Advances in Applied Mathematics 46 (2011) 424–456

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Branching rules for symmetric functions and $\mathfrak{sl}_n$ basic hypergeometric series ☆

Alain Lascoux [a], S. Ole Warnaar [b,*]

[a] CNRS, Institut Gaspard Monge, Université Paris-Est, Marne-La-Vallée, France
[b] School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

ARTICLE INFO

Article history:
Available online 4 November 2010

Dedicated to Dennis Stanton on the occasion of his 60th birthday

MSC:
05E05
33D52
33D67

Keywords:
Symmetric functions
Macdonald polynomials
Multiple basic hypergeometric series

ABSTRACT

A one-parameter rational function generalisation $R_\lambda(X;b)$ of the symmetric Macdonald polynomial and interpolation Macdonald polynomial is studied from the point of view of branching rules. We establish a Pieri formula, evaluation symmetry, principal specialisation formula and q-difference equation for $R_\lambda(X;b)$. Our main motivation for studying $R_\lambda(X;b)$ is that it leads to a new class of $\mathfrak{sl}_n$ basic hypergeometric series, generalising the well-known basic hypergeometric series with Macdonald polynomial argument. For these new series we prove $\mathfrak{sl}_n$ analogues of the q-Gauss and q-Kummer–Thomae–Whipple formulas. In a special limit, one of our results implies an elegant binomial formula for Jack polynomials, different to that of Kaneko, Lassalle, Okounkov and Olshanski.

1. Introduction

Let λ be a partition, i.e., $\lambda=(\lambda_1,\lambda_2,\dots)$ is a weakly decreasing sequence of nonnegative integers such that $|\lambda|:=\lambda_1+\lambda_2+\cdots$ is finite. Let the length $l(\lambda)$ of λ be the number of nonzero λ_i. For $x=(x_1,\dots,x_n)$ and $l(\lambda)\leqslant n$ the Schur function $s_\lambda(x)$ is defined as

$$s_\lambda(x):=\frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})}=\frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\Delta(x)}, \tag{1.1}$$

☆ Work supported by the ANR project MARS (BLAN06-2 134516) and the Australian Research Council.
* Corresponding author.
E-mail address: o.warnaar@maths.uq.edu.au (S.O. Warnaar).

0196-8858/$ – see front matter
doi:10.1016/j.aam.2010.01.012

where $\Delta(x) := \prod_{i<j}(x_i - x_j)$ is the Vandermonde product. If $l(\lambda) > n$ then $s_\lambda(x) := 0$. From its definition it is clear that $s_\lambda(x)$ is a symmetric polynomial in x of homogeneous degree $|\lambda|$, and that $\{s_\lambda \mid l(\lambda) \leqslant n\}$ forms a basis of the ring of symmetric functions $\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$.

A classical result for Schur functions is the combinatorial formula

$$s_\lambda(x) = \sum_T x^T. \tag{1.2}$$

Here the sum is over all semi-standard Young tableau T of shape λ, and x^T is shorthand for the monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$ with μ_i the number of squares of the tableau filled with the number i. One of the remarkable facts of (1.2) is that it actually yields a symmetric function.

The conventional way to view a semi-standard Young tableau of shape λ (and length at most n) as a filling of a Young diagram with the numbers $1, 2, \dots, n$ such that squares are strictly increasing along columns and weakly increasing along rows. Given two partitions (or Young diagrams) λ, μ write $\mu \preccurlyeq \lambda$ if $\mu \subseteq \lambda$ and $\lambda - \mu$ is a horizontal strip, i.e., if the skew diagram $\lambda - \mu$ contains at most one square in each column. Then an alternative viewpoint is to consider a Young tableau of shape λ as a sequence of partitions

$$0 = \lambda^{(0)} \preccurlyeq \lambda^{(1)} \preccurlyeq \cdots \preccurlyeq \lambda^{(n)} = \lambda, \tag{1.3}$$

where 0 denotes the empty partition. For example, for $n = 6$ the tableau

1	1	1	2	2	2	4	6
2	2	4	5	5	5		
4	5	6					
5	6						
6							

may be encoded as

$$0 \preccurlyeq (3) \preccurlyeq (6,2) \preccurlyeq (6,2) \preccurlyeq (7,3,1) \preccurlyeq (7,6,2,1) \preccurlyeq (8,6,3,2,1).$$

The above description implies that a recursive formulation of the Schur functions, equivalent to the combinatorial formula (1.2), is given by the *branching rule*

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu \preccurlyeq \lambda} x_n^{|\lambda - \mu|} s_\mu(x_1, \dots, x_{n-1}), \tag{1.4}$$

subject to the initial condition $s_\lambda(-) = \delta_{\lambda,0}$.

If we let $\mu \subseteq \lambda$ be a pair of partitions and define the skew Schur function $s_{\lambda/\mu}$ of a single variable z as

$$s_{\lambda/\mu}(z) := \begin{cases} z^{|\lambda-\mu|} & \text{if } \mu \preccurlyeq \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

then the branching rule for Schur functions takes the more familiar form

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} s_{\lambda/\mu}(x_n) s_\mu(x_1, \dots, x_{n-1}). \tag{1.5}$$

The Macdonald polynomials $P_\lambda(x) = P_\lambda(x; q, t)$ [13,14] are an important q, t-generalisation of the Schur functions, and the P_λ for $l(\lambda) \leqslant n$ form a basis of the ring $\Lambda_{n,\mathbb{F}} := \Lambda_n \otimes \mathbb{F}$, where $\mathbb{F} = \mathbb{Q}(q, t)$. A classical result in the theory is that the Macdonald polynomials satisfy a combinatorial formula not unlike that of the Schur functions;

$$P_\lambda(x) = \sum_T \psi_T x^T,$$

where $\psi_T = \psi_T(q, t) \in \mathbb{F}$ is a function that admits an explicit combinatorial description. Importantly, if T has no more than n rows it factorises as

$$\psi_T = \prod_{i=1}^{n} \psi_{\lambda^{(i)}/\lambda^{(i-1)}},$$

where, as before, $0 = \lambda^{(0)} \preccurlyeq \cdots \preccurlyeq \lambda^{(n)} = \lambda$ is the sequence of partitions representing T. Probably the simplest (albeit non-combinatorial) expression for $\psi_{\lambda/\mu}$ is [14, p. 342]

$$\psi_{\lambda/\mu} = \prod_{1 \leqslant i \leqslant j \leqslant l(\mu)} \frac{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})}, \tag{1.6}$$

where $f(a) = (at)_\infty/(aq)_\infty$ with $(a)_\infty = \prod_{i \geqslant 0}(1 - aq^i)$. (Note that $\psi_{\lambda/\mu} \in \mathbb{F}$ since $\mu \preccurlyeq \lambda$.) It follows from the above that the Macdonald polynomials, like the Schur functions, can be described by a simple branching rule. Namely,

$$P_\lambda(x_1, \ldots, x_n) = \sum_{\mu \preccurlyeq \lambda} x_n^{|\lambda - \mu|} \psi_{\lambda/\mu} P_\mu(x_1, \ldots, x_{n-1}), \tag{1.7}$$

subject to the initial condition $P_\lambda(-) = \delta_{\lambda,0}$. Again we may define a single-variable skew polynomial $P_{\lambda/\mu}(z) = P_{\lambda/\mu}(z; q, t)$ for $\mu \subseteq \lambda$

$$P_{\lambda/\mu}(z) := \begin{cases} z^{|\lambda - \mu|} \psi_{\lambda/\mu} & \text{if } \mu \preccurlyeq \lambda, \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

to turn the branching formula for the Macdonald polynomials into

$$P_\lambda(x_1, \ldots, x_n) = \sum_{\mu \subseteq \lambda} P_{\lambda/\mu}(x_n) P_\mu(x_1, \ldots, x_{n-1}). \tag{1.9}$$

In view of the above two examples of symmetric functions admitting a recursive description in the form of a branching formula, a natural question is

Can one find more general branching-type formulas that lead to symmetric functions?

To fully appreciate the question we should point out that it is not at all obvious that if one were to take (1.4) as the definition of the Schur functions, or (1.7) as the definition of the Macdonald polynomials, that the polynomials in question are symmetric in x.

Assuming throughout that $|q|<1$ let the (generalised) q-shifted factorials be defined as follows:

$$(b)_\infty=(b;q)_\infty:=\prod_{i=0}^{\infty}\bigl(1-bq^i\bigr), \tag{1.10a}$$

$$(b)_k=(b;q)_k:=\frac{(b)_\infty}{(bq^k)_\infty}, \tag{1.10b}$$

$$(b)_\lambda=(b;q,t)_\lambda:=\prod_{i=1}^{l(\lambda)}\bigl(bt^{1-i}\bigr)_{\lambda_i}, \tag{1.10c}$$

and let $(b_1,\dots,b_i)_k=(b_1)_k\cdots(b_i)_k$ and $(b_1,\dots,b_i)_\lambda=(b_1)_\lambda\cdots(b_i)_\lambda$. Then probably the best-known example of a branching rule generalising (1.9) and resulting in symmetric polynomials is

$$M_\lambda(x_1,\dots,x_n)=\sum_{\mu\subseteq\lambda}\frac{(t^{n-1}/x_n)_\lambda}{(t^{n-1}/x_n)_\mu}P_{\lambda/\mu}(x_n)M_\mu(x_1,\dots,x_{n-1}). \tag{1.11}$$

The $M_\lambda(x)=M_\lambda(x;q,t)$ are the interpolation Macdonald polynomials of Knop, Okounkov and Sahi [6,18,19,27], and (1.11) is [19, Proposition 5.3]. For comparison with [19], we have

$$M_\lambda(x_1,\dots,x_n)=t^{(n-1)|\lambda|}P^*_\lambda\bigl(t^{1-n}x_1,\dots,t^{-1}x_{n-1},x_n\bigr).$$

From (1.11) it is clear that the top-homogeneous component of $M_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ so that $\{M_\lambda \mid l(\lambda)\leqslant n\}$ forms an inhomogeneous basis of $\Lambda_{n,\mathbb{F}}$.

For $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_{n-1})$ such that $\lambda-\mu$ is a horizontal strip and such that $\lambda_1\leqslant m$ denote by $m^n-\lambda$ and $m^{n-1}-\mu$ the partitions $(m-\lambda_n,\dots,m-\lambda_1)$ and $(m-\mu_{n-1},\dots,m-\mu_1)$. Note that $(m^n-\lambda)-(m^{n-1}-\mu)$ is again a horizontal strip. It follows from (1.6) that

$$P_{(m^n-\lambda)/(m^{n-1}-\mu)}(1/z;1/q,1/t)=z^{-m}P_{\lambda/\mu}(z;1/q,1/t)=z^{-m}P_{\lambda/\mu}(z)$$

for $\lambda_1\leqslant m$. It also follows from (1.6) and (1.11) that

$$\frac{M_{m^n}(1/x_1,\dots,1/x_n;1/q,1/t)}{M_{m^{n-1}}(1/x_1,\dots,1/x_{n-1};1/q,1/t)}=x_n^{-m}(x_n;1/q)_m.$$

If we replace $(x,q,t)\mapsto(1/x,1/q,1/t)$ in (1.11) and then change $\lambda\mapsto(m-\lambda_n,\dots,m-\lambda_1)$ and $\mu\mapsto(m-\mu_{n-1},\dots,m-\mu_1)$ the branching rule for the interpolation Macdonald polynomials may thus be recast as

$$\begin{aligned}&\frac{M_{m^n-\lambda}(1/x_1,\dots,1/x_n;1/q,1/t)}{M_{m^n}(1/x_1,\dots,1/x_n;1/q,1/t)}\\&\quad=\sum_{\mu\subseteq\lambda}\frac{(q^{1-m}x_n/t)_\mu}{(q^{1-m}x_n)_\lambda}P_{\lambda/\mu}(x_n)\frac{M_{m^{n-1}-\mu}(1/x_1,\dots,1/x_{n-1};1/q,1/t)}{M_{m^{n-1}}(1/x_1,\dots,1/x_{n-1};1/q,1/t)}.\end{aligned} \tag{1.12}$$

In this paper we consider a rational function generalisation $R_\lambda(x;b)=R_\lambda(x;b;q,t)$ of the Macdonald polynomials and the Macdonald interpolation polynomials defined recursively by the branching rule

$$R_\lambda(x_1,\dots,x_n;b):=\sum_{\mu\subseteq\lambda}\frac{(bx_n/t)_\mu}{(bx_n)_\lambda}P_{\lambda/\mu}(x_n)R_\mu(x_1,\dots,x_{n-1};b). \tag{1.13}$$

Our interest in the functions $R_\lambda(x;b)$ is not merely that they provide another example of a class of symmetric functions defined by a simple branching formula. Indeed, since

$$\begin{aligned} R_\lambda\big(x_1,\dots,x_n;q^{1-m}\big) &= \frac{M_{m^n-\lambda}(1/x_1,\dots,1/x_n;1/q,1/t)}{M_{m^n}(1/x_1,\dots,1/x_n;1/q,1/t)} \\ &= \frac{M_{m^n-\lambda}(1/x_1,\dots,1/x_n;1/q,1/t)}{\prod_{i=1}^n x_i^m (x_i;1/q)_m}, \end{aligned} \tag{1.14}$$

the rational functions $R_\lambda(x;b)$ are not essentially different from the Macdonald interpolation polynomials. Moreover, it may be shown that the more general functions

$$R_\lambda(x_1,\dots,x_n;a,b) := \sum_{\mu\subseteq\lambda} \frac{(x_n/a)_\lambda (bx_n/t)_\mu}{(x_n/a)_\mu (bx_n)_\lambda} P_{\lambda/\mu}(a) R_\mu(x_1,\dots,x_{n-1};at,b) \tag{1.15}$$

are also symmetric, and arise as a limiting case (reducing BC_n symmetry to $\mathfrak{S}_n$ symmetry and breaking ellipticity) of Rains' BC_n symmetric abelian interpolation functions [24,25]. What makes the functions $R_\lambda(x;b)$ particularly interesting, however, is that they are the necessary building-block for generalising the $\mathfrak{sl}_n$ basic hypergeometric series with Macdonald polynomial argument. The later series were first introduced in full generality by Kaneko [4] and Macdonald [15], and studied or applied (sometimes in specialised form) in [1,3,5,7,16,17,28–33]. Many classical results for basic hypergeometric series admit generalisations to the Macdonald polynomial setting. For example, the $\mathfrak{sl}_n$ analogue of the q-binomial theorem reads [4,15]

$$\sum_\lambda (a)_\lambda \mathsf{P}_\lambda(X) = \prod_{x\in X} \frac{(ax)_\infty}{(x)_\infty}, \tag{1.16}$$

where $\mathsf{P}_\lambda(X)$ is a suitable normalisation of the Macdonald polynomial $P_\lambda(X)$ and X an arbitrary finite alphabet. Most other results for $\mathfrak{sl}_n$ basic hypergeometric series, however, require a specialisation of the alphabet X. For example, the q-Gauss sum for Macdonald polynomials [4]

$$\sum_\lambda \left(\frac{c}{ab}\right)^{|\lambda|} \frac{(a,b)_\lambda}{(c)_\lambda} \mathsf{P}_\lambda(X) = \prod_{x\in X} \frac{(cx/a,cx/b)_\infty}{(cx,cx/ab)_\infty} \tag{1.17}$$

only holds provided X is principally specialised as $X=\{1,t^{-1},\dots,t^{1-n}\}$. One of the main results of this paper is that if one lifts $\mathsf{P}_\lambda(X)$ to $\mathsf{R}_\lambda(X;c)$ (the latter a suitable normalisation of $R_\lambda(X;c)$) then an $\mathfrak{sl}_n$ q-Gauss sum holds for an arbitrary alphabet X:

$$\sum_\lambda \left(\frac{c}{ab}\right)^{|\lambda|} (a,b)_\lambda \mathsf{R}_\lambda(X;c) = \prod_{x\in X} \frac{(cx/a,cx/b)_\infty}{(cx,cx/ab)_\infty}. \tag{1.18}$$

2. Preliminaries on Macdonald polynomials

We begin with a remark about notation. If f is a symmetric function we will often write $f(X)$ with $X=\{x_1,\dots,x_n\}$ (and refer to X as an alphabet) instead of $f(x)$ with $x=(x_1,\dots,x_n)$, the latter notation being reserved for functions that are not (a priori) symmetric. Following this notation we also use $f(X+Y)$ where $X+Y$ denotes the (disjoint) union of the alphabets X and Y, and $f(X+z)$ where $X+z$ denotes the alphabet X with the single letter z added.

In the following we review some of the basics of Macdonald polynomial theory, most of which can be found in [13,14].

Let T_{q,x_i} be the q-shift operator acting on the variable x_i:

$$(T_{q,x_i} f)(x_1,\dots,x_n) = f(x_1,\dots,x_{i-1},qx_i,x_{i+1},\dots,x_n).$$

Then the Macdonald polynomials $P_\lambda(X) = P_\lambda(X;q,t)$ for $X = \{x_1,\dots,x_n\}$ are the unique polynomial eigenfunctions of the Macdonald operator

$$D_n(c) := \sum_{I\subseteq[n]} (-c)^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}, \tag{2.1}$$

where $[n] := \{1,\dots,n\}$. Explicitly,

$$D_n(c) P_\lambda(X) = P_\lambda(X) \prod_{i=1}^n \bigl(1 - cq^{\lambda_i} t^{n-i}\bigr). \tag{2.2}$$

For later reference we state the coefficient of c^1 of this equation separately; if

$$D_n^1 := \sum_{i=1}^n \biggl(\prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j}\biggr) T_{q,x_i}, \tag{2.3}$$

then

$$D_n^1 P_\lambda(X) = P_\lambda(X) \sum_{i=1}^n q^{\lambda_i} t^{n-i}. \tag{2.4}$$

For each square $s = (i,j) \in \mathbb{Z}^2$ in the (Young) diagram of a partition (i.e., for each $i \in \{1,\dots,l(\lambda)\}$ and $j \in \{1,\dots,\lambda_i\}$), the arm-length $a(s)$, arm-colength $a'(s)$, leg-length $l(s)$ and leg-colength $l'(s)$ are given by

$$a(s) = \lambda_i - j, \qquad a'(s) = j - 1$$

and

$$l(s) = \lambda'_j - i, \qquad l'(s) = i - 1,$$

where λ' is the conjugate of λ, obtained by reflecting the diagram of λ in the main diagonal. Note that the generalised q-shifted factorial (1.10c) can be expressed in terms of the colengths as

$$(b)_\lambda = \prod_{s\in\lambda} \bigl(1 - bq^{a'(s)} t^{-l'(s)}\bigr).$$

With the above notation we define the further q-shifted factorials $c'_\lambda = c'_\lambda(q,t)$, $c_\lambda = c_\lambda(q,t)$ and $b_\lambda = b_\lambda(q,t)$ as

$$c'_\lambda := \prod_{s\in\lambda} \bigl(1 - q^{a(s)+1} t^{l(s)}\bigr) \quad \text{and} \quad c_\lambda := \prod_{s\in\lambda} \bigl(1 - q^{a(s)} t^{l(s)+1}\bigr)$$

and

$$b_\lambda := \frac{c_\lambda}{c'_\lambda}.$$

Then the Macdonald polynomials $Q_\lambda(X) = Q_\lambda(X; q, t)$ are defined as

$$Q_\lambda(X) := b_\lambda P_\lambda(X).$$

We also need the skew Macdonald polynomials $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ defined for $\mu \subseteq \lambda$ by

$$P_\lambda(X+Y) = \sum_{\mu\subseteq\lambda} P_{\lambda/\mu}(Y) P_\mu(X), \tag{2.5a}$$

$$Q_\lambda(X+Y) = \sum_{\mu\subseteq\lambda} Q_{\lambda/\mu}(Y) Q_\mu(X). \tag{2.5b}$$

Note that $P_{\lambda/0} = P_\lambda$ and $Q_{\lambda/0} = Q_\lambda$, and that $P_{\lambda/\lambda} = Q_{\lambda/\lambda} = 1$. To simplify some later equations it will be useful to extend the definitions of $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ to all partition pairs λ, μ by setting $P_{\lambda/\mu} = Q_{\lambda/\mu} = 0$ if $\mu \not\subseteq \lambda$. From (2.1), (2.2) and (2.5) it follows that for a a scalar,

$$P_{\lambda/\mu}(aX) = a^{|\lambda-\mu|} P_{\lambda/\mu}(X), \tag{2.6a}$$

$$Q_{\lambda/\mu}(aX) = a^{|\lambda-\mu|} Q_{\lambda/\mu}(X), \tag{2.6b}$$

where $aX := \{ax \mid x \in X\}$.

For subsequent purposes it will be convenient to also introduce normalised (skew) Macdonald polynomials $\mathsf{P}_{\lambda/\mu}$ and $\mathsf{Q}_{\lambda/\mu}$ as

$$\mathsf{P}_{\lambda/\mu}(X) = t^{n(\lambda)-n(\mu)} \frac{c'_\mu}{c'_\lambda} P_{\lambda/\mu}(X) = t^{n(\lambda)-n(\mu)} \frac{c_\mu}{c_\lambda} Q_{\lambda/\mu}(X), \tag{2.7a}$$

$$\mathsf{Q}_{\lambda/\mu}(X) = t^{n(\mu)-n(\lambda)} \frac{c'_\lambda}{c'_\mu} Q_{\lambda/\mu}(X) = t^{n(\mu)-n(\lambda)} \frac{c_\lambda}{c_\mu} P_{\lambda/\mu}(X), \tag{2.7b}$$

where

$$n(\lambda) := \sum_{s\in\lambda} l'(s) = \sum_{i=1}^{l(\lambda)} (i-1)\lambda_i.$$

Note that no additional factors arise in the normalised form of (2.5):

$$\mathsf{P}_\lambda(X+Y) = \sum_\mu \mathsf{P}_{\lambda/\mu}(Y) \mathsf{P}_\mu(X),$$

$$\mathsf{Q}_\lambda(X+Y) = \sum_\mu \mathsf{Q}_{\lambda/\mu}(Y) \mathsf{Q}_\mu(X),$$

and that $\mathsf{P}_{\lambda/0} = \mathsf{P}_\lambda$ and $\mathsf{Q}_{\lambda/0} = \mathsf{Q}_\lambda$. If we define the structure constants $\mathsf{f}^\lambda_{\mu\nu} = \mathsf{f}^\lambda_{\mu\nu}(q,t)$ by

$$\mathsf{P}_\mu(X)\mathsf{P}_\nu(X) = \sum_\lambda \mathsf{f}^\lambda_{\mu\nu}\mathsf{P}_\lambda(X), \tag{2.8}$$

then

$$\mathsf{f}^\lambda_{\mu\nu} = t^{n(\mu)+n(\nu)-n(\lambda)} \frac{c'_\lambda}{c'_\mu c'_\nu} f^\lambda_{\mu\nu},$$

with $f^\lambda_{\mu\nu}$ the q,t-Littlewood–Richardson coefficients.

Below we make use of some limited λ-ring notation (see [8] for more details). Let p_r be the rth power-sum symmetric function

$$p_r(X) := \sum_{x\in X} x^r$$

and let $p_\lambda := \prod_{i\geqslant i} p_{\lambda_i}$. Then the $p_\lambda(X)$ form a basis of Λ_n where $n = |X|$. Given a symmetric function $f(X)$ we define

$$f\left[\frac{a-b}{1-t}\right] := \phi_{a,b}(f),$$

where $\phi_{a,b}$ is the evaluation homomorphism given by

$$\phi_{a,b}(p_r) = \frac{a^r - b^r}{1-t^r}. \tag{2.9}$$

In particular, $f[(1-t^n)/(1-t)] = f(1,t,\dots,t^{n-1})$ is known as the principal specialisation, which we will also denote as $f(\langle 0\rangle)$, and $f[1/(1-t)] = f(1,t,t^2,\dots)$. From [9, Eq. (6.24)]

$$\mathsf{Q}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] = a^{|\lambda-\mu|}\sum_\nu (b/a)_\nu \mathsf{f}^\lambda_{\mu\nu},$$

which, by $\mathsf{f}^\lambda_{0\nu} = \delta_{\lambda\nu}$, also implies that

$$\mathsf{Q}_\lambda\left[\frac{1-a}{1-t}\right] = (a)_\lambda \quad \text{and} \quad \mathsf{P}_\lambda\left[\frac{1-a}{1-t}\right] = t^{2n(\lambda)}\frac{(a)_\lambda}{c_\lambda c'_\lambda}. \tag{2.10}$$

This last equation yields the well-known principal specialisation formula

$$\mathsf{P}_\lambda\big(\langle 0\rangle\big) = t^{2n(\lambda)}\frac{(t^n)_\lambda}{c_\lambda c'_\lambda}.$$

The Cauchy identity for (skew) Macdonald polynomials is given by

$$\sum_\lambda \mathsf{P}_{\lambda/\mu}(X)\mathsf{Q}_{\lambda/\nu}(Y) = \left(\prod_{x\in X}\prod_{y\in Y}\frac{(txy)_\infty}{(xy)_\infty}\right)\sum_\lambda \mathsf{P}_{\nu/\lambda}(X)\mathsf{Q}_{\mu/\lambda}(Y).$$

The product on the right-hand side may alternatively be expressed in terms of the power-sum symmetric functions as

$$\exp\Biggl(\sum_{r=1}^{\infty}\frac{1}{r}\frac{1-t^r}{1-q^r}p_r(X)p_r(Y)\Biggr).$$

It thus follows from (2.9) and (2.10), as well as some elementary manipulations, that application of $\phi_{a,c}$ (acting on Y) turns the Cauchy identity into

$$\sum_{\lambda}\mathsf{Q}_{\lambda/\nu}\biggl[\frac{a-c}{1-t}\biggr]\mathsf{P}_{\lambda/\mu}(X)=\biggl(\prod_{x\in X}\frac{(cx)_\infty}{(ax)_\infty}\biggr)\sum_{\lambda}\mathsf{Q}_{\mu/\lambda}\biggl[\frac{a-c}{1-t}\biggr]\mathsf{P}_{\nu/\lambda}(X). \tag{2.11}$$

For $\mu=\nu=0$ (followed by the substitution $X\to X/a$ and then $a\to c/a$) this is the q-binomial identity for Macdonald polynomials (1.16). For later reference we also state the more general $(\mu,\nu)\mapsto(0,\mu)$ instance of (2.11)

$$\mathsf{P}_\mu(X)\prod_{x\in X}\frac{(bx)_\infty}{(ax)_\infty}=\sum_{\lambda}\mathsf{Q}_{\lambda/\mu}\biggl[\frac{a-b}{1-t}\biggr]\mathsf{P}_\lambda(X). \tag{2.12}$$

For reasons outlined below we will refer to this as a Pieri formula.

Let $\phi_{\lambda/\mu}=\phi_{\lambda,\mu}(q,t)$ and $\psi'_{\lambda/\mu}=\psi'_{\lambda,\mu}(q,t)$ be defined by

$$\phi_{\lambda/\mu}:=\frac{b_\lambda}{b_\mu}\psi_{\lambda/\mu}\quad\text{and}\quad\psi'_{\lambda,\mu}(q,t):=\psi_{\lambda'/\mu'}(t,q).$$

(For combinatorial expressions for all of $\psi_{\lambda/\mu}$, $\psi'_{\lambda/\mu}$ and $\phi_{\lambda/\mu}$, see [14].) Further let $g_{(r)}(X):=P_{(r)}(X)(t)_r/(q)_r$ and let $e_r(X)$ be the rth elementary symmetric function. Then the Macdonald polynomials $P_\lambda(X)$ satisfy the Pieri formulas

$$P_\mu(X)g_r(X)=\sum_{\substack{\lambda\succcurlyeq\mu\\|\lambda-\mu|=r}}\phi_{\lambda/\mu}P_\lambda(X), \tag{2.13a}$$

$$P_\mu(X)e_r(X)=\sum_{\substack{\lambda'\succcurlyeq\mu'\\|\lambda-\mu|=r}}\psi'_{\lambda/\mu}P_\lambda(X). \tag{2.13b}$$

Now observe that (2.12) for $b=at$ yields

$$\mathsf{P}_\mu(X)\prod_{x\in X}\frac{(atx)_\infty}{(ax)_\infty}=\sum_{\lambda}a^{|\lambda-\mu|}\mathsf{Q}_{\lambda/\mu}(1)\mathsf{P}_\lambda(X),$$

whereas for $a=bq$ it yields

$$\mathsf{P}_\mu(X)\prod_{x\in X}(1-bx)=\sum_{\lambda}b^{|\lambda-\mu|}\mathsf{Q}_{\lambda/\mu}\biggl[\frac{q-1}{1-t}\biggr]\mathsf{P}_\lambda(X).$$

Since

$$\sum_{r\geqslant 0} a^r g_r(X) = \prod_{x\in X} \frac{(atx)_\infty}{(ax)_\infty} \quad \text{and} \quad \sum_{r\geqslant 0} (-b)^r e_r(X) = \prod_{x\in X} (1-bx)$$

we therefore have

$$P_\mu(X) g_r(X) = \sum_{|\lambda-\mu|=r} Q_{\lambda/\mu}(1) P_\lambda(X)$$

and

$$P_\mu(X) e_r(X) = (-1)^r \sum_{|\lambda-\mu|=r} Q_{\lambda/\mu}\left[\frac{q-1}{1-t}\right] P_\lambda(X),$$

where we have also used (2.7b). Identifying

$$Q_{\lambda/\mu}(1) = \begin{cases} \phi_{\lambda/\mu} & \text{if } \mu \preccurlyeq \lambda, \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q_{\lambda/\mu}\left[\frac{q-1}{1-t}\right] = \begin{cases} (-1)^{|\lambda-\mu|} \psi'_{\lambda/\mu} & \text{if } \mu' \preccurlyeq \lambda', \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

these two formulas are equivalent to the Pieri rules of (2.13).

The skew polynomials can be used to define generalised q-binomial coefficients [18,11,12] as

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} := \mathsf{Q}_{\lambda/\mu}\left[\frac{1}{1-t}\right]. \tag{2.15}$$

In particular we have $\left[\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right] = 0$ if $\mu \not\subseteq \lambda$ and

$$\begin{bmatrix} (m) \\ (k) \end{bmatrix} = \prod_{i=1}^{k} \frac{1-q^{i+m-k}}{1-q^i} = \begin{bmatrix} m \\ k \end{bmatrix}$$

with on the right the classical q-binomial coefficients $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_q$.

If $\lambda_{(i)} := (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_i)$ then [11]

$$(1-q)t^{i-1} \begin{bmatrix} \lambda \\ \lambda_{(i)} \end{bmatrix} = \frac{c'_\lambda}{c'_{\lambda_{(i)}}} \psi'_{\lambda/\lambda_{(i)}}.$$

This, together with [11, Théorème 9, Bis]

$$(\omega_\lambda - \omega_\mu) \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = (1-q) \sum_{i=1}^{n} q^{-\lambda_i} t^{i-n} \begin{bmatrix} \lambda \\ \lambda_{(i)} \end{bmatrix} \begin{bmatrix} \lambda_{(i)} \\ \mu \end{bmatrix}, \tag{2.16}$$

where

$$\omega_\lambda = \omega_\lambda(q,t) := \sum_{i=1}^{n} q^{-\lambda_i} t^{i-n}, \tag{2.17}$$

provides a simple recursive method to compute the generalised q-binomial coefficients.

Lemma 2.1. *Assume that $l(\lambda), l(\mu) \leqslant n$. Then*

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \frac{s_\mu(\langle 0\rangle)}{s_\lambda(\langle 0\rangle)} \det_{1\leqslant i,j\leqslant n}\left(\begin{bmatrix} \lambda_i+n-i \\ \mu_j+n-j \end{bmatrix}\right) \quad \text{if } t=q \tag{2.18a}$$

and

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \sum_{u^+=\mu} \prod_{i=1}^{n} \begin{bmatrix} \lambda_i \\ u_i \end{bmatrix} \quad \text{if } t=1. \tag{2.18b}$$

In the above $\sum_{u^+=\mu}$ denotes a sum over compositions $u \in \mathbb{N}^n$ in the $\mathfrak{S}_n$ orbit of μ.

Proof of Lemma 2.1. Assume that $t=q$. Then (2.15) simplifies to

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = q^{n(\mu)-n(\lambda)} \frac{c_\lambda}{c_\mu} s_{\lambda/\mu}\left[\frac{1}{1-q}\right], \tag{2.19}$$

where we have also used (2.7b) and the fact that for $t=q$ the (skew) Macdonald polynomials reduce to the (skew) Schur functions. Let $h_r(X)$ be the rth complete symmetric function. By application of the Jacobi–Trudi identity [14, Eq. (I.5.4)]

$$s_{\lambda/\mu} = \det_{1\leqslant i,j\leqslant n}(h_{\lambda_i-\mu_j-i+j}) \quad \text{for } n \geqslant l(\lambda),$$

and the principal specialisation formula [14, p. 44]

$$s_\lambda\big(\langle 0\rangle\big) = \frac{q^{n(\lambda)}}{c'_\lambda} \prod_{i=1}^{n} \frac{(q)_{\lambda_i+n-i}}{(q)_{n-i}} \tag{2.20}$$

the generalised q-binomial coefficient (2.19) can be expressed as a determinant

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \frac{s_\mu(\langle 0\rangle)}{s_\lambda(\langle 0\rangle)} \det_{1\leqslant i,j\leqslant n}\left(\frac{(q;q)_{\lambda_i+n-i}}{(q,q)_{\mu_j+n-j}} h_{\lambda_i-\mu_j-i+j}\left[\frac{1}{1-q}\right]\right).$$

Since

$$h_r\left[\frac{1}{1-q}\right] = \frac{1}{(q)_r}$$

this establishes the first claim.

The second claim follows in analogous manner. Since making the substitution $t=1$ in the right-hand side of (2.15) is somewhat problematic it is best to first use the symmetry [18, Eq. (2.12)]

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} = \begin{bmatrix} \lambda' \\ \mu' \end{bmatrix}_{t^{-1},q^{-1}}. \tag{2.21}$$

Since

$$P_{\lambda/\mu}(X;1,t)=e_{\lambda'/\mu'}(X)$$

we therefore get

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q^{-1},1}=\begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}_{1,q}=q^{n(\mu')-n(\lambda')}\frac{c_{\lambda'}(1,q)}{c_{\mu'}(1,q)}e_{\lambda/\mu}\left[\frac{1}{1-q}\right].$$

Using

$$e_{\lambda/\mu}=\sum_{u^+=\mu}\prod_{i=1}^{n}e_{\lambda_i-u_i}\quad\text{for } n\geqslant l(\lambda),$$

and

$$e_r\left[\frac{1}{1-q}\right]=\frac{q^{\binom{r}{2}}}{(q;q)_r}$$

as well as $c_{\lambda'}(q,1)=\prod_i(q)_{\lambda_i}$, it follows that

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q^{-1},1}=\sum_{u^+=\mu}\prod_{i=1}^{n}q^{-\lambda_i u_i}\begin{bmatrix}\lambda_i\\ u_i\end{bmatrix}.$$

Finally replacing $q\mapsto 1/q$ yields the second claim. □

3. Symmetric functions and branching rules

In this section we consider the question posed in the introduction:

Can one find new(?) branching-type formulas, similar to (1.4), (1.7) *and* (1.11)*, that lead to symmetric functions?*

Assume that k is a fixed nonnegative integer, and let $\mathbf{a}=(a_1,a_2,\dots,a_k)$ denote a finite sequence of parameters. Then we are looking for branching coefficients $f_{\lambda/\mu}(z;\mathbf{a})$ such that

$$f_\lambda(x_1,\dots,x_n;\mathbf{a})=\sum_{\mu\subseteq\lambda}f_{\lambda/\mu}(x_n;\mathbf{a})f_\mu\bigl(x_1,\dots,x_{n-1};\mathbf{a}'\bigr),\tag{3.1a}$$

subject to the initial condition

$$f_\lambda(-;\mathbf{a})=\delta_{\lambda,0}\tag{3.1b}$$

defines a symmetric function. In the above $\mathbf{a}'=(a_1',\dots,a_k')=g(\mathbf{a})$. Of course, (3.1a) for $n=1$ combined with (3.1b) implies that

$$f_\lambda(z;\mathbf{a})=f_{\lambda/0}(z;\mathbf{a}).$$

If one wishes to only consider symmetric functions with the standard property

$$f_\lambda(x_1,\dots,x_n;\mathbf{a})=0 \quad \text{if } l(\lambda)>n, \tag{3.2}$$

then the additional condition

$$f_{\lambda/\mu}(z;\mathbf{a})=0 \quad \text{if } l(\lambda)-l(\mu)>1$$

must be imposed.

Because we assume the branching coefficients to be independent of n, it may perhaps seem we are excluding interesting classes of symmetric functions such as the interpolation Macdonald polynomials. As will be shown shortly, assuming n-independence is not actually a restriction, and (1.11) may easily be recovered as a special case of (3.1a).

Now let us assume that (3.1a) yields a symmetric function $f_\lambda(x_1,\dots,x_n;\mathbf{a})$ for all $n\leqslant N$. (For $N=0$ and $N=1$ this is obviously not an assumption.) Then,

$$\begin{aligned} f_\lambda(x_1,\dots,x_{n-1},y,z;\mathbf{a}) &= \sum_{\mu\subseteq\lambda} f_{\lambda/\mu}(z;\mathbf{a}) f_\mu\big(x_1,\dots,x_{n-1},y;\mathbf{a}'\big) \\ &= \sum_{\nu\subseteq\mu\subseteq\lambda} f_{\lambda/\mu}(z;\mathbf{a}) f_{\mu/\nu}\big(y;\mathbf{a}'\big) f_\nu\big(x_1,\dots,x_{n-1};\mathbf{a}''\big) \end{aligned}$$

is a symmetric function in $x_1,\dots,x_{n-1},y$ (for $n\leqslant N$). For it to also be a symmetric function in $x_1,\dots,x_{n-1},y,z$ we must have

$$f_\lambda(x_1,\dots,x_{n-1},y,z;\mathbf{a})=f_\lambda(x_1,\dots,x_{n-1},z,y;\mathbf{a}),$$

implying that for fixed λ

$$\begin{aligned} &\sum_{\nu\subseteq\mu\subseteq\lambda} f_{\lambda/\mu}(z;\mathbf{a}) f_{\mu/\nu}\big(y;\mathbf{a}'\big) f_\nu\big(x_1,\dots,x_{n-1};\mathbf{a}''\big) \\ &\quad = \sum_{\nu\subseteq\mu\subseteq\lambda} f_{\lambda/\mu}(y;\mathbf{a}) f_{\mu/\nu}\big(z;\mathbf{a}'\big) f_\nu\big(x_1,\dots,x_{n-1};\mathbf{a}''\big), \end{aligned}$$

where $\mathbf{a}'':=g(\mathbf{a}')$. Hence a sufficient condition for (3.1a) to yield a symmetric function is

$$\sum_{\nu\subseteq\mu\subseteq\lambda} f_{\lambda/\mu}(z;\mathbf{a}) f_{\mu/\nu}\big(y;\mathbf{a}'\big) = \sum_{\nu\subseteq\mu\subseteq\lambda} f_{\lambda/\mu}(y;\mathbf{a}) f_{\mu/\nu}\big(z;\mathbf{a}'\big) \tag{3.3}$$

for partitions λ, ν such that $\nu\subseteq\lambda$.

As a first example let us show how to recover the Macdonald interpolation polynomials of the introduction. To this end we take $\mathbf{a}=(a)$, $\mathbf{a}'=(a/t)$, and

$$f_{\lambda/\mu}(z;\mathbf{a})=f_{\lambda/\mu}(z;a)=\frac{(a/z)_\lambda}{(a/z)_\mu}P_{\lambda/\mu}(z).$$

Clearly, the resulting polynomials $f_\lambda(x;a)$ correspond to the interpolation polynomials after the specialisation $a=t^{n-1}$. To see that (3.3) is indeed satisfied we substitute the above choice for the branching coefficient (recall the convention that $P_{\lambda/\mu} := 0$ if $\mu \not\subseteq \lambda$) to obtain

$$\sum_{\mu} \frac{(a/z)_\lambda (a/ty)_\mu}{(a/z)_\mu (a/ty)_\nu} P_{\lambda/\mu}(z) P_{\mu/\nu}(y) = \sum_{\mu} \frac{(a/y)_\lambda (a/tz)_\mu}{(a/y)_\mu (a/tz)_\nu} P_{\lambda/\mu}(y) P_{\mu/\nu}(z). \tag{3.4}$$

The identity (3.4) is easily proved using Rains' Sears transformation for skew Macdonald polynomials [23, Corollary 4.9]

$$\sum_{\mu} \frac{(aq/b, aq/c)_\lambda (d,e)_\mu}{(aq/b, aq/c)_\mu (d,e)_\nu} P_{\lambda/\mu}\left[\frac{1-aq/de}{1-t}\right] P_{\mu/\nu}\left[\frac{aq/de - a^2q^2/bcde}{1-t}\right]$$
$$= \sum_{\mu} \frac{(aq/d, aq/e)_\lambda (b,c)_\mu}{(aq/d, aq/e)_\mu (b,c)_\nu} P_{\lambda/\mu}\left[\frac{1-aq/bc}{1-t}\right] P_{\mu/\nu}\left[\frac{aq/bc - a^2q^2/bcde}{1-t}\right]. \tag{3.5}$$

After simultaneously replacing $(a,b,c,d,e) \mapsto (c, a/tz, cqz/a, a/ty, cqy/a)$ and taking the $c \to \infty$ limit we obtain (3.4).

If, more generally, we let $(a,b,c,d,e) \mapsto (c, a/bz, cqz/a, a/by, cqy/a)$ in (3.5) and take the $c \to \infty$ limit we find that

$$\sum_{\mu} \frac{(a/z)_\lambda (a/by)_\mu}{(a/z)_\mu (a/by)_\nu} P_{\lambda/\mu}\left[\frac{z-bz}{1-t}\right] P_{\mu/\nu}\left[\frac{y-by}{1-t}\right]$$
$$= \sum_{\mu} \frac{(a/y)_\lambda (a/bz)_\mu}{(a/y)_\mu (a/bz)_\nu} P_{\lambda/\mu}\left[\frac{y-by}{1-t}\right] P_{\mu/\nu}\left[\frac{z-bz}{1-t}\right].$$

The Macdonald interpolation polynomials may thus be generalised by taking $\mathbf{a} = (a,b)$, $\mathbf{a}' = (a/b, b)$ and

$$f_{\lambda/\mu}(z;\mathbf{a}) = f_{\lambda/\mu}(z;a,b) = z^{|\lambda-\mu|} \frac{(a/z)_\lambda}{(a/z)_\mu} P_{\lambda/\mu}\left[\frac{1-b}{1-t}\right].$$

Proposition 3.1. *The polynomials $M_\lambda(x_1,\dots,x_n;a,b) = M_\lambda(x_1,\dots,x_n;a,b;q,t)$ defined by*

$$M_\lambda(x_1,\dots,x_n;a,b) = \sum_{\mu} x_n^{|\lambda-\mu|} \frac{(a/x_n)_\lambda}{(a/x_n)_\mu} P_{\lambda/\mu}\left[\frac{1-b}{1-t}\right] M_\mu(x_1,\dots,x_{n-1};a/b,b)$$

subject to $M_\lambda(-;a,b) = \delta_{\lambda,0}$ are symmetric. Moreover, the interpolation Macdonald polynomials correspond to

$$M_\lambda(x_1,\dots,x_n) = M_\lambda\big(x_1,\dots,x_n;t^{n-1},t\big).$$

The polynomials $M_\lambda(x_1,\dots,x_n;a,b)$ are an example of a class of symmetric functions for which $l(\lambda) > n$ does not imply vanishing. For example,

$$M_\lambda(z;a,b) = z^{|\lambda|}(a/z)_\lambda P_\lambda\left[\frac{1-b}{1-t}\right] = t^{n(\lambda)} z^{|\lambda|} \frac{(a/z,b)_\lambda}{c_\lambda}.$$

The next example corresponds to Okounkov's BC_n symmetric interpolation polynomials [20] (see also [21,23]).

Proposition 3.2. *If we take* $\mathbf{a}=(a,b)$, $\mathbf{a}'=(a/t,b/t)$ *and*

$$f_{\lambda/\mu}(z;\mathbf{a})=f_{\lambda/\mu}(z;a,b)=\frac{(a/z,bz)_\lambda}{(a/z,bz)_\mu}P_{\lambda/\mu}(1/b)$$

in (3.1a) *then the resulting functions* $f_\lambda(x;a,b)=f_\lambda(x;a,b;q,t)$ *are symmetric.*

Writing $O_\lambda(x;a,b)$ instead of $f_\lambda(x;a,b)$, the (Laurent) polynomials $O_\lambda(x;a,b)$ satisfy the symmetries

$$O_\lambda(x;a,b)=\left(\frac{a}{b}\right)^{|\lambda|}O_\lambda(1/x;b,a)=\left(\frac{a}{b}\right)^{|\lambda|}q^{2n(\lambda')}O_\lambda(1/x;1/a,1/b;1/q,1/t).$$

(This follows easily using that $P_\lambda(X;1/q,1/t)=P_\lambda(X,q,t)$.) Moreover, Okounkov's BC_n interpolation Macdonald polynomials $P^*_\lambda(x;q,t,s)$ follow as

$$P^*_\lambda\big(x_1,tx_2,\dots,t^{n-1}x_n;q,t,s\big)=q^{-n(\lambda')}O_\lambda\big(x;1,s^2t^{2(n-1)};q,t\big).$$

(Since $O_\lambda(ax;a,b/a)=a^{|\lambda|}O_\lambda(x;1,b)$ the $O_\lambda(x;a,b)$ are not more general than the $P^*_\lambda(x;q,t,s)$.)

Proof of Proposition 3.2. Substituting the claim in (3.3) and using (2.6) gives

$$\sum_\mu\frac{(a/z,bz)_\lambda(a/yt,byt)_\mu}{(a/z,bz)_\mu(a/yt,byt)_\nu}P_{\lambda/\mu}(1)P_{\mu/\nu}(t)=\sum_\mu\frac{(a/y,by)_\lambda(a/zt,bzt)_\mu}{(a/y,by)_\mu(a/zt,bzt)_\nu}P_{\lambda/\mu}(1)P_{\mu/\nu}(t).$$

This is (3.5) with $(a,b,c,d,e)\mapsto(ab/qt,bz/t,a/zt,a/yt,by/t)$. $\square$

Our final example will (in the limit) lead to the functions studied in the remainder of the paper.

Proposition 3.3. *If we take* $\mathbf{a}=(a,b)$, $\mathbf{a}'=(at,b)$ *and*

$$f_{\lambda/\mu}(z;\mathbf{a})=f_{\lambda/\mu}(z;a,b)=\frac{(z/a)_\lambda(bz/t)_\mu}{(z/a)_\mu(bz)_\lambda}P_{\lambda/\mu}(a)$$

in (3.1a) *then the resulting functions* $f_\lambda(x;a,b)=f_\lambda(x;a,b;q,t)$ *are symmetric.*

Proof. Substituting the claim in (3.3) and using (2.6) gives

$$\sum_\mu\frac{(z/a,by)_\lambda(bz/t,y/at)_\mu}{(z/a,by)_\mu(bz/t,y/at)_\nu}P_{\lambda/\mu}(1)P_{\mu/\nu}(t)=\sum_\mu\frac{(y/a,bz)_\lambda(by/t,z/at)_\mu}{(y/a,bz)_\mu(by/t,z/at)_\nu}P_{\lambda/\mu}(1)P_{\mu/\nu}(t).$$

This is (3.5) with $(a,b,c,d,e)\mapsto(byz/aqt,by/t,z/at,bz/t,y/at)$. $\square$

If we write $R_\lambda(x;a,b)$ instead of $f_\lambda(x;a,b)$ the symmetric functions of Proposition 3.3 correspond to the functions described by the branching rule (1.15) of the introduction. As already mentioned there, the $R_\lambda(x;a,b)$ are not new, and follow as a special limiting case of much more general functions studied by Rains [24,25]. More specifically, Rains defined a family of abelian interpolation functions

$$R^{*(n)}_\lambda(x;a,b)=R^{*(n)}_\lambda(x;a,b;q,t;p),$$

where $x=(x_1,\dots,x_n)$. The $R^{*(n)}_\lambda(x)$ are BC_n symmetric and, apart from parameters a, b, q, t, depend on an elliptic nome p. In [24, Theorem 4.16] Rains proved the branching rule

$$R^{*(n+1)}_\lambda(x_1,\dots,x_{n+1};a,b)=\sum_{\mu\subseteq\lambda}c^{(n)}_{\lambda\mu}(x_{n+1};a,b)R^{*(n)}_\mu(x_1,\dots,x_n;a,b),$$

where the branching coefficient $c^{(n)}_{\lambda\mu}(z;a,b)=c^{(n)}_{\lambda\mu}(z;a,b;q,t;p)$ is expressed in terms of the elliptic binomial coefficient $\left\langle{\lambda \atop \mu}\right\rangle_{[a,b](v_1,\dots,v_k)}$ (see [25, Eq. (4.2)]) as

$$c^{(n)}_{\lambda\mu}(z;a,b)=\left\langle{\lambda \atop \mu}\right\rangle_{[at^n/b,t](azt^n,at^n/z,pqa/tb)}. \tag{3.6}$$

If we define

$$\begin{aligned}R_\lambda(x;a,b)&=R_\lambda(x;a,b;q,t)\\&=\left(\frac{t^{1-n}}{b}\right)^{|\lambda|}P_\lambda\big(\langle 0\rangle\big)\lim_{p\to 0}R^{*(n)}_\lambda\big(p^{1/4}x;p^{1/4}at^{n-1},p^{3/4}b/q;1/q,1/t;p\big)\end{aligned}$$

and compute the corresponding limit of (3.6) we obtain the branching rule (1.15) with $n\mapsto n+1$.

4. The symmetric function $R_\lambda(x;b)$

Define

$$\tau_\lambda=(-1)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}.$$

In the remainder of the paper we consider the symmetric function

$$\begin{aligned}R_\lambda(X;b)&=\tau_\lambda^{-1}\lim_{a\to 0}R_\lambda(X;a,b)\\&=\tau_\lambda^{-1}\left(\frac{t^{1-n}}{b}\right)^{|\lambda|}P_\lambda\big(\langle 0\rangle\big)\quad\big(n:=|X|\big)\\&\quad\times\lim_{a\to 0}\lim_{p\to 0}R^{*(n)}_\lambda\big(p^{1/4}X;p^{1/4}at^{n-1},p^{3/4}b/q;1/q,1/t;p\big)\end{aligned}$$

which, alternatively, is defined by the branching rule (1.13). Because R_λ is a limiting case of the abelian interpolation function $R^{*(n)}_\lambda$ many properties of former follow by taking appropriate limits in the results of [24,25].

For example, if $D_n(b,c,d)$ is the generalised Macdonald operator

$$D_n(b,c,d)=\sum_{I\subseteq[n]}(-1)^{|I|}t^{\binom{|I|}{2}}\prod_{\substack{i\in I\\ j\notin I}}\frac{tx_i-x_j}{x_i-x_j}\prod_{j\notin I}(1-bx_j)\prod_{i\in I}\big(c-bdt^{1-n}x_i\big)T_{q,x_i},$$

then $R_\lambda(X;b)$ for $X=\{x_1,\dots,x_n\}$ satisfies two q-difference equations generalising (2.2):

$$D_n(b,c,1)R_\lambda(X;b)=R_\lambda(X;bq)\prod_{i=1}^n\big(1-cq^{\lambda_i}t^{n-i}\big) \tag{4.1}$$

and

$$D_n(b,c,d)R_\mu(X;b) = (c;t)_n \prod_{i=1}^n (1-bx_i) \\ \times \sum_{\lambda' \succcurlyeq \mu'} (-bd)^{|\lambda-\mu|} \frac{\tau_\lambda}{\tau_\mu} \frac{(cqt^{n-1})_\mu}{(ct^{n-1})_\lambda} \frac{(ct^{n-1}/d)_\lambda}{(ct^{n-1}/d)_\mu} \psi'_{\lambda/\mu} R_\lambda(X;b). \tag{4.2}$$

(For $d=1$ these two formulas agree by a limiting case of the Pieri formula (5.4).)

Below we will first prove a number of elementary properties of the functions $R_\lambda(X;b)$ using only the branching rule (1.13). Like the previous result, most of these can also be obtained by taking appropriate limits in results of Rains for the abelian interpolation functions $R_\lambda^{*(n)}(X;a,b)$, or by translating corresponding properties for the Macdonald interpolation polynomials. Then we give several deeper results for $R_\lambda(X;b)$ (such as Theorem 5.1 and Corollaries 5.2, 5.4 and 6.2) that, to the best of our knowledge, have no analogues for $R_\lambda^{*(n)}(X;a,b)$ or $R_\lambda(X;a,b)$ and do not follow from corresponding results for the Macdonald interpolation polynomials. First however we restate the branching rule (1.13) in the equivalent form

$$R_\lambda(X;b) = \sum_\mu \frac{(bz/t)_\mu}{(bz)_\lambda} P_{\lambda/\mu}(z) R_\mu(Y;b), \tag{4.3}$$

where $X = Y + z$.

When $X = \{z\}$ we find from (4.3) that

$$R_{(k)}(z;b) = \frac{z^k}{(bz)_k}. \tag{4.4}$$

From this it is clear that $R_{(k)}(cz;b) = c^k R_{(k)}(z;bc)$ and that in the $c \to \infty$ limit $R_{(k)}(cz;b)$ is given by $(-b)^{-k} q^{-\binom{k}{2}}$. It also shows that

$$R_{(k+1)}(z;b) = R_{(k)}(z;bq).$$

All three statements easily generalise to arbitrary X.

Lemma 4.1. *For c a scalar,*

$$R_\lambda(cX;b) = c^{|\lambda|} R_\lambda(X;bc).$$

Lemma 4.2. *For c a scalar and $n := |X|$,*

$$\lim_{c\to\infty} R_\lambda(cX;b) = \tau_\lambda^{-1} \left(\frac{t^{1-n}}{b}\right)^{|\lambda|} P_\lambda\big(\langle 0\rangle\big).$$

Lemma 4.3. *Let $n := |X|$ and let λ be a partition such that $l(\lambda) = n$. Define $\mu := (\lambda_1 - 1, \dots, \lambda_n - 1)$. Then*

$$R_\lambda(X;b) = R_\mu(X;bq) \prod_{x\in X} \frac{x}{1-bx}.$$

This last result allows the definition of $R_\lambda(X;b)$ to be extended to all weakly decreasing integer sequences $\lambda=(\lambda_1,\dots,\lambda_n)$.

Proof of Lemmas 4.1–4.3. By (4.4) all three statements are obviously true for X a single-letter alphabet, and we proceed by induction on n, the cardinality of X.

By (4.3),

$$R_\lambda(cX;b)=\sum_\mu \frac{(bcz/t)_\mu}{(bcz)_\lambda} P_{\lambda/\mu}(cz)R_\mu(cY;b). \tag{4.5}$$

Using (2.6) and the appropriate induction hypothesis this yields

$$R_\lambda(cX;b)=c^{|\lambda|}\sum_\mu \frac{(bcz/t)_\mu}{(bcz)_\lambda} P_{\lambda/\mu}(z)R_\mu(Y;bc)=c^{|\lambda|}R_\lambda(X;bc),$$

establishing the first lemma.

Taking the $c\to\infty$ limit on both sides of (4.5) and then using induction we get

$$\begin{aligned}
\lim_{c\to\infty} R_\lambda(cX;b) &= \sum_\mu b^{|\mu|-|\lambda|}t^{-|\mu|}\frac{\tau_\mu}{\tau_\lambda}P_{\lambda/\mu}(1)\lim_{c\to\infty}R_\mu(cY;b)\\
&=\tau_\lambda^{-1}\left(\frac{t^{1-n}}{b}\right)^{|\lambda|}\sum_\mu P_{\lambda/\mu}\bigl(t^{n-1}\bigr)P_\mu\bigl(t^{n-2},\dots,t,1\bigr)\\
&=\tau_\lambda^{-1}\left(\frac{t^{1-n}}{b}\right)^{|\lambda|}P_\lambda\bigl(\langle 0\rangle\bigr),
\end{aligned}$$

where the last equality follows from (2.5).

To prove the final lemma we consider (4.5) with $c=1$ and, in accordance with the conditions of Lemma 4.3, with $\lambda_n\geqslant 1$. Since $P_{\lambda/\nu}(a)$ vanishes unless $\lambda-\nu$ is a horizontal strip this implies that $\nu_{n-1}\geqslant 1$. The summand also vanishes if $l(\nu)>n-1$ so that we may assume that $l(\nu)=n-1$. Defining $\eta=(\nu_1-1,\dots,\nu_{n-1}-1)$ and $\mu=(\lambda_1-1,\dots,\lambda_n-1)$ and using induction, as well as

$$\frac{(bz/t)_\nu}{(bz)_\lambda}=\frac{1}{1-bz}\frac{(bzq/t)_\eta}{(bzq)_\mu} \quad \text{and} \quad P_{\lambda/\nu}(z)=zP_{\mu/\eta}(z),$$

we get

$$\begin{aligned}
R_\lambda(X;b)&=\frac{z}{1-bz}\left(\prod_{x\in Y}\frac{x}{1-bx}\right)\sum_{\eta\subseteq\mu}\frac{(bzq/t)_\eta}{(bzq)_\mu}P_{\mu/\eta}(z)R_\eta(Y;bq)\\
&=R_\mu(X;bq)\prod_{x\in X}\frac{x}{1-bx},
\end{aligned}$$

where in the final step we have used (4.3) and $X=Y+z$. □

Proposition 4.4 *(Principal specialisation). For λ such that $l(\lambda) \leqslant n$,*

$$R_\lambda\big(\langle 0\rangle; b\big) = \frac{P_\lambda(\langle 0\rangle)}{(bt^{n-1})_\lambda} = \frac{t^{n(\lambda)}(t^n)_\lambda}{(bt^{n-1})_\lambda c_\lambda}.$$

By Lemma 4.1 this may be stated slightly more generally as

$$R_\lambda\big(a\langle 0\rangle; b\big) = \frac{P_\lambda(a\langle 0\rangle)}{(abt^{n-1})_\lambda}. \tag{4.6}$$

Proof of Proposition 4.4. Iterating (1.13) using

$$\sum_\nu P_{\lambda/\nu}(X)P_{\nu/\mu}(Y) = P_{\lambda/\mu}(X+Y),$$

we obtain the generalised branching rule

$$R_\lambda\big(x_1,\dots,x_m,t^{n-1},\dots,t,1;b\big) = \sum_\mu \frac{(b/t)_\mu}{(bt^{n-1})_\lambda} P_{\lambda/\mu}\big(\langle 0\rangle\big)R_\mu(x_1,\dots,x_m;b),$$

for $l(\lambda) \leqslant n+m$. When $m=0$ this results in the claim. □

Proposition 4.5 *(Evaluation symmetry). For λ such that $l(\lambda) \leqslant n$ set*

$$\langle\lambda\rangle = \big(q^{\lambda_1}t^{n-1},\dots,q^{\lambda_{n-1}}t,1\big).$$

Then

$$\frac{R_\lambda(a\langle\mu\rangle;b)}{R_\lambda(a\langle 0\rangle;b)} = \frac{R_\mu(a\langle\lambda\rangle;b)}{R_\mu(a\langle 0\rangle;b)}.$$

Proof. Unlike almost all our other results, this does not easily follow from the branching rule and, somewhat against the spirit of the paper, we use the corresponding result for the Macdonald interpolation polynomials.

We may view the evaluation symmetry as a rational function identity in b. Hence it suffices to give a proof for $b = q^{1-m}$ where m runs over all integers such that $\lambda_1, \mu_1 \leqslant m$. By (1.14) it then follows that we need to prove that

$$\frac{M_{m^n-\lambda}(a\langle\mu\rangle)}{M_{m^n-\lambda}(a\langle 0\rangle)} = \frac{M_{m^n-\mu}(a\langle\lambda\rangle)}{M_{m^n-\mu}(a\langle 0\rangle)}\frac{(aqt^{n-1})_\mu}{(aqt^{n-1})_\lambda}\frac{(aq^{1-m}t^{n-1})_\lambda}{(aq^{1-m}t^{n-1})_\mu}.$$

Making the substitutions $\lambda \mapsto m^n - \lambda$, $\mu \mapsto m^n - \mu$ and $a \mapsto aq^{-m}t^{1-n}$ we get

$$\frac{M_\lambda(a/\langle\mu\rangle)}{M_\lambda(aq^{-m}t^{1-n}\langle 0\rangle)} = \frac{M_\mu(a/\langle\lambda\rangle)}{M_\mu(aq^{-m}t^{1-n}\langle 0\rangle)}\frac{(q^mt^{n-1}/a)_\mu}{(q^mt^{n-1}/a)_\lambda}\frac{(t^{n-1}/a)_\lambda}{(t^{n-1}/a)_\mu}q^{m(|\lambda|-|\mu|)}.$$

Finally, by the principal specialisation formula for the interpolation Macdonald polynomials [18]

$$\frac{M_\lambda(aq^{-m}t^{1-n}\langle 0\rangle)}{M_\lambda(a/\langle 0\rangle)} = \frac{(q^mt^{n-1}/a)_\lambda}{(t^{n-1}/a)_\lambda}q^{-m|\lambda|},$$

we end up with

$$\frac{M_\lambda(a/\langle\mu\rangle)}{M_\lambda(a/\langle 0\rangle)}=\frac{M_\mu(a/\langle\lambda\rangle)}{M_\mu(a/\langle 0\rangle)}.$$

This is the known evaluation symmetry of the interpolation Macdonald polynomials [18, Section 2]. □

It is clear from (4.3) that $R_\lambda(X;0)=P_\lambda(X)$ with on the right a Macdonald polynomial. The Macdonald polynomials in turn generalise the Jack polynomials $P^{(\alpha)}_\lambda(X)$, since $P^{(\alpha)}_\lambda(X)=\lim_{q\to 1}P_\lambda(X;q,q^{1/\alpha})$. Combining the last two equations it thus follows that

$$P^{(\alpha)}_\lambda(X)=\lim_{q\to 1}R_\lambda\big(X;0;q,q^{1/\alpha}\big).$$

Curiously, there is an alternative path from $R_\lambda(X;b)$ to the Jack polynomials as follows. For X an alphabet let

$$\hat{X}:=\left\{\frac{x}{1-x}\;\middle|\;x\in X\right\}.$$

Proposition 4.6. *We have*

$$P^{(\alpha)}_\lambda(\hat{X})=\lim_{q\to 1}R_\lambda\big(X;1;q,q^{1/\alpha}\big).$$

Proof. Let $X=Y+z$ be a finite alphabet.

Replacing $(b,t)\mapsto(1,q^{1/\alpha})$ in (4.3) and taking the $q\to 1$ limit yields

$$R^{(\alpha)}_\lambda(X)=\sum_\mu(1-z)^{|\mu|-|\lambda|}P^{(\alpha)}_{\lambda/\mu}(z)R^{(\alpha)}_\mu(Y),$$

where $P^{(\alpha)}_{\lambda/\mu}$ is a skew Jack polynomial and

$$R^{(\alpha)}_\lambda(X):=\lim_{q\to 1}R_\lambda\big(X;1;q,q^{1/\alpha}\big).$$

Using the homogeneity of $P^{(\alpha)}_{\lambda/\mu}$ the above can be rewritten as

$$R^{(\alpha)}_\lambda(X)=\sum_\mu P^{(\alpha)}_{\lambda/\mu}\left(\frac{z}{1-z}\right)R^{(\alpha)}_\mu(Y).$$

Comparing this with

$$P^{(\alpha)}_\lambda(X)=\sum_\mu P^{(\alpha)}_{\lambda/\mu}(z)P^{(\alpha)}_\mu(Y)$$

the proposition follows. □

5. Cauchy, Pieri and Gauss formulas for $R_\lambda(X;b)$

Probably our most important new results for $R_\lambda(X;b)$ are generalisation of the skew Cauchy identity (2.11), the Pieri formula (2.12) and the q-Gauss formula (1.18).

Before we get to these result we first need a few more definitions. First of all, in analogy with (2.7), we set

$$\mathsf{R}_\lambda(X;b) := t^{n(\lambda)} \frac{R_\lambda(X;b)}{c'_\lambda} \tag{5.1}$$

so that (4.3) becomes

$$\mathsf{R}_\lambda(X;b) = \sum_\mu \frac{(bz/t)_\mu}{(bz)_\lambda} \mathsf{P}_{\lambda/\mu}(z) \mathsf{R}_\mu(Y;b).$$

Furthermore, we also define the skew functions $\mathsf{R}_{\lambda/\mu}(X;b)$ by

$$\mathsf{R}_{\lambda/\mu}(X;b) := \sum_\nu \frac{(bz/t)_\nu}{(bz)_\lambda} \mathsf{P}_{\lambda/\nu}(z) \mathsf{R}_{\mu/\nu}(Y;b) \tag{5.2}$$

and

$$\mathsf{R}_{\lambda/\mu}(-;b) = \delta_{\lambda\mu}.$$

In other words,

$$\mathsf{R}_{\lambda/\mu}(X+Y;b) = \sum_\nu \mathsf{R}_{\lambda/\mu}(X;b) \mathsf{R}_{\mu/\nu}(Y;b)$$

and $\mathsf{R}_{\lambda/\mu}(X;0) = \mathsf{P}_{\lambda/\mu}(X)$.

Theorem 5.1 *(Skew Cauchy-type identity). Let $ab = cd$ and let X be a finite alphabet. Then*

$$\begin{aligned} &\sum_\lambda \frac{(b/c)_\lambda}{(b/c)_\nu} \mathsf{Q}_{\lambda/\nu}\left[\frac{a-c}{1-t}\right] \mathsf{R}_{\lambda/\mu}(X;b) \\ &\quad = \left(\prod_{x\in X} \frac{(cx,dx)_\infty}{(ax,bx)_\infty}\right) \sum_\lambda \frac{(b/c)_\mu}{(b/c)_\lambda} \mathsf{Q}_{\mu/\lambda}\left[\frac{a-c}{1-t}\right] \mathsf{R}_{\nu/\lambda}(X;d). \end{aligned} \tag{5.3}$$

Note that for $b=0$ the theorem simplifies to (2.11). We defer the proof of (5.3) till the end of this section and first list a number of corollaries.

Corollary 5.2 *(Pieri formula). Let $ab = cd$ and let X be a finite alphabet. Then*

$$\mathsf{R}_\mu(X;d) \prod_{x\in X} \frac{(cx,dx)_\infty}{(ax,bx)_\infty} = \sum_\lambda \frac{(b/c)_\lambda}{(b/c)_\mu} \mathsf{Q}_{\lambda/\mu}\left[\frac{a-c}{1-t}\right] \mathsf{R}_\lambda(X;b). \tag{5.4}$$

This follows from the theorem by taking $\mu = 0$ and then replacing ν by μ.

When $b, d \to 0$ Eq. (5.4) yields the Pieri formula (2.12) for Macdonald polynomials. When $b, c \to 0$ and $a \to 1$ such that $b/c = d$ Eq. (5.4) yields (after replacing $d \mapsto b$)

$$\sum_{\lambda} \frac{(b)_\lambda}{(b)_\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathsf{P}_\lambda(X) = \mathsf{R}_\mu(X; b) \prod_{x \in X} \frac{(bx)_\infty}{(x)_\infty}. \tag{5.5}$$

For $\mu = 0$ this is the q-binomial formula for Macdonald polynomials (1.16), and for $b = 0$ it is Lassalle's [11]

$$\sum_{\lambda} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathsf{P}_\lambda(X) = \mathsf{P}_\mu(X) \prod_{x \in X} \frac{1}{(x)_\infty}.$$

The Jack polynomial limit of (5.5) is of particular interest. To concisely state this we need some more notation. Let

$$\binom{\lambda}{\mu}^{(\alpha)} := \lim_{q \to 1} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q, q^{1/\alpha}}.$$

Further let

$$(b; \alpha)_\lambda := \prod_{i \geqslant 1} \big(b + (1-i)/\alpha\big)_{\lambda_i}$$

with $(b)_k = b(b+1) \cdots (b+k-1)$,

$$c'_\lambda(\alpha) := \prod_{s \in \lambda} \big(a(s) + 1 + l(s)/\alpha\big)$$

and

$$\mathsf{P}^{(\alpha)}_\lambda(X) := \frac{P^{(\alpha)}_\lambda(X)}{c'_\lambda(\alpha)}.$$

Using all of the above, replacing (b, q, t) in (5.5) by $(q^\beta, q, q^{1/\alpha})$ and taking the (formal) limit $q \to 1$ with the aid of Proposition 4.6, we arrive at the following identity.

Corollary 5.3 *(Binomial formula for Jack polynomials). For X a finite alphabet*

$$\sum_{\lambda} \frac{(\beta; \alpha)_\lambda}{(\beta; \alpha)_\mu} \binom{\lambda}{\mu}^{(\alpha)} \mathsf{P}^{(\alpha)}_\lambda(X) = \mathsf{P}^{(\alpha)}_\mu(\hat{X}) \prod_{x \in X} \frac{1}{(1-x)^\beta}.$$

This should be compared with the binomial formula of Kaneko, Lassalle, Okounkov and Olshanski [3,10,22]:

$$\frac{\mathsf{P}^{(\alpha)}_\lambda(x_1 + 1, \dots, x_n + 1)}{\mathsf{P}^{(\alpha)}_\lambda(1^n)} = \sum_{\mu} \binom{\lambda}{\mu}^{(\alpha)} \frac{\mathsf{P}^{(\alpha)}_\mu(x_1, \dots, x_n)}{\mathsf{P}^{(\alpha)}_\mu(1^n)},$$

where n is any integer such that $n \geqslant l(\lambda)$.

Another special case of (5.4) worth stating is the following multivariable extension of the ${}_1\phi_1$ summation [2, II.5], which follows straightforwardly by taking the $a, d \to 0$ limit,

$$\sum_{\lambda} c^{|\lambda-\mu|} \frac{(b/c)_\lambda}{(b/c)_\mu} \mathsf{Q}_{\lambda/\mu}\left[\frac{0-1}{1-t}\right] \mathsf{R}_\lambda(X; b) = \mathsf{P}_\mu(X) \prod_{x\in X} \frac{(cx)_\infty}{(bx)_\infty}. \tag{5.6}$$

This provides an expansion of the right-hand side different from (2.12).

If we let $\nu = 0$ in Theorem 5.1, use (2.10) and then replace $(a, b, c) \mapsto (c/ab, c, c/a)$, we obtain

$$\sum_{\lambda} \left(\frac{c}{ab}\right)^{|\lambda-\mu|} \frac{(a,b)_\lambda}{(a,b)_\mu} \mathsf{R}_{\lambda/\mu}(X; c) = \prod_{x\in X} \frac{(cx/a, cx/b)_\infty}{(cx, cx/ab)_\infty}.$$

For $\mu = 0$ we state this separately.

Corollary 5.4 *($\mathfrak{sl}_n$ q-Gauss sum). For X a finite alphabet*

$$\sum_{\lambda} \left(\frac{c}{ab}\right)^{|\lambda|} (a,b)_\lambda \mathsf{R}_\lambda(X; c) = \prod_{x\in X} \frac{(cx/a, cx/b)_\infty}{(cx, cx/ab)_\infty}. \tag{5.7}$$

Thanks to (4.4), the $X = \{1\}$ case of the above identity simplifies to the classical q-Gauss sum [2, (II.8)]

$$\sum_{k=0}^{\infty} \frac{(a,b)_k}{(q,c)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty}. \tag{5.8}$$

More generally, if we principally specialise $X = t^{1-n}\langle 0\rangle = \{1, t^{-1}, \dots, t^{1-n}\}$ and use (4.6) the $\mathfrak{sl}_n$ q-Gauss sum simplifies to Kaneko's q-Gauss sum for Macdonald polynomials [4, Proposition 5.4], given as (1.17) in the introduction:

$$\sum_{\lambda} \left(\frac{ct^{1-n}}{ab}\right)^{|\lambda|} \frac{(a,b)_\lambda}{(c)_\lambda} P_\lambda\big(\langle 0\rangle\big) = \prod_{i=1}^{n} \frac{(ct^{1-i}/a, ct^{1-i}/b)_\infty}{(ct^{1-i}, ct^{1-i}/ab)_\infty}.$$

As another consequence of the theorem we obtain an explicit expression for the Taylor series of $\mathsf{R}_\mu(X; b)$ in b. For $\mu \subseteq \lambda$ let $(1)_{\lambda/\mu}$ be defined as $(1)_{\lambda/\mu} = \lim_{a\to 1} (a)_\lambda/(a)_\mu$. That is,

$$(1)_{\lambda/\mu} = \prod_{s\in\lambda-\mu} \big(1 - q^{a'(s)} t^{-l'(s)}\big).$$

Corollary 5.5. *We have*

$$\mathsf{R}_\mu(X; b) = \sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|} (1)_{\lambda/\mu} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} \mathsf{P}_\lambda(X), \tag{5.9}$$

or, equivalently,

$$\big[b^r\big] \mathsf{R}_\mu(X; b) = \sum_{\substack{\lambda\supseteq\mu\\ |\lambda-\mu|=r}} (1)_{\lambda/\mu} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} \mathsf{P}_\lambda(X).$$

Proof. Replacing $(c, X) \mapsto (a/b, bX)$ in Lemma 4.1 and expressing the resulting identity in terms of the normalised function $\mathsf{R}_\lambda(X; b)$ (see (5.1)) we get

$$\mathsf{R}_\lambda(aX; b) = \left(\frac{a}{b}\right)^{|\lambda|} \mathsf{R}_\lambda(bX; a). \tag{5.10}$$

Combined with (5.5) this implies that

$$\mathsf{R}_\mu(aX; b) = \left(\frac{a}{b}\right)^{|\mu|} \left(\prod_{x \in X} \frac{(bx)_\infty}{(abx)_\infty}\right) \sum_\lambda b^{|\lambda|} \frac{(a)_\lambda}{(a)_\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathsf{P}_\lambda(X). \tag{5.11}$$

The summand vanishes unless $\mu \subseteq \lambda$ and so we may add this as a restriction in the sum over λ. Then the limit $a \to 1$ may be taken without causing ambiguities, and the claim follows. □

Corollary 5.5 implies the following simple expressions for $R_\lambda(X; b)$ when $t = q$ (Schur-like case) or $t = 1$ (monomial-like case).

Proposition 5.6. *Let $X = \{x_1, \dots, x_n\}$. Then*

$$R_\lambda(X; b) = \frac{1}{\Delta(x)} \det_{1 \leqslant i,j \leqslant n} \left(\frac{x_i^{\lambda_j + n - j}}{(bx_i)_{\lambda_j - j + 1}} \right) \quad \textit{if } t = q \tag{5.12a}$$

and

$$R_\lambda(X; b) = \sum_{u^+ = \lambda} \left(\prod_{i=1}^n \frac{x_i^{u_i}}{(bx_i)_{u_i}} \right) \quad \textit{if } t = 1. \tag{5.12b}$$

Proof. Since the two claims are proved in almost identical fashion we only present a proof of (5.12a). The only significant difference is that the omitted proof of (5.12b) uses (2.18b) instead of (2.18a).

Assume that $t = q$. Let $\nu = \lambda - (0, 1, \dots, n-1)$ and suppose that $\nu_n \geqslant 0$. Since for any $k \geqslant 0$, one has the expansion

$$\frac{x^{k+n-1}}{(bx)_k} = \sum_{r=0}^{\infty} x^{k+n-1+r} \begin{bmatrix} k + r - 1 \\ r \end{bmatrix},$$

the matrix $(x_i^{\nu_j + n - 1}/(bx_i)_{\nu_j})$ factorises into the product of rectangular matrices

$$\left(x_i^{n-1+r}\right)_{\substack{i=1,\dots,n \\ r=0,1,\dots}} \quad \text{and} \quad \left(b^{r-k} \begin{bmatrix} k-1 \\ r-k \end{bmatrix}\right)_{\substack{r=0,1,\dots \\ k=\nu_1,\dots,\nu_n}}.$$

According to Cauchy–Binet theorem, the determinant on the right-hand side of (5.12a) factorises into a sum of products of minors of these two matrices.

On the other hand, by (1.1) and (2.18a), the expansion (5.9) gives

$$R_\lambda(X; b)\Delta(x) = \sum_{\mu \supseteq \lambda} b^{|\mu - \lambda|} (1)_{\mu/\lambda} \det_{1 \leqslant i,j \leqslant n} \left(\frac{1}{(q)_{\mu_i - \lambda_j - i + j}} \right) \det_{1 \leqslant i,j \leqslant n} \left(x_i^{\mu_j + n - j}\right),$$

where we have also used (2.20) and (5.1). Using ν instead of λ, and $\eta = \mu - (0, 1, \dots, n-1)$, this becomes

$$R_\lambda(X;b)\Delta(x) = \sum_{\eta \supset \nu} \det_{1 \leqslant i,j \leqslant n} \left(\begin{bmatrix} \eta_i - 1 \\ \nu_j - 1 \end{bmatrix} b^{\eta_i - \nu_j} \right) \det_{1 \leqslant i,j \leqslant n} \bigl(x_i^{\eta_j}\bigr),$$

which is precisely the Cauchy–Binet expansion. The restriction $\nu_n \geqslant 0$ is lifted using Lemma 4.3. □

Recall that the Macdonald polynomials are the eigenfunctions of the operator D_n^1, see (2.4). Because $P_\lambda(X;q,t) = P_\lambda(X;q^{-1},t^{-1})$ this can also be stated as

$$\mathcal{D}_n^1 P_\lambda(X) = \omega_\lambda P_\lambda(X),$$

where $\mathcal{D}_n^1 := D_n^1(q^{-1};t^{-1})$ and ω_λ is given in (2.17).

The second consequence of Corollary 5.5 is a generalisation of this identity as follows. Let

$$A_i(x;t) := \prod_{\substack{j=1 \\ j \neq i}}^n \frac{tx_i - x_j}{x_i - x_j}$$

and

$$\mathcal{D}_n^1(b) := \sum_{i=1}^n A_i\bigl(x;t^{-1}\bigr)\left(\left(1 - \frac{bx_i}{q}\right)T_{q^{-1},x_i} + \frac{bx_i}{q}\right),$$

so that $\mathcal{D}_n^1(0) = \mathcal{D}_n^1$.

Theorem 5.7. *For $X = \{x_1, \dots, x_n\}$ we have*

$$\mathcal{D}_n^1(b) R_\lambda(X;b) = \omega_\lambda R_\lambda(X;b).$$

Proof. Define the operator $\mathcal{E}_n$ as

$$\mathcal{E}_n := \sum_{i=1}^n x_i A_i\bigl(x;t^{-1}\bigr)(T_{q^{-1},x_i} - 1).$$

Combining the $(q,t) \mapsto (q^{-1},t^{-1})$ instance of [11, Proposition 9] with (2.21) gives

$$\mathcal{E}_n \mathsf{P}_\lambda(X) = (1-q)\sum_{i=1}^n q^{-\lambda_i} t^{i-n} (1)_{\lambda^{(i)}/\lambda} \begin{bmatrix} \lambda^{(i)} \\ \lambda \end{bmatrix} \mathsf{P}_{\lambda^{(i)}}(X),$$

where $\lambda^{(i)} := (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_i)$.

Since

$$\mathcal{D}_n^1(b) = \mathcal{D}_n^1(0) - bq^{-1}\mathcal{E}_n$$

this implies that

$$\mathcal{D}_n^1(b)\mathsf{P}_\lambda(X) = \omega_\lambda \mathsf{P}_\lambda(X) - b(1-q)\sum_{i=1}^n q^{-\lambda_i-1}t^{i-n}(1)_{\lambda^{(i)}/\lambda}\begin{bmatrix}\lambda^{(i)}\\ \lambda\end{bmatrix}\mathsf{P}_{\lambda^{(i)}}(X).$$

By Corollary 5.5 we can now compute the action of $\mathcal{D}_n^1(b)$ on $\mathsf{R}_\lambda(X;b)$:

$$\begin{aligned}\mathcal{D}_n^1(b)\mathsf{R}_\mu(X;b) &= \sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|}(1)_{\lambda/\mu}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}\mathcal{D}_n^1(b)\mathsf{P}_\lambda(X)\\ &= \sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|}(1)_{\lambda/\mu}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}\omega_\lambda\mathsf{P}_\lambda(X)\\ &\quad - (1-q)\sum_{i=1}^n\sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|+1}q^{-\lambda_i-1}t^{i-n}(1)_{\lambda^{(i)}/\mu}\begin{bmatrix}\lambda^{(i)}\\ \lambda\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}\mathsf{P}_{\lambda^{(i)}}(X)\\ &= \sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|}(1)_{\lambda/\mu}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}\omega_\lambda\mathsf{P}_\lambda(X)\\ &\quad - (1-q)\sum_{i=1}^n\sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|}q^{-\lambda_i}t^{i-n}(1)_{\lambda/\mu}\begin{bmatrix}\lambda\\ \lambda_{(i)}\end{bmatrix}\begin{bmatrix}\lambda_{(i)}\\ \mu\end{bmatrix}\mathsf{P}_\lambda(X),\end{aligned}$$

where we have also used that $(1)_{\lambda/\mu}(1)_{\mu/\nu} = (1)_{\lambda/\nu}$ for $\nu\subseteq\mu\subseteq\lambda$. Recalling the recursion (2.16), the sum over i on the right can be performed to give

$$\mathcal{D}_n^1(b)\mathsf{R}_\mu(X;b) = \omega_\mu\sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|}(1)_{\lambda/\mu}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}\mathsf{P}_\lambda(X).$$

Again using (5.9) completes the proof. □

As a third and final application of Corollary 5.5 we are in a position to derive a simple expression for $D_n(b,c,d)P_\mu(x)$, to be compared with (2.2) (obtained for $b=0$) and (4.2).

Proposition 5.8. *For $X=\{x_1,\dots,x_n\}$ we have*

$$D_n(b,c,d)P_\mu(X) = (c;t)_n\sum_{\lambda'\succcurlyeq\mu'}(-b)^{|\lambda-\mu|}\frac{(cqt^{n-1})_\mu}{(ct^{n-1})_\lambda}\frac{(d)_\lambda}{(d)_\mu}\psi'_{\lambda/\mu}P_\lambda(X).$$

Proof. First we use (2.7b) and (2.14) to put the proposition in the form

$$D_n(b,c,d)\mathsf{P}_\mu(X) = (c;t)_n\sum_{\lambda\supseteq\mu} b^{|\lambda-\mu|}\frac{(cqt^{n-1})_\mu}{(ct^{n-1})_\lambda}\frac{(d)_\lambda}{(d)_\mu}\mathsf{Q}_{\lambda/\mu}\left[\frac{q-1}{1-t}\right]\mathsf{P}_\lambda(X). \tag{5.13}$$

To prove this we similarly normalise (4.2), so that

$$\begin{aligned}D_n(b,c,d)\mathsf{R}_\nu(X;b) &= (c;t)_n\prod_{i=1}^n(1-bx_i)\\ &\quad\times\sum_\mu (bd)^{|\mu-\nu|}\frac{\tau_\mu}{\tau_\nu}\frac{(cqt^{n-1})_\nu}{(ct^{n-1})_\mu}\frac{(ct^{n-1}/d)_\mu}{(ct^{n-1}/d)_\nu}\mathsf{Q}_{\mu/\nu}\left[\frac{q-1}{1-t}\right]\mathsf{R}_\mu(X;b).\end{aligned} \tag{5.14}$$

First we consider the right-hand side of this identity without the overall factor $(c;t)_n$ and with ct^{n-1} replaced by c:

$$g_\nu := \left(\prod_{i=1}^n (1-bx_i)\right) \sum_\mu (bd)^{|\mu-\nu|} \frac{\tau_\mu}{\tau_\nu} \frac{(cq)_\nu}{(c)_\mu} \frac{(c/d)_\mu}{(c/d)_\nu} \mathsf{Q}_{\mu/\nu}\left[\frac{q-1}{1-t}\right] \mathsf{R}_\mu(X;b).$$

Eliminating $\mathsf{R}_\mu(X;b)$ using (5.9) yields

$$\begin{aligned} g_\nu &= \prod_{i=1}^n (1-bx_i) \\ &\quad \times \sum_{\substack{\lambda,\mu \\ \mu\subseteq\lambda}} b^{|\lambda-\nu|} d^{|\mu-\nu|} \frac{\tau_\mu}{\tau_\nu} \frac{(cq)_\nu}{(c)_\mu} \frac{(c/d)_\mu}{(c/d)_\nu} (1)_{\lambda/\mu} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} \mathsf{Q}_{\mu/\nu}\left[\frac{q-1}{1-t}\right] \mathsf{P}_\lambda(X). \end{aligned}$$

To simplify what follows it is helpful to first consider the more general function $g_\nu(e,f,g)$ defined as

$$\begin{aligned} g_\nu(e,f,g) &:= \left(\prod_{i=1}^n \frac{(bx_i)_\infty}{(bgqx_i)_\infty}\right) \sum_{\lambda,\mu} b^{|\lambda-\nu|} (dg/e)^{|\mu-\nu|} \frac{(df/e)_\nu}{(df/e)_\lambda} \frac{(cq)_\nu}{(cg)_\mu} \frac{(e,c/d)_\mu}{(e,c/d)_\nu} \frac{(f)_\lambda}{(f)_\mu} \\ &\quad \times \mathsf{Q}_{\lambda/\mu}\left[\frac{1-dgq/e}{1-t}\right] \mathsf{Q}_{\mu/\nu}\left[\frac{q-f}{1-t}\right] \mathsf{P}_\lambda(X). \end{aligned}$$

Obviously,

$$g_\nu = \lim_{e\to\infty} g_\nu(e,1,1).$$

By the Pieri formula (2.12)

$$\begin{aligned} g_\nu(e,f,g) &= \sum_{\eta,\lambda,\mu} (bg)^{|\eta-\nu|} (d/e)^{|\mu-\nu|} (q/f)^{|\eta-\mu|} \frac{(df/e)_\nu}{(df/e)_\lambda} \frac{(cq)_\nu}{(cg)_\mu} \frac{(e,c/d)_\mu}{(e,c/d)_\nu} \frac{(f)_\lambda}{(f)_\mu} \\ &\quad \times \mathsf{Q}_{\eta/\lambda}\left[\frac{f-f/gq}{1-t}\right] \mathsf{Q}_{\lambda/\mu}\left[\frac{f/gq-df/e}{1-t}\right] \mathsf{Q}_{\mu/\nu}\left[\frac{q-f}{1-t}\right] \mathsf{P}_\eta(X). \end{aligned}$$

Thanks to Rains' generalised q-Pfaff–Saalschütz sum [23, Corollary 4.9]

$$\sum_\mu \frac{(a)_\mu (c)_\lambda}{(a)_\nu (c)_\mu} \mathsf{Q}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] \mathsf{Q}_{\mu/\nu}\left[\frac{b-c}{1-t}\right] = \frac{(b)_\lambda}{(b)_\nu} \mathsf{Q}_{\lambda/\nu}\left[\frac{a-c}{1-t}\right] \tag{5.15}$$

with $(a,b,c;\lambda,\mu,\nu) \mapsto (f, f/gq, df/e; \eta,\lambda,\mu)$ we can perform the sum over λ. Then renaming the summation index η as λ results in

$$\begin{aligned} g_\nu(e,f,g) &= \sum_{\lambda,\mu} (bgq)^{|\lambda-\nu|} \frac{(df/e)_\nu}{(df/e)_\lambda} \frac{(cq)_\nu}{(cg)_\mu} \frac{(e,c/d)_\mu}{(e,c/d)_\nu} \frac{(f/gq)_\lambda}{(f/gq)_\mu} \\ &\quad \times \mathsf{Q}_{\lambda/\mu}\left[\frac{1-d/e}{1-t}\right] \mathsf{Q}_{\mu/\nu}\left[\frac{d/e-df/eq}{1-t}\right] \mathsf{P}_\lambda(X). \end{aligned}$$

In the Sears transformation (3.5) all occurrences of P may be replaced by Q. Applying this normalised form of (3.5) with $(a,b,c,d) \mapsto (c/q, cgq/f, 1/g, c/d)$, we obtain

$$g_\nu(e,f,g) = \sum_{\lambda,\mu} (bgq)^{|\lambda-\nu|} \frac{(df/e, cq)_\nu}{(df/e, cg)_\lambda} \frac{(d, c/e)_\lambda}{(d, c/e)_\mu} \frac{(1/g, cgq/f)_\mu}{(1/g, cgq/f)_\nu} \times \mathsf{Q}_{\lambda/\mu}\left[\frac{1-f/q}{1-t}\right] \mathsf{Q}_{\mu/\nu}\left[\frac{f/q - df/yq}{1-t}\right] \mathsf{P}_\lambda(X).$$

It may not seem that we have achieved much, but letting $f = g = 1$ and then taking the large e limit we find

$$g_\nu = \sum_{\substack{\lambda,\mu \\ \mu\subseteq\lambda}} b^{|\lambda-\nu|} \frac{(cq)_\mu}{(c)_\lambda} \frac{(d)_\lambda}{(d)_\mu} (1)_{\mu/\nu} \begin{bmatrix} \mu \\ \nu \end{bmatrix} \mathsf{Q}_{\lambda/\mu}\left[\frac{q-1}{1-t}\right] \mathsf{P}_\lambda(X).$$

This essentially establishes the claim. Indeed, by an appeal to (5.9) the left-hand side of (5.14) is

$$D_n(b,c,d)\mathsf{R}_\nu(X;b) = \sum_\mu b^{|\mu-\nu|} (1)_{\mu/\nu} \begin{bmatrix} \mu \\ \nu \end{bmatrix} D_n(b,c,d)\mathsf{P}_\mu(X).$$

This must be equal to $g_\nu|_{c\mapsto ct^{n-1}}$ times $(c;t)_n$, thus proving (5.13). □

We remark that by a slight variation of the above proof it may also be shown that (4.1) is the $d = 1$ instance of

$$D_n(b,c,d)\left(\mathsf{R}_\mu(X;bd) \prod_{i=1}^n \frac{(bdx_i)_\infty}{(bx_i)_\infty}\right) = \mathsf{R}_\mu(X;bdq) \prod_{i=1}^n \bigl(1 - cq^{\mu_i} t^{n-i}\bigr) \frac{(bdqx_i)_\infty}{(bqx_i)_\infty}.$$

5.1. Proof of Theorem 5.1

To prove the theorem we first prepare the following result.

Proposition 5.9. *For μ, ν partitions*

$$\sum_\lambda \frac{(a)_\lambda}{(c)_\lambda} \mathsf{P}_{\lambda/\mu}\left(\frac{c}{ab}\right) \mathsf{Q}_{\lambda/\nu}\left[\frac{1-b}{1-t}\right] = \frac{(a)_\mu}{(c/t)_\mu} \frac{(a)_\nu}{(c/b)_\nu} \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty} \sum_\lambda \frac{(c/bt)_\lambda}{(a)_\lambda} \mathsf{P}_{\nu/\lambda}\left(\frac{c}{ab}\right) \mathsf{Q}_{\mu/\lambda}\left[\frac{1-b}{1-t}\right].$$

Note that for $\nu = 0$ this is

$$\sum_\lambda \frac{(a,b)_\lambda}{(c)_\lambda} \mathsf{P}_{\lambda/\mu}\left(\frac{c}{ab}\right) = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty} \frac{(a,b)_\mu}{(c/t)_\mu},$$

which, for $\mu = 0$, simplifies to the q-Gauss sum (5.8).

Proof of Proposition 5.9. Key is the Cauchy-type identity for skew Macdonald polynomials due to Rains [26, Corollary 3.8]:

$$\begin{aligned}
&\frac{1}{Z}\sum_{\lambda} q^{|\lambda|}\frac{(a,b)_\lambda}{(e,f)_\lambda}\mathsf{P}_{\lambda/\mu}\biggl[\frac{1-c}{1-t}\biggr]\mathsf{Q}_{\lambda/\nu}\biggl[\frac{1-d}{1-t}\biggr]\\
&\quad=\biggl(\frac{q}{c}\biggr)^{|\mu|}\frac{(a,b)_\mu}{(e/c,f/c)_\mu}\biggl(\frac{q}{d}\biggr)^{|\nu|}\frac{(a,b)_\nu}{(e/d,f/d)_\nu}\\
&\qquad\times\sum_{\lambda}\biggl(\frac{cd}{q}\biggr)^{|\lambda|}\frac{(e/cd,f/cd)_\lambda}{(a,b)_\lambda}\mathsf{P}_{\nu/\lambda}\biggl[\frac{1-c}{1-t}\biggr]\mathsf{Q}_{\mu/\lambda}\biggl[\frac{1-d}{1-t}\biggr],
\end{aligned}\tag{5.16}$$

provided that the sum on the left terminates and the balancing condition $abcdq = eft$ holds. The prefactor Z refers to the sum on the left for $\mu = \nu = 0$, i.e., to

$$\begin{aligned}
Z&=\sum_{\lambda} q^{|\lambda|}\frac{(a,b)_\lambda}{(e,f)_\lambda}\mathsf{P}_{\lambda}\biggl[\frac{1-c}{1-t}\biggr]\mathsf{Q}_{\lambda}\biggl[\frac{1-d}{1-t}\biggr]\\
&=\sum_{\lambda} q^{|\lambda|}\frac{(a,b,c,d)_\lambda}{(e,f)_\lambda}\mathsf{P}_{\lambda}\biggl[\frac{1}{1-t}\biggr].
\end{aligned}$$

Since $(t)_\lambda$ vanishes if $l(\lambda) > 1$, and since

$$(t)_{(k)}\mathsf{P}_{(k)}\biggl[\frac{1}{1-t}\biggr]=\frac{1}{(q)_k}$$

if follows that for $(b,c) = (q^{-N}, t)$

$$Z=\sum_{k=0}^{N}\frac{(a,d,q^{-N})_k q^k}{(q,e,adq^{1-N}/e)_k}=\frac{(e/a,e/d)_N}{(e,e/ad)_N},$$

where the second equality follows from the q-Pfaff–Saalschütz sum [2, Eq. (II.12)].

To make the same $(b,c) = (q^{-N}, t)$ specialisation in the right-hand side of (5.16) we first replace the sum over λ by $\lambda \subseteq \nu$ (using the fact that $P_{\nu/\lambda}(X) = 0$ if $\lambda \nsubseteq \nu$). Then

$$\frac{(b)_\nu}{(b)_\lambda}=\prod_{s\in\nu-\lambda}\bigl(1-bq^{a'(s)}t^{-l'(s)}\bigr)$$

is well defined for $b = q^{-N}$. It thus follows that for $(b,c) = (q^{-N}, t)$ (5.16) simplifies to

$$\begin{aligned}
&\sum_{\lambda} q^{|\lambda|}\frac{(a,q^{-N})_\lambda}{(e,f)_\lambda}\mathsf{P}_{\lambda/\mu}(1)\mathsf{Q}_{\lambda/\nu}\biggl[\frac{1-d}{1-t}\biggr]\\
&\quad=\biggl(\frac{q}{t}\biggr)^{|\mu|}\frac{(a,q^{-N})_\mu}{(e/t,f/t)_\mu}\biggl(\frac{q}{d}\biggr)^{|\nu|}\frac{(a,q^{-N})_\nu}{(e/d,f/d)_\nu}\frac{(e/a,e/d)_N}{(e,e/ad)_N}\\
&\qquad\times\sum_{\lambda\subseteq\nu}\biggl(\frac{dt}{q}\biggr)^{|\lambda|}\frac{(e/dt,f/dt)_\lambda}{(a,q^{-N})_\lambda}\mathsf{P}_{\nu/\lambda}(1)\mathsf{Q}_{\mu/\lambda}\biggl[\frac{1-d}{1-t}\biggr]
\end{aligned}$$

for $adq^{1-N}=ef$. Eliminating f, taking the limit $N\to\infty$ and using (2.6) this results in the claim with $(b,c)\mapsto(d,e)$. □

We are now prepared to prove Theorem 5.1.

Proof of Theorem 5.1. We proceed by induction on $n:=|X|$. For $n=0$ we get the tautology

$$\frac{(b/c)_\mu}{(b/c)_\nu}\mathsf{Q}_{\mu/\nu}\left[\frac{a-c}{1-t}\right]=\frac{(b/c)_\mu}{(b/c)_\nu}\mathsf{Q}_{\mu/\nu}\left[\frac{a-c}{1-t}\right].$$

For $n\geqslant 1$ we compute the sum on the right-hand side of (5.3), assuming the formula is true for the alphabet Y. (Recall that $X=Y+z$.) Using the branching rule (5.2) and

$$\mathsf{Q}_{\lambda/\nu}\left[\frac{a-c}{1-t}\right]\mathsf{P}_{\lambda/\eta}(z)=a^{|\eta-\nu|}\mathsf{Q}_{\lambda/\nu}\left[\frac{1-c/a}{1-t}\right]\mathsf{P}_{\lambda/\eta}(az) \tag{5.17}$$

we obtain

$$\begin{aligned}S_{\mu,\nu}(X;a,b,c)&:=\sum_\lambda\frac{(b/c)_\lambda}{(b/c)_\nu}\mathsf{Q}_{\lambda/\nu}\left[\frac{a-c}{1-t}\right]\mathsf{R}_{\lambda/\mu}(X;b)\\&=\sum_{\lambda,\eta}a^{|\eta-\nu|}\frac{(b/c)_\lambda(bz/t)_\eta}{(b/c)_\nu(bz)_\lambda}\mathsf{Q}_{\lambda/\nu}\left[\frac{1-c/a}{1-t}\right]\mathsf{P}_{\lambda/\eta}(az)\mathsf{R}_{\eta/\mu}(Y;b).\end{aligned}$$

The sum over λ can be transformed by Proposition 5.9 with $(a,b,c)\mapsto(b/c,c/a,bz)$ and $\mu\mapsto\eta$. As a result

$$\begin{aligned}S_{\mu,\nu}(X;a,b,c)&=\frac{(cz,dz)_\infty}{(az,bz)_\infty}\sum_{\lambda,\eta}\frac{(b/c)_\eta(dz/t)_\lambda}{(b/c)_\lambda(dz)_\nu}\mathsf{Q}_{\eta/\lambda}\left[\frac{a-c}{1-t}\right]\mathsf{P}_{\nu/\lambda}(z)\mathsf{R}_{\eta/\mu}(Y;b)\\&=\frac{(cz,dz)_\infty}{(az,bz)_\infty}\sum_{\lambda}\frac{(dz/t)_\lambda}{(dz)_\nu}\mathsf{P}_{\nu/\lambda}(z)S_{\mu,\lambda}(Y,a,b,c),\end{aligned}$$

where once again we have used (5.17), and where $d=ab/c$. By the induction hypothesis, $S_{\mu,\lambda}(Y,a,b,c)$ may be replaced by the right-hand side of (5.3) with $(X,\lambda,\nu)\mapsto(Y,\eta,\lambda)$, leading to

$$S_{\mu,\nu}(X;a,b,c)=\prod_{x\in X}\frac{(cx,dx)_\infty}{(ax,bx)_\infty}\sum_{\lambda,\eta}\frac{(b/c)_\mu(dz/t)_\lambda}{(b/c)_\eta(dz)_\nu}\mathsf{Q}_{\mu/\eta}\left[\frac{a-c}{1-t}\right]\mathsf{P}_{\nu/\lambda}(z)\mathsf{R}_{\lambda/\eta}(Y;d).$$

One final application of (5.2) results in

$$S_{\mu,\nu}(X;a,b,c)=\prod_{x\in X}\frac{(cx,dx)_\infty}{(ax,bx)_\infty}\sum_{\eta}\frac{(b/c)_\mu}{(b/c)_\eta}\mathsf{Q}_{\mu/\eta}\left[\frac{a-c}{1-t}\right]\mathsf{R}_{\nu/\eta}(X;d),$$

which is the right-hand side of (5.3) with $\lambda\mapsto\eta$. □

6. Transformation formulas for $\mathfrak{sl}_n$ basic hypergeometric series

In this final section we prove a number of additional identities for basic hypergeometric series involving the function $\mathsf{R}_\lambda(X;b)$. For easy comparison with known results for one-variable basic hypergeometric series we define

$$ {}_r\Phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z;X\right]:=\sum_\lambda \frac{(a_1,\dots,a_r)_\lambda}{(b_1,\dots,b_{s-1})_\lambda}\bigl((-1)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\bigr)^{s-r+1}z^{|\lambda|}\mathsf{R}_\lambda(X;b_s), $$

where X is a finite alphabet. There is some redundancy in the above definition since, by Lemma 4.1, $z^{|\lambda|}\mathsf{R}_\lambda(X;b_s)=\mathsf{R}_\lambda(zX;b_s/z)$. Recalling (4.4) it follows that for $X=t^{1-n}\langle 0\rangle=\{1,t^{-1},\dots,t^{1-n}\}$ the above series simplifies to a principally specialised basic hypergeometric series with Macdonald polynomial argument:

$$ \begin{aligned} {}_r\Phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};zt^{n-1};t^{1-n}\langle 0\rangle\right]&=\sum_\lambda \frac{(a_1,\dots,a_r)_\lambda}{(b_1,\dots,b_s)_\lambda}\bigl((-1)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\bigr)^{s-r+1}\mathsf{P}_\lambda\bigl(\langle 0\rangle\bigr)\\ &=:{}_r\Phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\langle 0\rangle\right], \end{aligned} $$

see [4,15].

Since $\mathsf{R}_\lambda(X;b)$ vanishes if $l(\lambda)>|X|$, and recalling $\mathsf{R}_{(k)}(z;b)=z^k/(bz)_k$ (see (4.4)), it follows that

$$ \begin{aligned} {}_r\Phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z;\{1\}\right]&=\sum_{k=0}^\infty \frac{(a_1,\dots,a_r)_k}{(b_1,\dots,b_s)_k}\bigl((-1)^kq^{\binom{k}{2}}\bigr)^{s-r+1}z^k\\ &={}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right], \end{aligned} $$

with on the right the standard notation for one-variable basic hypergeometric series (with dependence on the base q suppressed). The reader is warned that only in this degenerate case do the "lower-parameters" $b_1,\dots,b_s$ enjoy full $\mathfrak{S}_s$ symmetry. Using the above notation the q-Gauss sum (5.7) may be stated as

$$ {}_2\Phi_1\left[\begin{matrix}a,b\\ c\end{matrix};\frac{c}{ab};X\right]=\prod_{x\in X}\frac{(cx/a,cx/b)_\infty}{(cx,cx/ab)_\infty}. \tag{6.1} $$

To generalise this result we first prove the following transformation formula.

Theorem 6.1. *For $f=de/bc$ and μ a partition,*

$$ \begin{aligned} &\sum_\lambda \frac{(a,b)_\lambda}{(d)_\lambda}\mathsf{Q}_{\lambda/\mu}\left[\frac{f/a-cf/a}{1-t}\right]\mathsf{R}_\lambda(X;e)\\ &\quad=\frac{(b)_\mu}{(d/c)_\mu}\left(\prod_{x\in X}\frac{(fx,ex/a)_\infty}{(ex,fx/a)_\infty}\right)\sum_\lambda\frac{(a,d/c)_\lambda}{(d)_\lambda}\mathsf{Q}_{\lambda/\mu}\left[\frac{e/a-cf/a}{1-t}\right]\mathsf{R}_\lambda(X;f). \end{aligned} $$

Note that the substitution $(b, c, e, f) \mapsto (d/c, d/b, f, e)$ interchanges the left- and right-hand sides.

Proof of Theorem 6.1. We replace $\mu \mapsto \nu$ and then rename the summation index λ on the right as μ. By the Pieri formula of Theorem 5.2 (with $(a, b, c) \mapsto (bc/d, e, a)$) the term

$$\mathsf{R}_\mu(X; f) \prod_{x \in X} \frac{(fx, ex/a)_\infty}{(ex, fx/a)_\infty}$$

on the right-hand side can be expanded as

$$\sum_\lambda \frac{(a)_\lambda}{(a)_\mu} \mathsf{Q}_{\lambda/\mu}\left[\frac{f/a - e/a}{1-t}\right] \mathsf{R}_\lambda(X; e),$$

resulting in

$$\begin{aligned}
&\sum_\lambda \frac{(a, b)_\lambda}{(d)_\lambda} \mathsf{Q}_{\lambda/\nu}\left[\frac{f/a - cf/a}{1-t}\right] \mathsf{R}_\lambda(X; e) \\
&\quad = \frac{(b)_\nu}{(d/c)_\nu} \sum_{\lambda,\mu} \frac{(a)_\lambda (d/c)_\mu}{(d)_\mu} \mathsf{Q}_{\lambda/\mu}\left[\frac{f/a - e/a}{1-t}\right] \mathsf{Q}_{\mu/\nu}\left[\frac{e/a - cf/a}{1-t}\right] \mathsf{R}_\lambda(X; e).
\end{aligned}$$

By (5.15) with $(a, b, c, d) \mapsto (d/c, b, d, e/ab)$ the sum over μ can be carried out, completing the proof. □

For $\mu = 0$ Theorem 6.1 simplifies to a multiple analogue of the q-Kummer–Thomae–Whipple formula [2, Eq. (III.9)] (corresponding to the formula below when $X = \{1\}$).

Corollary 6.2 *($\mathfrak{sl}_n$ q-Kummer–Thomae–Whipple formula). For $f = de/bc$,*

$${}_3\Phi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; \frac{f}{a}; X\right] = \left(\prod_{x \in X} \frac{(fx, ex/a)_\infty}{(ex, fx/a)_\infty}\right) {}_3\Phi_2\left[\begin{matrix} a, d/b, d/c \\ d, f \end{matrix}; \frac{e}{a}; X\right]. \tag{6.2}$$

For $c = d$ this reduces to the q-Gauss sum (6.1).

If we let $c, d \to 0$ in Theorem 6.1 such that $d/c = bf/e$ and then replace $(e, f) \mapsto (c, az)$, we find

$$\begin{aligned}
&\sum_\lambda z^{|\lambda - \mu|} \frac{(a, b)_\lambda}{(a, b)_\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathsf{R}_\lambda(X; c) \\
&\quad = \left(\prod_{x \in X} \frac{(cx/a, azx)_\infty}{(cx, zx)_\infty}\right) \sum_\lambda \left(\frac{c}{a}\right)^{|\lambda - \mu|} \frac{(a, abz/c)_\lambda}{(a, abz/c)_\mu} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mathsf{R}_\lambda(X; az).
\end{aligned}$$

For $\mu = 0$ this is a multiple analogue of Heine's ${}_2\phi_1$ transformation [2, Eq. (III.2)].

Corollary 6.3 *($\mathfrak{sl}_n$ Heine transformation). We have*

$${}_2\Phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; z; X\right] = \left(\prod_{x \in X} \frac{(cx/a, azx)_\infty}{(cx, zx)_\infty}\right) {}_2\Phi_1\left[\begin{matrix} a, abz/c \\ az \end{matrix}; \frac{c}{a}; X\right].$$

Acknowledgment

We thank Eric Rains for helpful comments on a preliminary draft of this paper.

References

[1] T.H. Baker, P.J. Forrester, Transformation formulas for multivariable basic hypergeometric series, Methods Appl. Anal. 6 (1999) 147–164.
[2] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Encyclopedia Math. Appl., vol. 96, Cambridge University Press, Cambridge, 2004.
[3] J. Kaneko, Selberg integrals and hypergeometric functions associated with Jack polynomials, SIAM J. Math. Anal. 24 (1993) 1086–1110.
[4] J. Kaneko, q-Selberg integrals and Macdonald polynomials, Ann. Sci. Éc. Norm. Super. 29 (1996) 583–637.
[5] J. Kaneko, A triple product identity for Macdonald polynomials, J. Math. Anal. Appl. 200 (1996) 355–367.
[6] F. Knop, Symmetric and non-symmetric quantum Capelli polynomials, Comment. Math. Helv. 72 (1997) 84–100.
[7] A. Korányi, Hua-type integrals, hypergeometric functions and symmetric polynomials, in: International Symposium in Memory of Hua Loo Keng, vol. II, Springer, Berlin, 1991, pp. 169–180.
[8] A. Lascoux, Symmetric Functions and Combinatorial Operators on Polynomials, CBMS Reg. Conf. Ser. Math., vol. 99, American Mathematical Society, Providence, Rhode Island, 2003.
[9] A. Lascoux, E.M. Rains, S.O. Warnaar, Nonsymmetric interpolation Macdonald polynomials and $\mathfrak{gl}_n$ basic hypergeometric series, Transform. Groups 14 (2009) 613–647.
[10] M. Lassalle, Une formule du binôme généralisée pour les polynômes de Jack, Rendus Acad. Sci. Paris Sér. I 310 (1990) 253–256.
[11] M. Lassalle, Coefficients binomiaux généralisés et polynômes de Macdonald, J. Funct. Anal. 158 (1998) 289–324.
[12] M. Lassalle, Quelques valeurs prises par les polynômes de Macdonald décalés, Ann. Inst. Fourier 49 (1999) 543–561.
[13] I.G. Macdonald, A new class of symmetric functions, Sém. Lothar. Combin. B20a (1988), 41 pp.
[14] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford University Press, New York, 1995.
[15] I.G. Macdonald, Hypergeometric series II, unpublished manuscript.
[16] S.C. Milne, Summation theorems for basic hypergeometric series of Schur function argument, in: A.A. Gonchar, E.B. Saff (Eds.), Progress in Approximation Theory, in: Springer Ser. Comput. Math., vol. 19, Springer, New York, 1992, pp. 51–77.
[17] K. Mimachi, M. Noumi, A reproducing kernel for nonsymmetric Macdonald polynomials, Duke Math. J. 91 (1998) 621–634.
[18] A. Okounkov, Binomial formula for Macdonald polynomials and applications, Math. Res. Lett. 4 (1997) 533–553.
[19] A. Okounkov, (Shifted) Macdonald polynomials: q-integral representation and combinatorial formula, Compos. Math. 112 (1998) 147–182.
[20] A. Okounkov, BC-type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials, Transform. Groups 3 (1998) 181–207.
[21] A. Okounkov, Combinatorial formula for Macdonald polynomials and generic Macdonald polynomials, Transform. Groups 8 (2003) 293–305.
[22] A. Okounkov, G. Olshanski, Shifted Jack polynomials, binomial formula, and applications, Math. Res. Lett. 4 (1997) 69–78.
[23] E.M. Rains, BC_n-symmetric polynomials, Transform. Groups 10 (2005) 63–132.
[24] E.M. Rains, BC_n-symmetric abelian functions, Duke Math. J. 135 (2006) 99–180.
[25] E.M. Rains, Transformations of elliptic hypergeometric integrals, Ann. of Math. (2) 171 (2010) 169–243.
[26] E.M. Rains, Elliptic Littlewood identities, arXiv:0806.0871.
[27] S. Sahi, Interpolation, integrality, and a generalization of Macdonald's polynomials, Int. Math. Res. Not. IMRN (1996) 457–471.
[28] S.O. Warnaar, q-Selberg integrals and Macdonald polynomials, Ramanujan J. 10 (2005) 237–268.
[29] S.O. Warnaar, Bisymmetric functions, Macdonald polynomials and $\mathfrak{sl}_3$ basic hypergeometric series, Compos. Math. 144 (2008) 271–303.
[30] S.O. Warnaar, A Selberg integral for the Lie algebra A_n, Acta Math. 203 (2009) 269–304.
[31] S.O. Warnaar, The $\mathfrak{sl}_3$ Selberg integral, Adv. Math. 224 (2010) 499–524.
[32] Z.M. Yan, Generalized hypergeometric functions, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990) 349–354.
[33] Z.M. Yan, A class of generalized hypergeometric functions in several variables, Canad. J. Math. 44 (1992) 1317–1338.