THE PRODUCT OF PARTIAL THETA FUNCTIONS

GEORGE E. ANDREWS AND S. OLE WARNAAR

Abstract. In this paper, we prove a new identity for the product of two partial theta functions. An immediate corollary is Warnaar’s generalization of the Jacobi triple product identity.

1. Introduction

In [5, Eq. (1.7)], one of the authors proved the following generalization of Jacobi’s triple product identity:

\begin{equation}
1 + \sum_{n=1}^{\infty} (-1)^n q^{n(n-1)/2} (a^n + b^n) = (q)_\infty (a)_\infty (b)_\infty \sum_{n=0}^{\infty} \frac{(ab/q)_n q^n}{(q)_n (a)_n (b)_n (ab)_n},
\end{equation}

where

\begin{equation}
(a; q)_n = (a)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).
\end{equation}

The celebrated Jacobi triple product identity [3, p. 12, Eq. (1.6.1)]

\begin{equation}
\sum_{n=-\infty}^{\infty} (-1)^n a^n q^{n(n-1)/2} = (q)_\infty (a)_\infty (q/a)_\infty
\end{equation}

follows immediately from (1.1) upon setting \( b = q/a \) and noting that the sum on the right-hand side of (1.1) reduces to 1 in this instance.

Sums of the form

\begin{equation}
\sum_{n=0}^{\infty} (-1)^n a^n q^{n(n-1)/2}
\end{equation}

are called partial theta functions owing to the fact that the sum in (1.3) is usually referred to as a complete theta function or just a theta function. Partial theta functions appear often in Ramanujan’s Lost Notebook [4]. An extensive explication of Ramanujan’s discoveries was given in [2]. This was further elaborated on in [5], with (1.1) playing a central role.

Our object here is to prove the following theorem for the product of partial theta functions.
Theorem 1.1. We have

\[
\left( \sum_{n=0}^{\infty} (-1)^{n} a^{n} q^{n(n-1)/2} \right) \left( \sum_{n=0}^{\infty} (-1)^{n} b^{n} q^{n(n-1)/2} \right)
= (q)_{\infty}(a)_{\infty}(b)_{\infty} \sum_{n=0}^{\infty} \frac{(abq^{n-1})_{n} q^{n}}{(q)_{n}(a)_{n}(b)_{n}}.
\]

Section 2 will be devoted to the proof of Theorem 1.1. In Section 3 we deduce the generalized triple product identity (1.1) from Theorem 1.1 and in the conclusion we discuss the relationship of Theorem 1.1 to other theorems in \(q\)-hypergeometric series.

2. PROOF OF THEOREM 1.1

In the following we employ standard notation for basic hypergeometric series, see e.g., [3, p. 4, Eq. (1.2.22)].

We begin by noting that the substitutions and limits \(z \to z/a, b \to q\) followed by \(a \to \infty, c \to 0\) followed by \(z \to a\) in Heine’s first and second transformation [1, p. 19, Cor. 2.3 and p. 39]

\[
2_{\phi 1}(a,b;c,q,z) = (b)_{\infty}(az)_{\infty} 2_{\phi 1}(c/b,z;az,q,b)
= (c/a,az)_{\infty} 2_{\phi 1}(abz/c,a;az,q,c/a)
\]

imply the identities

\[
\sum_{n=0}^{\infty} (-1)^{n} a^{n} q^{n} = (q)_{\infty}(a)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n}}{(q)_{n}(a)_{n}}
= (a)_{\infty} \sum_{m=0}^{\infty} \frac{a^{m} q^{m^{2}}}{(q)_{m}(a)_{m}}.
\]

Therefore,

\[
\left( \sum_{n=0}^{\infty} (-1)^{n} a^{n} q^{n} \right) \left( \sum_{n=0}^{\infty} (-1)^{n} b^{n} q^{n} \right)
= (q)_{\infty}(a)_{\infty}(b)_{\infty} \sum_{n,m=0}^{\infty} \frac{b^{m} q^{n+m^{2}}}{(q)_{n}(a)_{n}(q)_{m}(b)_{m}} \quad \text{(by (2.1a) and (2.1b))}
= (q)_{\infty}(a)_{\infty}(b)_{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{b^{m} q^{n+(m-n)^{2}}}{(q)_{n}(a)_{n}(q)_{m-n}(b)_{m-n}}
= (q)_{\infty}(a)_{\infty}(b)_{\infty} \sum_{m=0}^{\infty} \frac{b^{m} q^{m^{2}}}{(q)_{m}(b)_{m}} \sum_{n=0}^{m} \frac{(q^{1-m/b})_{n}(q^{-m})_{n}}{(q)_{n}(a)_{n}} q^{n}.
\]

The sum over \(n\) may be performed by the \(q\)-Chu–Vandermonde sum [3, Eq. (1.5.3)]

\[
2_{\phi 1}(a, q^{-n}; b, q) = \frac{(b/a)_{n}}{(b)_{n}} a^{n},
\]
resulting in
\[
\left( \sum_{n=0}^{\infty} (-1)^n a^n q^{\frac{n(n+1)}{2}} \right) \left( \sum_{n=0}^{\infty} (-1)^n b^n q^{\frac{n(n+1)}{2}} \right) = (q)_\infty(a)_\infty(b)_\infty \sum_{m=0}^{\infty} \frac{(abq^{m-1})_m q^m}{(q)_m(a)_m(b)_m}.
\]

3. Proof of identity \[1.1\]

**Theorem 3.1.** Identity \[1.1\] is valid.

**Proof.** Define
\[
L(a, b) := (q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(ab/q)_{2n} q^n}{(q)_n(a)_n(b)_n(ab)_n}.
\]
Then
\[
(q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(ab/q)_{2n} q^n}{(q)_n(a)_n(b)_n(ab)_n} = (q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(1 - ab/q)(abq^n)_{n-1} q^n}{(q)_n(a)_n(b)_n}
\]
\[
= (q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \left( (1 - abq^{n-1}) - (ab/q)(1 - q^n) \right)(abq^n)_{n-1} q^n
\]
\[
= (q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(abq^n)_{n-1} q^n}{(q)_n(a)_n(b)_n} - ab (q)_\infty(a)_\infty(b)_\infty \sum_{n=1}^{\infty} \frac{(abq^n)_{n-1} q^{n-1}}{n}.
\]
\[
= L(a, b) - ab (q)_\infty(aq)_\infty(bq)_\infty \sum_{n=0}^{\infty} \frac{(abq^{n+1})_{n+1} q^n}{(q)_n(aq)_n(bq)_n}
\]
\[
= L(a, b) - ab L(aq, bq)
\]
\[
= \left( \sum_{n=0}^{\infty} (-1)^n a^n q^{\frac{n(n+1)}{2}} \right) \left( \sum_{n=0}^{\infty} (-1)^n b^n q^{\frac{n(n+1)}{2}} \right) - \left( \sum_{n=0}^{\infty} (-1)^n a^{n+1} q^{\frac{n(n+2)}{2}} \right) \left( \sum_{n=0}^{\infty} (-1)^n b^n q^{\frac{n(n+2)}{2}} \right) 
\]
(by Theorem \[1.1\])
\[
= \left( \sum_{n=0}^{\infty} (-1)^n a^n q^{\frac{n(n+1)}{2}} \right) \left( \sum_{n=0}^{\infty} (-1)^n b^n q^{\frac{n(n+1)}{2}} \right) - \left( \sum_{n=1}^{\infty} (-1)^n a^n q^{\frac{n(n+1)}{2}} \right) \left( \sum_{n=1}^{\infty} (-1)^n b^n q^{\frac{n(n+1)}{2}} \right)
\]
\[
= 1 + \sum_{n=1}^{\infty} (-1)^n a^n q^{\frac{n(n+1)}{2}} + \sum_{n=1}^{\infty} (-1)^n b^n q^{\frac{n(n+1)}{2}}.
\]
\[\square\]
4. Conclusion

There are numerous corollaries that follow from Theorem 1.1. The most important is Theorem 3.1 of Section 3. As shown in [5], there are extensive implications of Theorem 3.1.

In light of the fact that (1.3) may be rewritten as

\[(q)_{\infty} = \sum_{n=0}^{\infty} (-1)^n a^n q^{(2)}_n = (q/a) \sum_{n=0}^{\infty} (-1)^n (q^2/a)^n q^{(2)}_n,
\]

we see that

\[(q^2)_{\infty} = (q/a) \sum_{n=0}^{\infty} (-1)^n b^n q^{(2)}_n = (q/b) \sum_{n=0}^{\infty} (-1)^n (q^2/b)^n q^{(2)}_n = L(a,b) - (q/a) L(q^2/a,b) - (q/b) L(a,q^2/b) + (q^2/ab) L(q^2/a,q^2/b).
\]

Noting that in \(q\)-hypergeometric series notation

\[L(a,b) = (q)_{\infty} (q/ab)_{\infty} \times 4\phi_3 \left( \frac{(ab/q)^{1/2}, -(ab/q)^{1/2}, (ab)^{1/2}, -(ab)^{1/2}}{a,b,ab/q} ; q, q \right),
\]

we see that we have an identity between four \(4\phi_3\)’s and the infinite product on the left-hand side of (4.2).

Also, one can make use of Gauss’ formula [11, p. 23, Eq. (2.2.12)]

\[\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)^{\infty}}{(q;q^2)^{\infty}}
\]

and \((q;q^2)^{\infty}(-q;q^2) = 1\) [11, p. 5, Eq. (1.2.5)] to deduce from Theorem 1.1 with \(b = -q\), that

\[\sum_{n=0}^{\infty} \frac{(-aq^n;q)_n q^n}{(q^2;q^2)_n (a)_n} = \frac{(-q;q)^{\infty}}{(a)^{\infty}} \sum_{n=0}^{\infty} (-1)^n a^n q^{(2)}_n.
\]

Or, we can use a further instance of Jacobi’s identity [11, p.23, Eq. (2.2.12)]

\[1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = \frac{(q)^{\infty}}{(-q;q)^{\infty}}
\]

to infer that

\[\sum_{n=0}^{\infty} \frac{(aq^{2n-1};q^2)_n q^{2n}}{(q)_{2n} (a;q^2)_n} = \frac{1}{2(a;q^2)^{\infty}} \left( \frac{1}{(-q;q)^{\infty}} + \frac{1}{(q)^{\infty}} \right) \sum_{n=0}^{\infty} (-1)^n a^n q^{2n-n}.
\]

Finally, we point out that Theorem 1.1 may also be deduced from Theorem 3.1. To achieve this one merely iterates the functional equation

\[L(a,b) - ab L(aq,bq) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{(2)}_n (a^n + b^n),
\]

which we see from the proof of Theorem 3.1 is an assertion equivalent to Theorem 3.1.
References


