$$\sum_{n\geqslant 0} (-1)^n q^{(n+a/b)^2}, \quad q = \mathrm{e}^{2\mathrm{i}\pi z},$$

for coprime integers $0 < a < b$ and b even, is a strong, weight-1/2 quantum modular form for $z \in \mathcal{Q}_{a,b}$ with quantum set $\mathcal{Q}_{a,b} := \{h/k \in \mathbb{Q} : \gcd(h,k) = 1,\ h > 0,\ b \mid 2h,\ b \nmid h,\ k \equiv a \pmod{b},\ k \geqslant a\}$. It was later pointed out by Bringmann, Creutzig and Rolen [30] that the condition $a < b$ may be dropped. In the same paper, they also showed that the Fourier coefficients of certain Jacobi forms of negative index can be decomposed in terms of partial theta functions. This was further extended to more general negative-index Jacobi forms in [32]. In Section 3 we discussed the use of the partial theta function identity (3.8) in the asymptotic expansions (3.10) and (3.11). Bringmann and Milas [31] have shown that a similar use of (3.8) allows for the radial limits of the false theta functions

$$F_{j,p}(z)=\bigg(\sum_{n\geqslant 0}-\sum_{n<0}\bigg)q^{(n+j/(2p))^2}$$

(for integers j,p such that $1\leqslant j\leqslant p$ and $p\geqslant 2$) to be explicitly computed. This shows that that $F_{j,p}(z)$ is a strong quantum modular form for z in the quantum set $\mathcal{Q}_p=\{h/k\in\mathbb{Q}:\gcd(h,k)=1\,2p\nmid k\}$.

Some of the partial theta function identities discussed in Section 3 also bear a close relationship with mock modular forms, including Ramanujan's mock theta functions. For example, for $a=-1$ the identity (1.6) simplifies to

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_n^2}=2\sum_{n=0}^{\infty}\Big(\frac{-12}{n}\Big)q^{(n^2-1)/24}-\frac{1}{(-q;q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}, \tag{4.3}$$

where $(\frac{\cdot}{\cdot})$ is the Kronecker symbol, and where both q-series on the right are false theta series. As discussed in greater detail in [35,41,64,113,123], the left-hand side is Ramanujan's famous third order mock theta function $f(q):=\sum_{n\geqslant 0}q^{n^2}/(-q;q)_n^2$ for $|q|>1$, i.e., the left-hand side of (4.3) is $f(q^{-1})$ for $|q|<1$. Furthermore, the first false theta series on the right (including the prefactor 2) can be rewritten as

$$f^\star(q^{-1}):=1-\sum_{n\geqslant 1}\frac{(-1)^n q^{\binom{n}{2}}}{(-q;q)_n},\quad\text{where}\quad f^\star(q)=1-\sum_{n\geqslant 1}\frac{(-1)^n q^n}{(-q;q)_n}=f(q).$$

In other words, although $f(q)=f^\star(q)$ inside the unit disc, for $|q|>1$ their difference is a false theta series, as can be seen immediately from Ramanujan's identity (1.6).

Identities related to the fifth order mock theta function $f_0(q):=\sum_{n\geqslant 0}q^{n^2}/(-q;q)_n$ for $|q|>1$ arise from (1.6) in a similar manner. In the Lost Notebook, Ramanujan stated ten identities relating the fifth order mock theta functions [122]. Andrews and Garvan [17] proved the equivalence within two sets of five of these, resulting in what they called the 'first and second mock theta conjecture', subsequently proved by Hickerson [65]. The first mock theta conjecture is

$$f_0(q)=\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q,q^4;q^5)_\infty}-2\Phi(q^2),$$

where

$$\Phi(q):=-1+\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}.$$

As for $f(q)$, the function $\Phi(q)$ is well-defined for $|q|>1$, and defining $\Phi^*(q):=\Phi(1/q)$, we have

$$\Phi^*(q)=-1-\sum_{n=0}^{\infty}\frac{q^{5n+1}}{(q^4;q^5)_{n+1}(q;q^5)_n}.$$

Zwegers [151] observed that replacing $q \mapsto q^5$ in (1.6) and then specialising $a = q$ or $a = 1/q$ leads to the following pair of identities for $\Phi^*(q)$:

$$\begin{aligned} -\Phi^*(q) &= \sum_{\substack{n=1 \\ n\equiv 1 \pmod 5}}^{\infty} \left(\frac{12}{n}\right) q^{(n^2-1)/120} - \frac{1}{(q,q^4;q^5)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2} \\ &= -\sum_{\substack{n=1 \\ n\equiv 4 \pmod 5}}^{\infty} \left(\frac{12}{n}\right) q^{(n^2-1)/120} + \frac{1}{(q,q^4;q^5)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(5n+1)/2}, \end{aligned}$$

where $|q| < 1$. These identities were applied by Lawrence and Zagier [103] in their work on quantum invariants of 3-manifolds, resulting in a 'strange'-type formula for the Witten–Reshetikhin–Turaev invariant of the Poincaré homology sphere. In much the same way, (1.6) for $q \mapsto q^5$ and $a = q^{\pm 2}$ implies two identities for the function $\Psi^*(q) = \Psi(1/q)$, where $\Psi(q)$ is featured in the second mock theta conjecture.

We finally remark that the universal mock theta functions of Gordon–McIntosh [60] and Hickerson [66] can all be expressed as simple sums over partial theta functions. For example, the universal mock theta function $g(z;q)$ is given by $zg(z;q) = R(z;q)/(1-z)-1$ with $R(z;q)$ Dyson's rank generating function [57]

$$R(z;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq,q/z;q)_n}.$$

By modifying (3.6) and again using the Bailey chain, Ji and Zhao [69] used this to show that Garvan's Rogers–Hecke type formulas for the universal mock theta functions [58] can be embedded into infinite series of such identities, much in the spirit of (3.7).

4.4. **Partial theta functions and combinatorics.** All of Ramanujan's partial theta function identities have been proven analytically through the use of basic hypergeometric functions and Bailey pairs. Finding combinatorial proofs is generally much harder, but significant progress has been made in this direction. We already mentioned Kim's and Levande's combinatorial proofs of (3.2). In his paper Kim also proved (4.2) combinatorially. Further combinatorial proofs, including proofs of (3.8) and (3.9), as well as applications to other areas in combinatorics, such as integer partitions, unimodal sequences and derangements, may be found in [3–6, 10, 24, 40, 42, 70, 77, 78, 105, 110, 131, 143, 144]. We single out [3], in which Alladi gave a particularly elegant application of Ramanujan's identity (3.9), closely resembling Euler's pentagonal number theorem [8]. If $r_{\mathrm{e/o}}(n)$ is the set of strict (or distinct) partitions of n with smallest part odd and number of parts even/odd, and $R_{\mathrm{e/o}}(n) := |r_{\mathrm{e/o}}(n)|$, then (3.9) implies that

$$R_{\mathrm{e}}(n) - R_{\mathrm{o}}(n) = \begin{cases} (-1)^k & \text{if } n = k^2, \\ 0 & \text{otherwise.} \end{cases}$$

For example, $r_{\mathrm{e}}(9) = \{(8,1),(6,3)\}$ and $r_{\mathrm{o}}(9) = \{(9),(6,2,1),(5,3,1)\}$ so that $R_{\mathrm{e}}(9) - R_{\mathrm{o}}(9) = -1$, whereas $r_{\mathrm{e}}(10) = \{(9,1),(7,3),(4,3,2,1)\}$ and $r_{\mathrm{o}}(10) = \{(7,2,1),(6,3,1),(5,4,1)\}$ so that $R_{\mathrm{e}}(10) - R_{\mathrm{o}}(10) = 0$.

4.5. **The Tschakaloff function.** Throughout this section, $q = r/s$ for r and s nonzero integers.

As mentioned previously, in the literature on the arithmetics of partial theta functions, these functions are better known as Tschakaloff functions. One of Tschakaloff's main theorems [136] is the irrationality of the Tschakaloff function

$$\mathcal{T}_q(x)=\sum_{n=0}^{\infty}x^n q^{\binom{n}{2}}$$

for $x\in\mathbb{Q}^*$ and $|s|>|r|^{(3+\sqrt{5})/2}$.[7] He also proved the more general result that the linear combination

$$A_1\mathcal{T}_q(x_1)+\cdots+A_m\mathcal{T}_q(x_m)$$

is irrational [137] provided $|s|>|r|^{(2m+1+\sqrt{4m^2+1})/2}$ and $\{A_i\}_{i=1}^m$ and $\{x_i\}_{i=1}^m$ are sets of nonzero rational numbers such that none of the ratios x_i/x_j are an integral power of q. In particular, this result implies that $1,\mathcal{T}_q(x_1),\dots,\mathcal{T}_q(x_m)$ are linearly independent over $\mathbb{Q}$. A quantative version of Tschakaloff's first irrationality result is due to Bundschuh [37], see also [36, 39, 126, 134, 150] for related results. Assuming once again that $x\in\mathbb{Q}^*$, he showed that if $\gamma:=\log|r|/\log|s|$ satisfies $0<\gamma<2/(3+\sqrt{5})$ then for every $\epsilon>0$ there exists a $b_\epsilon>0$ such that

$$\Big|\mathcal{T}_q(x)-\frac{a}{b}\Big|>|b|^{-1-\frac{1+\sqrt{5}}{2-(3+\sqrt{5})\gamma}-\epsilon}$$

for all $a,b\in\mathbb{Z}$ and $|b|\geqslant b_\epsilon$. This shows that for such q and x, the irrational number $\mathcal{T}_q(x)$ is not Liouville. For K a quadratic field, Bézivin [26] has further shown that for $\gamma>14$ the Tschakaloff function $\mathcal{T}_q(x)$ for $x\in K^*$ is not also in K. For more on the arithmetic properties of the Tschakaloff function we refer to the survey by Bundschuh [38] and to [7, 27, 44, 49, 74, 82, 99, 145–147].

4.6. **Generalised partial theta functions.** The definition of partial theta functions can be extended to more general rank-r lattices Λ endowed with a positive definite bilinear form $\langle\cdot,\cdot\rangle:\Lambda\times\Lambda\mapsto\mathbb{Q}$, such as the root or weight lattices from Lie theory. For example [47][8]

$$\Theta_{\mathrm{p}}(x;q)=\sum_{\lambda\in\Lambda_+}q^{\frac{1}{2}\langle\lambda,\lambda\rangle}\,\mathrm{e}^{2\pi\mathrm{i}\langle x,\lambda\rangle},\tag{4.4}$$

where Λ_+ is the positive (or strictly positive) cone in Λ relative to some fixed basis $\alpha_1,\dots,\alpha_r$ and $x\in\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$. For the codimension-1 lattice in $\mathbb{Z}^{r+1}$ spanned by $\alpha_i=\epsilon_i-\epsilon_{i+1}$ (ϵ_i the ith standard unit vector in $\mathbb{Z}^{r+1}$) and $x=x_1\omega_1+\cdots+x_r\omega_r$ with $\langle\omega_i,\alpha_j\rangle=\delta_{ij}$, this yields

$$\Theta_{\mathrm{p}}(x;q)=\sum_{n_1,\dots,n_r\geqslant 0}q^{\frac{1}{2}\sum_{i,j}C_{ij}n_in_j}\,\mathrm{e}^{2\pi\mathrm{i}\sum_i x_in_i},$$

where $C_{ij}=\langle\alpha_i,\alpha_j\rangle$ is the Cartan matrix of $\mathfrak{sl}(r+1,\mathbb{C})$. Since $C_{ii}=2$, this simplifies to (1.1) for $r=1$. Creutzig and Milas [46, 47] have made a detailed study of the modular behaviour of a regularised version of (4.4), which includes a regularisation of the rank-1 partial theta function. They have also shown how (4.4) and 'Kostant type' analogues of (4.4) arise in the representation theory of logarithmic W-algebras [46, 47, 112]. Further applications of partial theta functions in representation theory may be found in [28], where these functions are used to give fermionic formulas for q-multiplicities of low-level Demazure modules for $\mathfrak{sl}_2[t]$.

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[7] The constant $(3+\sqrt{5})/2$ is not optimal and for sharper bounds, see e.g., [26, 44].

[8] For $x=0$ these are essentially the 'cone theta functions' of [54].

A very different kind of generalisation of partial theta functions is due to Kim and Lovejoy [79]. Hecke (or Hecke–Rogers) type indefinite theta functions [62] are functions of the form

$$\Bigg(\sum_{r,s\geqslant 0}-\sum_{r,s<0}\Bigg)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

for $ac<b^2$, and play an important role in the theory of mock theta functions. Kim and Lovejoy defined partial indefinite theta series to be double-sums over indefinite quadratic forms with a more restricted summation range, typically $r,s\geqslant 0$ subject to additional parity conditions. Using this definition they generalised many of Ramanujan's partial theta function identities to identities for partial indefinite theta series. The indefinite analogue of (1.6), for example, is

$$\sum_{n=0}^{\infty}\frac{(q;q)_{2n}q^n}{(aq,q/a;q)_n}=(1-a)\sum_{\substack{r,s\geqslant 0\\ r\equiv s \pmod 2}}(-1)^r a^{(r+s)/2}q^{3rs/2+r/2+s} \\ +\frac{a(q;q)_\infty}{(aq,q/a;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n}q^{3n(n+1)/2}.$$

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School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia
Email address: o.warnaar@maths.uq.edu.au