Ramanujan's ${}_1\psi_1$ summation

S. Ole Warnaar

Acknowledgements I thank Dick Askey, Bruce Berndt, Susanna Fishel, Jeff Lagarias and Michael Schlosser for their helpful correspondence.

Notation. It is impossible to give an account of the ${}_1\psi_1$ summation without introducing some q-series notation. To keep the presentation as simple as possible, we assume that $0 < q < 1$. Suppressing q-dependence, we define two q-shifted factorials: $(a)_\infty := \prod_{k=0}^\infty (1-aq^k)$ and $(a)_z := (a)_\infty/(aq^z)_\infty$ for $z \in \mathbb{C}$. Note that $1/(q)_n = 0$ if n is a negative integer. For $x \in \mathbb{C} - \{0,-1,\dots\}$, the q-gamma function is defined as $\Gamma_q(x) := (q)_{x-1}/(1-q)^{x-1}$.

Ramanujan's ${}_1\psi_1$ summation. Ramanujan recorded his now famous ${}_1\psi_1$ summation as item 17 of Chapter 16 in the second of his three notebooks [13, p. 32], [46]. It was brought to the attention of the wider mathematical community in 1940 by Hardy, who included it in his twelfth and final lecture on Ramanujan's work [31]. Hardy remarked that the result constituted "a remarkable formula with many parameters". Instead of presenting the ${}_1\psi_1$ sum as given by Ramanujan and Hardy, we will state its modern form:

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (q/az)_\infty (b/a)_\infty (q)_\infty}{(z)_\infty (b/az)_\infty (q/a)_\infty (b)_\infty}, \qquad |b/a| < |z| < 1, \tag{1}$$

where it is understood that $a, q/b \notin \{q, q^2, \dots\}$. Characteristically, Ramanujan did not provide a proof of (1). Neither did Hardy, who however remarked that it could be "deduced from one which is familiar and probably goes back to Euler". The result to which Hardy was referring is another famous identity—known as the q-binomial theorem—corresponding to (1) with $b = q$: $\sum_{n=0}^\infty z^n (a)_n/(q)_n = (az)_\infty/(z)_\infty$ and valid for $|z| < 1$. Although not actually due to Euler, the q-binomial theorem is certainly classic. It seems to have appeared first and without proof (for $a = q^{-N}$) in Rothe's 1811 book "*Systematisches Lehrbuch der Arithmetik*", and in the 1840s many mathematicians of note, such as Cauchy (1843), Eisenstein (1846), Heine (1847) and Jacobi (1847) published proofs. The first proof of the ${}_1\psi_1$ sum is due to Hahn in 1949 [30] and, as hinted by Hardy, uses the q-binomial theorem. After Hahn a large number of alternative proofs of (1) were found, including one probabilistic and three combinatorial proofs [2,3,5,16–20,23,24,32,34,35,44,48,50,53]. The proof from the book, which again relies on the q-binomial theorem, was discovered by Ismail [32] and is short enough to include here. Assuming $|z| < 1$ and $|b| < \min\{1, |az|\}$, both sides are analytic functions of b. Moreover, they coincide when $b = q^{k+1}$ with $k = 0, 1, 2, \dots$ by the q-binomial theorem with $a \mapsto aq^{-k}$. Since 0 is the accumulation point of this sequence of b's the proof is done.

Apart from the q-binomial theorem, the ${}_1\psi_1$ sum generalises another classic identity, known as the Jacobi triple-product identity: $\sum_{n=-\infty}^\infty (-1)^n z^n q^{\binom{n}{2}} = (z)_\infty (q/z)_\infty (q)_\infty =: \theta(z)$. This result plays a central role in the theory of theta and elliptic functions.

The ${}_1\psi_1$ sum as discrete beta integral. As pointed out by Askey [8, 9], the ${}_1\psi_1$ summation may be viewed as a discrete analogue of Euler's beta integral. First define the Jackson or q-integral $\int_0^{c\cdot\infty} f(t)\mathrm{d}_q t := (1-q)\sum_{n=-\infty}^\infty f(cq^n) cq^n$. Replacing $(a, b, z) \mapsto (-c, -cq^{\alpha+\beta}, q^\alpha)$ in (1) then gives

$$\int_0^{c\cdot\infty} \frac{t^{\alpha-1}}{(-t)_{\alpha+\beta}} \mathrm{d}_q t = c^\alpha \frac{\theta(-cq^\alpha)}{\theta(-c)} \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}, \tag{2}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$. For real, positive c the limit $q \to 1$ can be taken, resulting in the beta integral modulo the substitution $t \mapsto t/(1-t)$. Askey further noted in [8] that the specialisation $(\alpha, \beta) \mapsto (x, 1-x)$ in (2) (so that $0 < \mathrm{Re}(x) < 1$) may be viewed as a q-analogue of Euler's reflection formula.

Simple applications of the ${}_1\psi_1$ sum. There are numerous easy applications of the ${}_1\psi_1$ sum. For example, Jacobi's well-known four- and six-square theorems as well as a number of similar results readily follow from (1), see e.g., [1,14,15,21,22,25]. To give a flavour of how the ${}_1\psi_1$ implies these types of results we shall sketch a proof of the four-square theorem. Let $r_s(n)$ be the number of representations of n as the sum of s squares. The generating function $R_s(q) := \sum_{n\geq 0} r_s(n)(-q)^n$ is given by $\big(\sum_{m=-\infty}^{\infty}(-1)^m q^{m^2}\big)^s$. By the triple-product identity this is also $\big((q)_\infty/(-q)_\infty\big)^s$. Any identity that allows the extraction of the coefficient of $(-q)^n$ results in an explicit formula for $r_s(n)$. Back to (1), replace $(b,z)\mapsto(aq,b)$ and multiply both sides by $(1-b)/(1-ab)$. By the geometric series this yields

$$1+\frac{(1-a)(1-b)}{1-ab}\sum_{k,n=1}^{\infty}q^{kn}(a^kb^n-a^{-k}b^{-n})=\frac{(abq)_\infty(q/ab)_\infty(q)_\infty^2}{(aq)_\infty(q/a)_\infty(bq)_\infty(q/b)_\infty}, \tag{3}$$

which may also be found in Kronecker's 1881 paper "*Zur Theorie der elliptischen Functionen*". For $a,b\to -1$ the right side gives $R_4(q)$ whereas the left side becomes

$$1-8\sum_{m=1}^{\infty}q^m\sum_{\substack{n,k=1\\nk=m}}^{\infty}n(-1)^{n+k}=1+8\sum_{m=1}^{\infty}(-q)^m\sum_{\substack{d\geq 1\\4\nmid d|m}}d.$$

Hence $r_4(n)=8\sum_{d\geq 1;\ 4\nmid d|n}d$. This result of Jacobi implies Lagrange's theorem that every positive integer is a sum of four squares. By taking $a,b^2\to -1$ in (3) the reader will have little trouble showing that $r_2(n)=4(d_1(n)-d_3(n))$, with $d_k(n)$ the number of divisors of n of the form $4m+k$. This is a result of Gauss and Lagrange which implies Fermat's two-square theorem.

Other simple but important applications of the ${}_1\psi_1$ sum concern orthogonal polynomials. In [11] it was employed by Askey and Wilson to compute a special case—corresponding to the continuous q-Jacobi polynomials—of the Askey–Wilson integral, and in [10] Askey gave an elementary proof of the full Askey–Wilson integral using the ${}_1\psi_1$ sum. The sum also implies the norm evaluation of the weight functions of the q-Laguerre polynomials [45]. These are a family of orthogonal polynomials with discrete measure μ on $[0,c\cdot\infty)$ given by $\mathrm{d}_q\mu(t)=t^\alpha/(-t)_\infty\,\mathrm{d}_q t$. The normalisation $\int \mathrm{d}_q\mu(t)$ thus follows from the q-beta integral (2) in the limit of large β.

Generalisations in one dimension. There exist several generalisations of Ramanujan's sum containing one additional parameter. In his work on partial theta functions Andrews [4] obtained a generalisation in which each product of four infinite products on the right-hand side is replaced by six such products. Another example is the curious identity of Guo and Schlosser, which is no longer hypergeometric in nature [27]:

$$\sum_{k=-\infty}^{\infty}\frac{(a)_k(1-ac_kq^k)(c_kq)_\infty(b/ac_k)_\infty}{(b)_k(1-azq^k)(ac_k)_\infty(q/ac_k)_\infty}c_k^k=\frac{1}{(1-z)}\frac{(q)_\infty(b/a)_\infty}{(q/a)_\infty(b)_\infty},$$

where $c_k:=z(1-aczq^k)/(1-azq^k)$ and $|b/ac|<|z|<1$. For $c=1$ this is (1).

As discovered by Schlosser [49], a quite different extension of the ${}_1\psi_1$ sum arises by considering non-commutative variables. Let R be a unital Banach algebra with identity 1, central elements b and q, and norm $\|\cdot\|$. Write a^{-1} for the inverse of an invertible element $a\in R$. Let $\prod_{i=m}^n a_i$ stand for 1 if $n=m-1$, $a_m\cdots a_n$ if $n\geq m$ and $a_{m-1}^{-1}\cdots a_{n+1}^{-1}$ if $n<m-1$, and define

$$\begin{pmatrix}a_1,\dots,a_r\\b_1,\dots,b_r\end{pmatrix};z\Big)_k^{\pm}:=\prod_i\Big[z\prod_{s=1}^{r}(1-a_sq^{i-1})(1-b_sq^{i-1})^{-1}\Big],$$

where $k\in\mathbb{Z}\cup\{\infty\}$, $a_1,\dots,a_r,b_1,\dots,b_r\in R$, $\prod_i=\prod_{i=1}^k$ in the $+$ case and $\prod_i=\prod_{i=k}^1$ in the $-$ case. Subject to $\max\{\|q\|,\|z\|,\|ba^{-1}z^{-1}\|\}<1$, the following non-commutative ${}_1\psi_1$ sum holds:

$$\sum_{k=-\infty}^{\infty}\left(\begin{matrix}a\\b\end{matrix};z\right)_k^{+}=\left(\begin{matrix}za\\z\end{matrix};1\right)_\infty^{-}\left(\begin{matrix}qa^{-1}z^{-1}\\qza^{-1}z^{-1}\end{matrix};1\right)_\infty^{+}\left(\begin{matrix}bza^{-1}z^{-1},q\\ba^{-1}z^{-1},b\end{matrix};1\right)_\infty^{-}.$$

Higher-dimensional generalisations. Various authors have generalised (1) to multiple ${}_1\psi_1$ sums. Below we state a generalisation due to Gustafson and Milne [28,41] which is labelled by the A-type root system. Similar such ${}_1\psi_1$ sums are given in [6,7,29,43,47]. More involved multiple ${}_1\psi_1$ sums with a Schur or Macdonald polynomial argument can be found in [12,36,42,52]. For $r=(r_1,\dots,r_n)\in\mathbb{Z}^n$ denote $|r|:=r_1+\dots+r_n$. Then

$$\sum_{r\in\mathbb{Z}^n} z^{|r|}\prod_{1\le i<j\le n}\frac{x_iq^{r_i}-x_jq^{r_j}}{x_i-x_j}\prod_{i,j=1}^n\frac{(a_jx_{ij})_{r_i}}{(b_jx_{ij})_{r_i}}=\frac{(az)_\infty(q/az)_\infty}{(z)_\infty(b/az)_\infty}\prod_{i,j=1}^n\frac{(b_jx_{ij}/a_i)_\infty(qx_{ij})_\infty}{(qx_{ij}/a_i)_\infty(b_jx_{ij})_\infty},$$

where $a:=a_1\cdots a_n$, $b:=q^{1-n}b_1\cdots b_n$, $x_{ij}:=x_i/x_j$ and $|b/a|<|z|<1$. Milne first proved this for $b_1=\dots=b_n$ [41] and shortly thereafter Gustafson established the full result [28]. We have already seen that the ${}_1\psi_1$ sum implies the Jacobi triple-product identity. The latter is the $\mathrm{A}_1^{(1)}$ case of Macdonald's generalised Weyl denominator identities for affine root systems [38]. To obtain further Macdonald identities from the Gustafson–Milne sum one replaces $z\to z/a$ before letting $a_1,\dots,a_n\to\infty$ and $b_1,\dots,b_n\to 0$. Extracting the coefficient of z^0 (on the right this requires the triple-product identity) results in the Macdonald identity for $\mathrm{A}_{n-1}^{(1)}$.

Higher-dimensional generalisations of a special case of the ${}_1\psi_1$ sum can be given for all affine root systems. A full description is beyond this note, and we will only sketch the simplest case. The reader is referred to [26,38–40] for the full details. In [39] Macdonald gave the following multivariable extension of the product formula for the Poincaré polynomial of a Coxeter group

$$W(\boldsymbol{t}):=\sum_{w\in W}\prod_{\alpha\in R^+}\frac{1-t_\alpha\,\mathrm{e}^{w(\alpha)}}{1-\mathrm{e}^{w(\alpha)}}=\prod_{\alpha\in R^+}\frac{1-t_\alpha\boldsymbol{t}^{\mathrm{ht}(\alpha)}}{1-\boldsymbol{t}^{\mathrm{ht}(\alpha)}}. \tag{4}$$

Here R is a reduced, irreducible finite root system in a Euclidean space V, R^+ the set of positive roots, W the Weyl group and t_α for $\alpha\in R^+$ a set of formal variables constant along Weyl orbits. The symbol $\boldsymbol{t}^{\mathrm{ht}(\alpha)}$ stands for $\prod_{\beta\in R^+}t_\beta^{(\beta,\alpha)/\|\alpha\|^2}$ with $(\cdot,\cdot)$ the W-invariant positive definite bilinear form on V. If all t_α are set to t then $\boldsymbol{t}^{\mathrm{ht}(\alpha)}=t^{\mathrm{ht}(\alpha)}$ with $\mathrm{ht}(\alpha)$ the usual height function on R, in which case $W(\boldsymbol{t})$ reduces to the classical Poincaré polynomial $W(t)$. Now let S be a reduced, irreducible affine root system of type $S=S(R)$ [38]. In analogy with the finite case, assume that t_a for $a\in S$ is constant along orbits of the affine Weyl group W of S. Then Macdonald generalised (4) to [40]

$$\sum_{w\in W}\prod_{a\in S^+}\frac{1-t_a\,\mathrm{e}^{w(a)}}{1-\mathrm{e}^{w(a)}}=\prod_{\alpha\in R^+}\frac{(t_\alpha\boldsymbol{t}^{\mathrm{ht}(\alpha)})_\infty(\boldsymbol{t}^{\mathrm{ht}(\alpha)}q^{\chi(\alpha\in B)}/t_\alpha)_\infty}{(\boldsymbol{t}^{\mathrm{ht}(\alpha)})_\infty^2}, \tag{5}$$

where B is a base for R. The parameter q on the right is fixed by $q=\prod_{a\in B(S)}\exp(n_a a)$, where $B(S)$ is a basis for S and the n_a are the labels of the extended Dynkin diagrams given in [38]. If R is simply-laced then $t_a=t$. In the case of $S(R)=\mathrm{A}_1^{(1)}$, $q=\exp(a_0+a_1)$ so that after replacing $\exp(a_1)$ by x we obtain the ${}_1\psi_1$ sum (1) with $(a,b,z)\to(x/t,tx,t)$. This is not the end of the story concerning root systems and the ${}_1\psi_1$ sum. Identity (5) can be rewritten as [40]

$$\sum_{\gamma\in Q^\vee}\prod_{\alpha\in R}\frac{(q\,\mathrm{e}^\alpha)_{(\alpha,\gamma)}}{(t_\alpha q\,\mathrm{e}^\alpha)_{(\alpha,\gamma)}}=\prod_{\alpha\in R^+}\frac{(t_\alpha\boldsymbol{t}^{\mathrm{ht}(\alpha)}q)_\infty(\boldsymbol{t}^{\mathrm{ht}(\alpha)}q^{\chi(\alpha\in B)}/t_\alpha)_\infty}{(\boldsymbol{t}^{\mathrm{ht}(\alpha)}q)_\infty(\boldsymbol{t}^{\mathrm{ht}(\alpha)})_\infty}\,\frac{(q\,\mathrm{e}^\alpha)_\infty(q\,\mathrm{e}^{-\alpha})_\infty}{(t_\alpha q\,\mathrm{e}^\alpha)_\infty(t_\alpha q\,\mathrm{e}^{-\alpha})_\infty}, \tag{6}$$

where $Q^\vee$ is the coroot lattice. Interestingly, for $t_\alpha=t$ this was also found by Fishel, Grojnowski and Teleman [26] by computing the generating function of the q-weighted Euler characteristics of certain Dolbeault cohomologies. For $R=\mathrm{A}_{n-1}$, $Q^\vee=Q=\sum_{i=1}^n r_i\epsilon_i$ with $|r|=0$, $R=\{\epsilon_i-\epsilon_j:\ 1\le i\neq j\le n\}$ and $t^{\mathrm{ht}(\epsilon_i-\epsilon_j)}=t^{j-i}$. By fairly elementary manipulations the identity

(6) may then be transformed into the multiple ${}_1\psi_1$ sum

$$\sum_{r\in\mathbb{Z}^n} z^{|r|}\frac{(a)_{|r|}}{(b)_{|r|}}\prod_{1\le i<j\le n}\frac{x_iq^{r_i}-x_jq^{r_j}}{x_i-x_j}\,\frac{(t^{-1}x_{ij})_{r_i-r_j}}{(tqx_{ij})_{r_i-r_j}}\,t^{r_i-r_j}q^{-r_j} = \frac{(az)_\infty(q/az)_\infty(b/a)_\infty(tq)_\infty}{(z)_\infty(b/az)_\infty(q/a)_\infty(b)_\infty}\prod_{i=1}^{n-1}\frac{(t^{i+1}q)_\infty}{(t^i)_\infty}\prod_{i,j=1}^{n}\frac{(qx_{ij})_\infty}{(tqx_{ij})_\infty},$$

for $|b/a|<|z|<1$ and $|t|<1$. This is the only result in this survey that is new.

We finally remark that all higher-dimensional ${}_1\psi_1$ sums admit representations as discrete Selberg-type integrals. The most important such integrals are due to Aomoto [6, 7] and Ito [33], and are closely related to (5). Further examples may be found in [37, 51].

References

1. C. Adiga, *On the representations of an integer as a sum of two or four triangular numbers*, Nihonkai Math. J. **3** (1992), 125–131.
2. C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson, *Chapter 16 of Ramanujan's second notebook: theta-functions and q-series*, Mem. Amer. Math. Soc. **53** (1985), no. 315.
3. G. E. Andrews, *On Ramanujan's* ${}_1\psi_1(a;b;z)$, Proc. Amer. Math. Soc. **22** (1969), 552–553.
4. G. E. Andrews, *Ramanujan's "lost" notebook. I. Partial θ-functions*, Adv. in Math. **41** (1981), 137–172.
5. G. E. Andrews and R. Askey, *A simple proof of Ramanujan's* ${}_1\psi_1$, Aequationes Math. **18** (1978), 333–337.
6. K. Aomoto, *On a theta product formula for the symmetric A-type connection function*, Osaka J. Math. **32** (1995), 35–39.
7. K. Aomoto, *On elliptic product formulas for Jackson integrals associated with reduced root systems*, J. Algebraic Combin. **8** (1998), 115–126.
8. R. Askey, *The q-gamma and q-beta functions*, Applicable Anal. 8 (1978/79), 125–141.
9. R. Askey, *Ramanujan's extensions of the gamma and beta functions* Amer. Math. Monthly **87** (1980), 346–359.
10. R. Askey, *An elementary evaluation of a beta type integral*, Indian J. Pure Appl. Math. 14 (1983), 892–895.
11. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), no. 319.
12. T. H. Baker and P. J. Forrester, *Transformation formulas for multivariable basic hypergeometric series*, Methods Appl. Anal. **6** (1999), 147–164.
13. B. C. Berndt, *Ramanujan's notebooks. Part III*, Springer-Verlag, New York, 1991.
14. S. Bhargava and C. Adiga, *Simple proofs of Jacobi's two and four square theorems*, Int. J. Math. Ed. Sci. Tech. **19** (1988), 779–782.
15. S. H. Chan, *An elementary proof of Jacobi's six squares theorem*, Amer. Math. Monthly **111** (2004), 806–811.
16. S. H. Chan, *A short proof of Ramanujan's famous* ${}_1\psi_1$ *summation formula*, J. Approx. Theory **132** (2005), 149–153.
17. S. H. L. Chen, W. Y. C. Chen, A. M. Fu and W. J. T. Zang, *The Algorithm Z and Ramanujan's* ${}_1\psi_1$ *summation*, Ramanujan J. **25** (2011), 37–47.
18. W. Y. C. Chen and E. X. W. Xia, *The q-WZ method for infinite series*, J. Symbolic Comput. **44** (2009), 960–971.
19. X. Chen, Q. Liu and X. Ma, *A short proof of Ramanujan's* ${}_1\psi_1$ *formula*, J. Math. Res. Exposition **28** (2008), 46–48.
20. W. Chu, *Abel's lemma on summation by parts and Ramanujan's* ${}_1\psi_1$*-series identity*, Aequationes Math. **72** (2006), 172–176.
21. S. Cooper, *On sums of an even number of squares, and an even number of triangular numbers: an elementary approach based on Ramanujan's* ${}_1\psi_1$ *summation formula*, Contemp. Math. **291**, (2001), 115–137.
22. S. Cooper and H. Y. Lam, *Sums of two, four, six and eight squares and triangular numbers: an elementary approach*, Indian J. Math. **44** (2002), 21–40.
23. S. Corteel, *Particle seas and basic hypergeometric series*, Adv. Appl. Math. **31** (2003), 199–214.
24. S. Corteel and J. Lovejoy, *Frobenius partitions and the combinatorics of Ramanujan's* ${}_1\psi_1$ *summation*, J. Combin. Theory Ser. A **97** (2002), 177–183.
25. N. J. Fine, *Basic hypergeometric series and applications*, Mathematical Surveys and Monographs, Vol. 27, AMS, Providence, RI, 1988.
26. S. Fishel, I. Grojnowski and C. Teleman, *The strong Macdonald conjecture and Hodge theory on the loop Grassmannian*, Ann. of Math. (2) **168** (2008), 175–220.
27. V.J. Guo and M. J. Schlosser, *Curious extensions of Ramanujan's* ${}_1\psi_1$ *summation formula*, J. Math. Anal. Appl. **334** (2007), 393–403.
28. R. A. Gustafson, *Multilateral summation theorems for ordinary and basic hypergeometric series in* $U(n)$, SIAM J. Math. Anal. **18** (1987), 1576–1596.
29. R. A. Gustafson and C. Krattenthaler, *Heine transformations for a new kind of basic hypergeometric series in* $U(n)$, J. Comput. Appl. Math. **68** (1996), 151–158.
30. W. Hahn, *Beiträge zur Theorie der Heineschen Reihen*, Math. Nachr. **2** (1949), 340–379.
31. G. H. Hardy, *Ramanujan. Twelve lectures on subjects suggested by his life and work*, Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1940.
32. M. E. H. Ismail, *A simple proof of Ramanujan's* ${}_1\psi_1$ *sum*, Proc. Amer. Math. Soc. **63** (1977), 185–186.
33. M. Ito, *On a theta product formula for Jackson integrals associated with root systems of rank two*, J. Math. Anal. Appl. **216** (1997), 122–163
34. M. Jackson, *On Lerch's transcendent and the basic bilateral hypergeometric series* ${}_2\psi_2$, J. London Math. Soc. **25** (1950), 189–196.
35. K. W. J. Kadell, *A probabilistic proof of Ramanujan's* ${}_1\psi_1$ *sum*, Siam. J. Math. Anal. **18** (1987), 1539–1548.
36. J. Kaneko, *A* ${}_1\psi_1$ *summation theorem for Macdonald polynomials*, Ramanujan J. **2** (1996), 379–386.
37. J. Kaneko, *q-Selberg integrals and Macdonald polynomials*, Ann. Sci. École Norm. Sup. (4) **29** (1996), 583–637.
38. I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Invent. Math. **15** (1972), 91–143.
39. I. G. Macdonald, *The Poincaré series of a Coxeter group*, Math. Ann. **199** (1972), 161–174.
40. I. G. Macdonald, *A formal identity for affine root systems*, in *Lie groups and symmetric spaces*, pp. 195–211, Amer. Math. Soc. Transl. Ser. 2, 210, AMS, Providence, RI, 2003.
41. S. C. Milne, *A* $U(n)$ *generalization of Ramanujan's* ${}_1\psi_1$ *summation*, J. Math. Anal. Appl. **118** (1986), 263–277.
42. S. C. Milne, *Summation theorems for basic hypergeometric series of Schur function argument*, in *Progress in approximation theory*, pp. 51–77, Springer Ser. Comput. Math. 19, Springer, New York, 1992.
43. S. C. Milne and M. Schlosser, *A new* A_n *extension of Ramanujan's* ${}_1\psi_1$ *summation with applications to multilateral* A_n *series*, Rocky Mountain J. Math. **32** (2002), 759–792.
44. K. Mimachi, *A proof of Ramanujan's identity by use of loop integrals*, SIAM J. Math. Anal. **19** (1988), 1490–1493.
45. D. S. Moak, *The q-analogue of the Laguerre polynomials*, J. Math. Anal. Appl. **81** (1981), 20–47.
46. S. Ramanujan, *Notebooks*, Tata Institute of Fundamental Research, Bombay, 1957.

47. M. Schlosser, *Summation theorems for multidimensional basic hypergeometric series by determinant evaluations*, Discrete Math. **210** (2000), 151–169.
48. M. Schlosser, *Abel–Rothe type generalizations of Jacobi's triple product identity*, in *Theory and applications of special functions*, pp. 383–400, Dev. Math., 13, Springer, New York, 2005.
49. M. Schlosser, *Noncommutative extensions of Ramanujan's* ${}_1\psi_1$ *summation*, Electron. Trans. Numer. Anal. **24** (2006), 94–102.
50. K. Venkatachaliengar and S. Cooper *Development of Elliptic Functions According to Ramanujan*, Monographs in Number Theory, Vol. 6, World Sientific, Singapore, 2012.
51. S. O. Warnaar, *q-Selberg integrals and Macdonald polynomials*, Ramanujan J. **10** (2005), 237–268.
52. S. O. Warnaar, *The* $\mathfrak{sl}_3$ *Selberg integral*, Adv. Math. **224** (2010), 499–524.
53. A. J. Yee, *Combinatorial proofs of Ramanujan's* ${}_1\psi_1$ *summation and the q-Gauss summation*, J. Combin. Theory Ser. A. **105** (2004), 63–77.

School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia
Work supported by the Australian Research Council